MATRICULATION ALGEBRA

WITH NUMEROUS EXAMPLES

For the use of Schools and Colleges

Based on the New Regulations of the Calcutta University

BY

GAURI SANKAR DE, M.A.,

PROFESSOR OF MATHEMATICS, GENERAL ASSEMBLY'S INSTITUTION
AND FELLOW OF THE CALCUTTA UNIVERSITY.

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over the other with a line between them, in the form of a *fraction*. The result is called the **quotient**.

Thus, $a \div b$ and $\frac{a}{b}$ both mean that the number denoted by a is to be divided by the number denoted by b. If $a=6$ and $b=2$, then $a \div b$ or $\frac{a}{b}=3$. Also $\frac{a+b}{c-d}$ means that the sum of the numbers denoted by a and b is to be divided by the difference of the numbers denoted by c and d; that, if $a=8$, $b=4$, $c=5$, and $d=2$, then $\frac{a+b}{c-d}=\frac{8+4}{5-2}=\frac{12}{3}=4$.

Sometimes the symbol / is used to denote the operation of division. Thus $8/5=8 \div 5=\frac{8}{5}$.

14. Any collection of algebraical symbols (letters and figures), together with the signs, is called an **algebraical expression**, or briefly an **expression**.

Thus, $5ab+ad-3bc$ is an *expression*.

15. The **numerical value** of an algebraical expression is the number found when a certain value is given to each letter, and the operations carried out as represented by the signs. It is worth re remembering that if one of the quantities forming a product is 0, the whole product is 0, and that 0 divided by any quantity which is not zero is 0.

16. The *numerical values* of the following expressions will illustrate the use of the *principal signs* explained above.

Ex. 1. If $a=8$, $b=4$, $c=3$, $d=2$, then the numerical value of

(1) $4a+3b=4\times 8+3\times 4=32+12=44$.

(2) $6a+9b-4c=6\times 8+9\times 4-4\times 3=48+36-12=84-12=72$.

(3) $7a-3b+4c-2d=7\times 8-3\times 4+4\times 3-2\times 2=56-12+12-4$
$=68-16=52$.

Ex. 2. If $a=1$, $b=2$, $c=3$, $d=4$, $e=0$, then the value of

(1) $6ac-3bc+7ad=6\times 1\times 3-3\times 2\times 3+7\times 1\times 4$
$=18-18+28=28$.

(2) $6abc+3bcd-6b-2ade=6\times 1\times 2\times 3+3\times 2\times 3\times 4-6\times 2$
$-2\times 1\times 4\times 0=36+72-12-0=108-12=96$.

Ex. 3. If $a=24$, $b=8$, $c=4$, $d=6$, then the value of

(1) $\frac{9bcd}{4a}=\frac{9\times 8\times 4\times 6}{4\times 24}=\frac{1728}{96}=18$.

(2) $\dfrac{3a+2b}{5c-3d}=\dfrac{3\times24+2\times8}{5\times4-3\times6}=\dfrac{72+16}{20-18}=\dfrac{88}{2}=44.$

Exercise I.

1. If $a=$ [illegible], $b=2$, $c=4$, $d=8$, find the numerical values of :—

(1) $5bd$. (2) $4ac$. (3) $7abc$. (4) $7a+3b$.

(5) $9b-2c+d$. (6) $6a-2b-3c$. (7) $5c-7b$. (8) $15a-2b-5d$.

2. If $a=1$, $b=2$, $c=3$, $d=4$, find the values of :—

(1) $3bc-2ad$. (2) $ab+bc+3cd$.

(3) $16ac-3bc+7ad$. (4) $2bc+6ad-2dc$.

(5) $9ac+5ab-2ad-4bc$. (6) $3ac-bc-cd+4ad$.

(7) $3ac-2bc-cd+4acd$. (8) $3bc+2abd-bd-2abc$.

3. If $a=24$, $b=8$, $c=4$, $d=6$, $e=2$, find the values of :—

(1) $\dfrac{6c}{d}$. (2) $\dfrac{10a}{4e}$. (3) $\dfrac{ac}{bd}$. (4) $\dfrac{18be}{3cd}$. (5) $\dfrac{6de}{ac}$.

(6) $\dfrac{10b-2c}{2d+3c}$. (7) $\dfrac{15c-3b}{4d-3e}$. (8) $\dfrac{6a}{4c}-\dfrac{9c}{2d}$. (9) $\dfrac{3a}{d}+\dfrac{12b}{a}$.

4. If $a=6$, $b=5$, $c=4$, $d=3$, $e=2$, $f=1$, and $g=0$, find the numerical values of the following expressions :—

(1) $3b-4a-6c+7d+2e-4g$. (2) $-3a+2b+3c-2e+f$.

(3) $ab+5bc-4de+5fg$. (4) $4ag-3bf+4ce-ad$.

(5) $-3ab-2ac+4bc-abc$. (6) $abcd-2bcde+3cdef-4defg$.

5. If $a=3$, $b=4$, $c=5$, $x=2$, $y=8$, find the values of :—

(1) $\dfrac{abc}{3x}+\dfrac{5xy}{2c}$. (2) $\dfrac{12xy}{a+c}+\dfrac{6ab}{2x}$.

(3) $\dfrac{8a-b}{x+y}+\dfrac{10x-y}{c-a}$. (4) $\dfrac{18ax}{a+b+c}+\dfrac{4by}{a-b+c}$.

(5) $\dfrac{6ab}{y}-\dfrac{12xy}{a}+\dfrac{3bc}{x}$. (6) $\dfrac{2abc}{x}+\dfrac{6xy}{ab}+\dfrac{4bc}{y}$.

(7) $\dfrac{2a-4b+6c}{5x-y}-\dfrac{6a-4b}{2x-y+c}$. (8) $\dfrac{4a+4b+4c}{2x+y}+\dfrac{20x-4y}{4b-4a}$.

6. Find the value of $\dfrac{3a+2b}{b}-\dfrac{2b+3c}{7a}+\dfrac{2ab-c}{a}-\dfrac{5ad}{a+c}$,

when $a=1$, $b=3$, $c=5$ and $d=0$.

17. The number, whether positive or negative, prefixed to any algebraical quantity showing how many times that quantity is to be added to itself, is called its **numerical coefficient.**

Thus, in $3a$, 3 is the *numerical coefficient* of a and $3a$ means $a+a+a$; in $-7ax$, -7 is the *coefficient* of ax.

18. If no number is expressed, the coefficient is understood, being 1; thus a means $1a$; ab means $1ab$.

19. The term **literal coefficient** is sometimes applied to a coefficient which is represented by a letter or letters instead of a number.

Thus, in ab, a is the *literal coefficient* of b; and in abc, a may be termed the *coefficient* of bc, or b the *coefficient* of ac, or c the *coefficient* of ab.

20 When any quantity is multiplied by *itself* any number of times, the product is called a **power** of the quantity, and is briefly expressed by writing down the quantity, with a small figure above it to the right denoting the number of times it is repeated.

Thus, instead of $a \times a$, we write a^2, and a^2 is called the *second power* of a. Similarly, we write a^3 to denote $a \times a \times a$, and a^3 is called the *third power* of a, and so on.

21 The small figure of the above Art. in any case is called the **index** or **exponent** of the corresponding power.

Thus, in a^3 the *index* is 3; in a^4 the *exponent* is 4; in a^n the *exponent* is n.

22. When a quantity has no exponent, 1 is always understood; thus, $a=a^1$, and denotes that a is taken once as a factor.

23. The student must carefully notice the distinction between a *coefficient* and an *exponent*.

Thus, $3a$ means *three times a*; here 3 is a *coefficient*. But a^3 means *a times a times a*; here 3 is an *exponent*. Thus, $3a=a+a+a$, but $a^3=a \times a \times a$.

24. The second power of a, written a^2, is often called the **square** of a, or *a squared*; the third power of a, written a^3, is often called the **cube** of a, or *a cubed*; the fourth power of a, written a^4, is called *a to the fourth*, and so on.

25. Those parts of an expression, which are connected by the signs + or −, are called its **terms**, and the expression itself is said to be **simple** or **compound**, according as it contains one or more terms.

Thus, a^2, $2ab$ and $-5b^2$, are each *simple* quantities, and $a^2+2ab-5b^2$ is a *compound* quantity, whose *terms* are a^2, $+2ab$ and $-5b^2$.

26. A quantity of *one* term is called a **monomial**, of *two* terms, a **binomial**, of three, a **trinomial**, &c., and, generally of *more than two* terms, a **multinomial.**

Thus, a, ab, a *monomial*; $a+2b$, a *binomial*; $2a+b-c$, a *trinomial*, and $a+2b-3c+4d$, a *multinomial.*

27. Those parts of an expression which are connected by *Multiplication* are called its **factors.**

Thus, the factors of a^2 are a and a, those of $5ab$ are 5, a and b, those of $-3b^2$ are -3, b and b, or, as we should rather say, -3 and b^2. Of course we might include 1 as a factor in each case; thus, since $a^2=1\times a^2$, the factors of a^2 are 1 and a^2, and so of the rest.

Ex. 1. If $a=3$, $b=4$, $c=5$, $d=6$, the numerical values of

(1) $3c^2-2a^3=3\times 5^2-2\times 3^3=75-54=21.$

(2) $4a^3+3a^2b^2-3c^3=4\times 3^3+3\times 3^2\times 4^2-3\times 5^3=108+432-375$
$=540-375=165.$

Ex. 2. If $a=1$, $b=2$, $c=3$, then the values of

(1) $\dfrac{b+c^3}{c^2-b^3}=\dfrac{2+3^3}{3^2-2^3}=\dfrac{2+27}{9-8}=\dfrac{29}{1}=29.$

(2) $\dfrac{a^2+3ab+ac+2b^2-2bc}{a+a^2-b^2+2bc-c^2}=\dfrac{1^2+3\times 1\times 2+1\times 3+2\times 2^2-2\times 2\times 3}{1+1^2-2^2+2\times 2\times 3-3^2}$

$=\dfrac{1+6+3+8-12}{1+1-4+12-9}=\dfrac{18-12}{14-13}=\dfrac{6}{1}=6.$

Exercise II.

1. If $a=3$, $b=4$, $c=5$, $d=6$, find the numerical values of

(1) $5a^2b^3$. (2) $3a^3a^3$. (3) $7ab^3c^2$. (4) $3a^2b^2+4c^2d^2$.

(5) $5b^2+2c^2+3d^2$. (6) $5a^2b^2c^2+4a^2d^2$. (7) $9b^2+2a^2c^2-3d^2$.

2. If $a=1$, $b=3$, $c=5$, $d=0$, find the values of

(1) $a^2+2b^2+3c^2+4d^2$. (2) $3a^2b+2b^2c-2a^2c+3b^2d$.

(3) $a^3-3a^2c+3ac^2-c^3$. (4) $a^4-4a^3b+6a^2b^2-4ab^3+b^4$.

(5) $\dfrac{a^3-c^2+2bc}{b^2-3a}$. (6) $\dfrac{a^2+b^2+c^2}{b+d+2}$. (7) $\dfrac{a^3-b^3+c^3}{5c+b^2-a^2}$.

(8) $\dfrac{a^3+b^3+c^3}{a^3+3a}$. (9) $\dfrac{3a^2+5b^2+2c^2}{a^2+d^2+1}$. (10) $\dfrac{a^2b+2a^2c+a^4d}{4a^2+b^3}$.

3. Find the value of $4abc^2-3a^2bc+\dfrac{2ab^2c}{2a+b+c}$,

when $a=1$, $b=3$, $c=5$.

4. Find the value of $\dfrac{a^2+b^2}{b+c}+\dfrac{a^2+c^2}{b^2+d}+\dfrac{b^2+c^2}{a+d}$,

when $a=1$, $b=2$, $c=3$, $d=6$.

5. If $x=1, y=2, z=3, p=4, q=6$, find the values of

(1) $\dfrac{x^3+y^3}{x+y}-\dfrac{q^3-p^3}{q-p}$. (2) $\dfrac{x^4-4x^3z+6x^2z^2-4xz^3+z^4}{y^4-4y^3z+6y^2z^2-4yz^3+z^4}$.

(3) $\dfrac{x^2+2xy+y^2}{x+y}-\dfrac{y^2+2yz+z^2}{y+z}+\dfrac{z^2+2pz+p^2}{z+p}$.

(4) $\dfrac{28}{x^2+y^2+z^2}+\dfrac{24}{p^2-z^2-y^2}+\dfrac{16}{x^2+q^2-z^2-p^2}$.

28. **Brackets**, ( ), { }, [ ], are employed to shew that all the quantities within them are to be treated as though forming but one quantity.

Thus, $a+(b+c)$ means that the sum of b and c is to be added to a, and $a-(b+c)$ means that the sum of b and c is to be taken from a. Again, $a-(b-c)$ is not the same as $a-b-c$; for, in this last both b and c are to be subtracted, whereas in the former it is the quantity, $b-c$, which is to be subtracted. Also $a(b+c)$ means that the sum of b and c is to be multiplied by a. Likewise $(a+b)(x-y)$ means that the sum of a and b is to be multiplied by the difference of x and y.

Hence, if $a=4$, $b=3$, $c=1$, we have

(1) $a-b-c=4-3-1=0$, and $a-(b-c)=4-(3-1)=4-2=2$.

(2) $4a-3b+2c=16-9+2=9$, and

$4a-(3b+2c)=16-(9+2)=16-11=5$.

(3) $2a+b-c=8+3-1=10$, $2(a+b)-c=2.7-1=14-1=13$,

and $2(a+b-c)=2(4+3-1)=2(7-1)=2\times 6=12$.

29. Sometimes, instead of brackets, a line is used, called a **vinculum**, and drawn above the quantities that are connected; thus $a-\overline{b-c}$ is the same as $a-(b-c)$.

30. The line, which separates the numerator and denominator of a fraction, is also a species of *vinculum*, corresponding, in fact, in *Division* to the bracket in *Multiplication*.

Thus, $\dfrac{a+b+c}{4}$ implies that the *whole* quantity $a+b+c$ is to be divided by 4, and might have been written $\frac{1}{4}(a+b+c)$.

Ex. 1. If $a=8$, $b=4$, $c=5$, $d=6$, $e=2$, then the values of

(1) $(a+b)^2-(c+d)^2=(8+4)^2-(5+6)^2=12.12-11.11=144-121=23.$

(2) $(d+e)^3=(6+2)^3=8.8.8=512.$

(3) $(a-b)^2(d-e)^3=(8-4)^2(6-2)^3=4^2\times4^3=16\times64=1024.$

Exercise III.

1. If $a=6$, $b=4$, $c=1$, $d=3$, $e=2$, find the values of

(1) $a+(b-c)$. (2) $a-(b+c)$. (3) $2a-b-c-d$.

(4) $2a-b-(d-c)$. (5) $2a-(b-c-d)$. (6) $9a-(3b+2c)$.

(7) $2(a+b+d)-e$. (8) $2(a-b)+d-c$. (9) $(b+c)-(a-d)$.

2. If $a=8$, $b=4$, $c=5$, $d=6$, $e=2$, find the values of

(1) $(b+d)(a-c)$. (2) $4a(b^2+c^2)$. (3) $3a^2(b+d)+2c(b^2-e^2)$.

(4) $(a+b)^2(c+d)^2$. (5) $(b+c+d)^2$. (6) $(a+b)^3-(c+d)^3$.

(7) $3b^2(a+d)+2a(a^2-d^2)$. (8) $4(a^2-b^2)(a-d)+3(c^2+d^2)(b-e)$.

3. If $a=10$, $b=4$, $c=3$, find the numerical values of

(1) $2b+\overline{a-3c}$. (2) $4a+b-\overline{3a+c}$. (3) $a+2b-c+\overline{a-2b+c}$.

4. Find the value of $3b(a^2b+cd^2)-e(ab^2+c^2d)$,

when $a=8$, $b=4$, $c=5$, $d=6$, $e=2$.

5. If $a=0$, $b=2$, $c=4$, $d=6$, find the values of

(1) $3a+(2b-c)^2+\{c^2-(2a+3b)\}+\{3c-(2a+3b)\}^2$.

(2) $3b+(2c-d)^2+\{3b-(2c-d)\}^2-\{3b-(2c-d)^2\}$.

(3) $\{a+(b+c)^2-d\}\{(a+b)^2+(d-$

31. The **square root** of a quantity is that quantity whose *square power* is equal to the given quantity.

Thus, the *square root* of 9 is 3, since $3^2=9$; the *square root* of a^2 is a; of 25 is 5.

So also the **cube, fourth,** &c. root of a quantity is that quantity whose *cube, fourth,* &c. *power* is equal to the given one.

Thus, the *cube root* of 125 is 5, since $5^3=125$; the *cube root* of a^3 is a; of $(a-b)^3$ is $(a-b)$.

32. The symbol used to denote a root is $\sqrt{}$ (a corruption of *r*, the first letter of the word *radix*), which, with the proper index on the left side of it, is set before the quantity whose root is expressed.

Thus, $\sqrt[2]{a^2}=a$, $\sqrt[3]{125}=5$, $\sqrt[5]{3125}=5$, $\sqrt[3]{a^3}=a$, &c

The index, however, is generally omitted in denoting the *square* root; thus $\sqrt{a}$ is written instead of $\sqrt[2]{a}$.

Ex. 1. Find the value of $\sqrt{(3abc)}$, when $a=2$, $b=3$, $c=8$.

$$\sqrt{(3abc)}=\sqrt{(3\times2\times3\times8)}=\sqrt{(9\times16)}=3\times4=12.$$

Ex. 2. Find the value of $c\sqrt{(b^2-3c)}+b\sqrt{(b^2+3c)}-\sqrt{(ac+1)}$, when $a=6$, $b=5$, $c=8$.

$$\begin{aligned}\text{The Exp.}&=8\sqrt{(5^2-3.8)}+5\sqrt{(5^2+3.8)}-\sqrt{(6.8+1)}\\&=8\sqrt{(25-24)}+5\sqrt{(25+24)}-\sqrt{(48+1)}\\&=8\times\sqrt{1}+5\sqrt{49}-\sqrt{49}=8\times1+5\times7-7=8+35-7=36.\end{aligned}$$

Ex. 3. Find the value of $4\sqrt{(a+b+2c+d)}+2\sqrt{(d-b)}+3\sqrt{(2c+3d-1)}$, when $a=0$, $b=2$, $c=4$, $d=6$.

$$\begin{aligned}\text{The Exp.}&=4\sqrt{(0+2+8+6)}+2\sqrt{(6-2)}+3\sqrt{(8+18-1)}\\&=4\sqrt{(16)}+2\sqrt{4}+3\sqrt{(25)}=4\times4+2\times2+3\times5\\&=16+4+15=35.\end{aligned}$$

Exercise IV.

1. If $a=4$, $b=5$, $c=9$, $d=2$, find the values of

(1) $\sqrt{(5abc)}$. (2) $4\sqrt{(a^2b^2)}$. (3) $\sqrt{(a^3b^4c)}$. (4) $\sqrt[3]{(9c^2)}$.
(5) $b\sqrt{(ad+2c-1)}$. (6) $\sqrt{(3ac+2bd-2a+1)}$. (7) $\sqrt{\{a(c-b)\}}$.

2. If $a=25$, $b=9$, $c=4$, $d=1$, find the values of

(1) $\sqrt{a}+2\sqrt{b}+3\sqrt{c}+4\sqrt{d}$. (2) $\sqrt{(4a)}+\sqrt{(9b)}+\sqrt{(16c)}-\sqrt{(25d)}$.
(3) $3\sqrt{a}+2\sqrt{(4b)}-4\sqrt{(9c)}+\sqrt{(16d)}$.
(4) $\sqrt[3]{(5a)}+2\sqrt[3]{(3b)}-\sqrt[3]{(2c)}+4\sqrt[3]{d}$. (5) $\sqrt{a^2}-2\sqrt[3]{b^3}+3\sqrt[4]{c^4}-4\sqrt[5]{d}$.
(6) $\sqrt{(bc)}+3\sqrt{(acd)}-4\sqrt{(b^2d)}+\sqrt{(c^2d^3)}$.

3. If $a=0$, $b=2$, $c=4$, $d=6$, find the values of

(1) $2\sqrt{(d-b)}+3\sqrt{(3d+2c-1)}+4\sqrt{(a+b+2c+d)}$.
(2) $3\sqrt[3]{(2b^3-a)}+2\sqrt[3]{(b^2+c^2+7)}-\sqrt[3]{\{2(b+c)^2-(b+d)^2\}}$.

4. Find the value of $\sqrt{\left(\frac{2a}{b}\right)}+\frac{a+\sqrt{c^2}}{\sqrt{d^2}}+2\sqrt{a^2}$,

when $a=2$, $b=4$, $c=3$, $d=5$.

5. If $a=2$, $b=0$, $c=3$, $d=4$, find the numerical values of

(1) $\sqrt{\{(8a+b)(b+bc)\}}+\sqrt{\{(5c-2d)(2b+7d)\}}$.

(2) $5\sqrt{\left(\frac{a^2+b^2}{a^2c^2}+\frac{c^2d^2-1}{c^2}\right)}+2\sqrt{\left\{\frac{36(c^2+d^2)}{c^2d^2}\right\}}$

---

33. It is very necessary that the student should learn at once to distinguish between *terms* and *factors*.

Thus, $3a+b-c$ is a *compound* quantity of three *terms*, $3a$, b, and $-c$; $3(a+b)-c$ is one of two *terms* only, $3(a+b)$ and $-c$, of which the former, $3(a+b)$, consists of two *factors*, 3 and $a+b$. the factor, $a+b$, being itself a compound quantity of two terms; and so also $3(a+b-c)$ is a *simple* quantity or *single* term, of two factors, 3 and $(a+b-c)$, of which the latter is itself a *compound* quantity of three *terms*.

Hence, **terms** are the quantities which make up an expression by way of *Addition* or *Subtraction*, whereas **factors**, by way of *Multiplication*.

34. Each of the letters which occur in a product is called a **dimension** of the product, and the number of letters is called the **degree** of the product.

Thus, a^3b^2c or $a\times a\times a\times b\times b\times c$ is said to be of six *dimensions*, or of the sixth *degree*.

A numerical coefficient is not reckoned; thus, a^3b^2c and $3a^3b^3$ are both of six *dimensions*.

35. An expression is said to be **homogeneous** when all its terms are of the same dimensions.

Thus, $3a^3+4a^2b-7abc$ is *homogeneous*, for each term is of three dimensions, but $3a^4+4ab^2+5a^2bc^2$ is not *homogeneous*.

36. Algebraical quantities are said to be **like** or **unlike**, according as they contain the *same* or *different* combinations of letters.

Thus, a and $3a$, $-5a^2b$ and $7a^2b$, $3a^2bc$ and $-a^2bc$, are pairs of *like* quantities; a^3 and a^2, $3ab$ and $-7a$, $5ab^2$ and $5a^2b$, of *unlike* quantities.

37. The sign $\sim$ is used to denote that the less of two quantities is to be taken from the greater, when it is not known which is the greater.

Thus, $a\sim b$ denotes the *difference* between a and b.

38. The sign $\therefore$ stands for **then** or **therefore**, and $\because$ for **since** or **because**.

II. SUBSTITUTIONS.

Ex. 1. If $a=4$, $b=3$, $c=2$, $d=1$, find the values of

(1) $\frac{3}{4}a^2bc^3$. (2) $\dfrac{abc(a+b-c)+bcd(b+c-d)}{ad(a-d)+bc(b-c)}$.

(1) The Exp. $=\frac{3}{4}.4^2.3.2^3=\frac{3}{4}\times16\times3\times8=3\times4\times3\times8=288$.

(2) The Exp. $=\dfrac{4\cdot3\cdot2(4+3-2)+3.2.1(3+2-1)}{4.1(4-1)+3.2(3-2)}$

$$=\frac{24(7-2)+6(5-1)}{4\times3+6\times1}=\frac{24\times5+6\times4}{12+6}=\frac{120+24}{18}=\frac{144}{18}=8.$$

Ex. 2. If $a=4$, $b=3$, $c=2$, $d=1$, find the values of

(1) $\dfrac{a^2+b^2+c^2}{a^2}+\dfrac{a^2-ab(a-b)}{2ab}-\dfrac{4(a^2+b^2)}{a^2c^2}$.

(2) $\dfrac{ab^2+ad^2}{abd}-\dfrac{c^2-d^2+1}{(a+b)(c-d)}$.

(1) The Exp. $=\dfrac{4^2+3^2+2^2}{4^2}+\dfrac{4^2-4.3(4-3)}{2.4.3}-\dfrac{4(4^2+3^2)}{4^2.2^2}$

$$=\frac{16+9+4}{16}+\frac{16-12.1}{24}-\frac{4(16+9)}{16\,4}=\frac{29}{16}+\frac{16-12}{24}-\frac{4.25}{64}.$$

$$=\frac{29}{16}+\frac{4}{24}-\frac{100}{64}=1\frac{13}{16}+\frac{1}{6}-1\frac{9}{16}=\frac{1}{4}+\frac{1}{6}=\frac{5}{12}.$$

(2) The Exp. $=\dfrac{4\cdot3^2+4.1^2}{4\cdot3.1}-\dfrac{2^2-1^2+1}{(4+3)(2-1)}=\dfrac{4\cdot9+4.1}{12}-\dfrac{4-1+1}{7.1}$

$$=\frac{36+4}{12}-\frac{4}{7}=\frac{40}{12}-\frac{4}{7}=\frac{10}{3}-\frac{4}{7}=\frac{58}{21}=2\frac{16}{21}.$$

Ex. 3. If $a=5$, $b=2$, $c=3$, find the value of

$$\sqrt{\{\sqrt{(2a^2-3b^2-b)}+\sqrt{(3c^2-2b^2+2c)}+\sqrt{(abc-5)}\}}.$$

The Exp. $=\sqrt{\{\sqrt{(2.5^2-3.2^2-2)}+\sqrt{(3.3^2-2.2^2+2.3)}+\sqrt{(5.2.3-5)}\}}$

$=\sqrt{\{\sqrt{(50-12-2)}+\sqrt{(27-8+6)}+\sqrt{(30-5)}\}}$

$=\sqrt{\{\sqrt{(50-14)}+\sqrt{(33-8)}+\sqrt{(25)}\}}$

$=\sqrt{\{\sqrt{(36)}+\sqrt{(25)}+\sqrt{(25)}\}}=\sqrt{\{6+5+5\}}=\sqrt{(16)}=4.$

Ex. 4. If $x=\frac{1}{2}$, $y=\frac{1}{3}$, $z=\frac{1}{4}$, find the numerical value of

$$3xy+\left\{\left(\frac{z}{1-6yz}+\frac{2}{z}\right)\div\left(\frac{4x^2-\sqrt{z}}{4x^2+\sqrt{z}}\right)\right\}.$$

$$\text{The Exp.} = 3\cdot\tfrac{1}{2}\cdot\tfrac{1}{3} + \left\{\left(\frac{\frac{1}{4}}{1-6\cdot\frac{1}{3}\cdot\frac{1}{4}} + \frac{2}{\frac{1}{4}}\right) \div \left(\frac{4\cdot(\frac{1}{2})^2 - \sqrt{\frac{1}{4}}}{4\cdot(\frac{1}{2})^2 + \sqrt{\frac{1}{4}}}\right)\right\}$$

$$= \tfrac{1}{2} + \left(\frac{\frac{1}{4}}{1-\frac{1}{2}} + \frac{2}{\frac{1}{4}}\right) \div \left(\frac{1-\frac{1}{2}}{1+\frac{1}{2}}\right) = \tfrac{1}{2} + \left(\frac{\frac{1}{4}}{\frac{1}{2}} + \frac{2}{\frac{1}{4}}\right) \div \left(\frac{\frac{1}{2}}{\frac{3}{2}}\right)$$

$$= \tfrac{1}{2} + (\tfrac{1}{4}\times 2 + 2\times 4) \div (\tfrac{1}{2}\times\tfrac{2}{3}) = \tfrac{1}{2} + (\tfrac{1}{2}+8) \div \tfrac{1}{3}$$

$$= \tfrac{1}{2} + 8\tfrac{1}{2} \div \tfrac{1}{3} = \tfrac{1}{2} + \tfrac{17}{2}\times 3 = \tfrac{1}{2} + \tfrac{51}{2} = \tfrac{52}{2} = 26.$$

Ex. 5. Shew that $(b+c+d)(b+c-d)(b+d-c)(c+d-b) = 4b^2c^2 - \{d^2-(b^2+c^2)\}^2$, when $b=2$, $c=3$, $d=4$.

The left side $=(2+3+4)(2+3-4)(2+4-3)(3+4-2) = 9.1.3.5 = 135$.

The right side $= 4.2^2.3^2 - \{4^2-(2^2+3^2)\}^2 = 4.4.9 - \{16-(4+9)\}^2$

$= 144 - \{16-13\}^2 = 144 - 3^2 = 144 - 9 = 135.$

Exercise V.

1. If $a=3$, find the numerical values of

$3a$; a^3 ; $3a^3$; $(3a)^2$; $3+a$; $3-a$; $3/a$.

2. When $x=2, y=3$, find the values of

$3x+2y$; $3x-2y$; $(3x+2y)^2$; $3x^2+2y^2$; $(3x)^2+(2y)^2$.

3. When $a=9$, $b=3$, find the values of

a^3-3ab^2 ; $a^2(a-3b)(a+3b)$; $a^3(a-b)^2$.

4. If $x=4, y=6, z=8, p=3, q=7, w=1$, find the values of

(1) $\dfrac{x}{y}+\dfrac{y}{z}+\dfrac{2z}{p}$. (2) $\dfrac{2z-2q}{y}+\dfrac{2p+3w}{3y-2z}-\dfrac{3p-x}{3p+w}-\dfrac{p}{x}+\dfrac{y}{z}$.

(3) $2x-3y+4z-\dfrac{2q+2x-z+2y}{2px}$. (4) $\dfrac{2y}{x}+\dfrac{3y-x}{y}+\dfrac{(x+y)(y-x)}{xy}$.

(5) $\dfrac{xy+pz+wq}{pz-xy+wq}+\dfrac{pqw-xyz+z}{2q-2px}-\dfrac{pq}{xyw}$.

5. If $a=0$, $b=1$, $c=2$, $x=3, y=4, z=5$, find the value of

$a^3+3a^2b+2b^2c+7ax^2+cxy+3byz$.

6. If $a=4$, $b=2$, $c=0$, $x=5, y=3, z=1$, find the value of

$2cx-bxz+2axy+3byz-4ac$.

7. If $a=5$, $b=2$, $c=1$, find the values of

(1) $\dfrac{(2a+b)^2\times(2a-2)^2\div(a-3)^2\times(b-1)^2}{ab}$.

(2) $\dfrac{(a+3)^2}{ab}+\dfrac{(a^2-3)^2}{ab^2}+\dfrac{2(a+b)^2}{a-b}-\dfrac{a^2b^2}{ab(b-c)}$.

(3) $\dfrac{a^2+b^2-c^2}{2ab+2a-b}+\dfrac{(2a+b+c)^2}{(a^2-6b)^2}+\left(\dfrac{a+1}{a^2-1}\right)^2\times\dfrac{5a+3c+1}{a^2-c+5}$.

(4) $a^2+a^2b^2+2+\dfrac{a+a^3b+b^3}{2b^3}-\dfrac{a}{4b}$.

8. Find the values of :—

(1) $\dfrac{11a}{8a-7b}+\dfrac{7c}{11a-3b}-\dfrac{10b}{7c-5a}$, when $a=4$, $b=3$, $c=5$.

(2) $\dfrac{a^2+b^2+c^2}{b+c}-\dfrac{a^2+b^2-c^2}{a+b+c}$, when $a=4$, $b=5$, $c=6$.

(3) $(a^2+b^2)(a-b)^2-b(c-b)^3+(a-c)^4$, when $a=4$, $b=1$, $c=3$.

(4) $x(y+z)+(x^2+y^2+z^2)+(x+y+z)^2$, if $x=1$, $y=2$, $z=3$.

(5) $\dfrac{a^2(b^2+c^2+3d)}{b+c}+\dfrac{2a(c+2d+2e)}{de}+\dfrac{bcd(1-c)}{b+c}$,

if $a=0$, $b=\frac{1}{2}$, $c=\frac{1}{3}$, $d=\frac{1}{4}$, $e=\frac{1}{5}$.

(6) $\dfrac{a^2-a-2}{8}-\sqrt{(a+2)}+4$, when $a=7$.

(7) $\sqrt{\left\{\dfrac{(a+b+c)(a-b+c)}{a+b-c}\right\}}$, when $a=4$, $b=5$, $c=6$.

(8) $\dfrac{a^3-b^3}{\sqrt{(a^2+b^2)}}-\dfrac{a^2-b^2}{\sqrt{(a-b)}}$, when $a=4$, $b=3$.

(9) $a-\{\sqrt{(a+1)}+2\}-\dfrac{a-\sqrt{(2a)}}{\sqrt{(a-4)}}$, when $a=4$.

(10) $\dfrac{ab-cd+d^2}{2ab+cd-d^2}+\dfrac{a-b+c}{a+b-c}$, when $a=2$, $b=4$, $c=3$.

(11) $\sqrt{\left\{\dfrac{3a-4b+6d}{6a+2c-11d}\times\dfrac{8b-7a+4d}{8a-3c+10d}\right\}}$, when $a=4$, $b=5$, $c=9$, $d=2$.

(12) $\sqrt{\left\{\dfrac{b^2+a^2+d^2}{b^2-a^2-d^2}-\dfrac{c(c^2-a^2-b^2)}{c^2+a^2-b^2}\right\}}$, when $a=4$, $b=5$, $c=9$, $d=2$.

9. If $a=1$, $b=2$, $c=3$, find the values of

(1) b^c. (2) a^c+b^a. (3) c^{a+2}. (4) 5^{a+b}. (5) 8^b.

(6) $\dfrac{a^c+b^c}{a^2-ab+b^2}$. (7) $\dfrac{a^a+b^b+c^c}{3a^2+b^2+c^2}$. (8) $\dfrac{c^c-b^c}{b^b+bc+c^b}$.

10. If $a=1$, $b=2$, $c=3$, $d=4$, shew that the numerical values of the following are equal.

(1) $\{d-(c-b+a)\}\{(d+c)-(b+a)\}$ and $d^2-(b^2+c^2)+a^2+2(bc-ad)$.

(2) $\{(b+c)-(d-a)\}^2+\{(c+d)-(b-a)\}^2+\{(b+d)-(c-a)\}^2$
$+(b+c+d-a)^2$ and $4(a^2+b^2+c^2+d^2)$.

(3) $d^2-(2d-c)c+\{2(d-c)+b\}b-\{2(d-c+b)-a\}c$ and
$\{(d-a)-(c-b)\}^2$.

(4) $\{(a+d)-(c-b)\}\{(a+c+d-b\}\{c-(d-a-b)\}\{(b+c+d-a)\}$ and
$4(ad+bc)^2-\{(a^2+d^2)-(b^2+c^2)\}^2$.

11. Find the value of $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$,
when $a=2$, $b=3$, $c=5$.

12. Find the value of $a^3-b^3-c^3-3abc$,
when $a=\cdot 05$, $b=\cdot 035$, $c=\cdot 015$.

13. Find the value of, when $a=\cdot 05$, $b=\cdot 03$, $c=\cdot 02$

$$\frac{1}{b^2+c^2-a^2}+\frac{1}{c^2+a^2-b^2}+\frac{1}{a^2+b^2-c^2}.$$

14. What are the numbers 3 and 4 called respectively in $4a^3$?

15. Find the value of $\frac{x-1}{x+1}+\frac{x+3}{x-3}-2.\frac{x+2}{x-2}$, when $x=5$.

CHAPTER II.

SYMBOLICAL EXPRESSIONS AND NEGATIVE QUANTITIES.

I. SYMBOLICAL EXPRESSIONS.

39. The student who has already become familiar in Arithmetic, must carefully notice the following examples of the expression in Algebraic symbols.

Since 4 Rupees $=(16\times 4)$ annas,

$\therefore$ a Rupees $=(16\times a)$ annas $=16a$ annas.

Similarly ; a Rupees $=192a$ pies.

Again, 240 annas $=(240\div 16)$ Rupees

$\therefore$ a annas $=(a\div 16)$ Rupees $=a/16$ Rupees.

x Rupees $+y$ annas $=(16x+y)$ annas.

If 2 be taken from 5 the result is $5-2$, that is 3.

So if y be taken from x the result is $x-y$.

The number which is 3 greater than 5 is $5+3$.

So the number which is 3 greater than x is $x+3$.

5 oranges at 2 pice each, cost (5×2) pice.

$\therefore$ x oranges at y pice each cost $(x \times y)$ pice $= xy$ pice.

A man walking 6 miles an hour, walks (4×6) miles in 4 hours.

Thus, he walks $(x \times 6)$ or $6x$ miles, in x hours.

Again, he walks 26 miles in $\frac{26}{6}$ hours.

$\therefore$ he walks x miles in $\dfrac{x}{6}$ hours.

Exercise VI.

1. By how much is x greater than y ?

2 What is the number which is 5 less than x ?

3. A has x Rupees, B has y annas, and c has z pice ; how many pice have they between them ?

4. If x be a whole number, what are the numbers next above and next below it ?

5. Ram had x oranges, out of which he gave Jadu y ; how many oranges has Ram now ?

6. Bepin has a marbles more than Sham, who has b marbles ; how many marbles has Bepin ?

7. Express a Rupees (i) in annas, (ii) pies, (iii) in pice, (iv) in quarter-rupees, (v) in two-anna pieces.

8. Express b yards (i) in feet, (ii) in inches.

9. Express b inches (i) in feet, (ii) in yards.

10. Each of a boys has x marbles ; how many marbles have they altogether ?

11. By how much is x greater than 13 ?

12. By how much is 13 greater than x ?

13. If I give 2 rupees to each boy, how many rupees do I give to x boys ? How many pice do I give them ?

14. If a book costs a shillings, how many can I buy for five shillings ? How many for five pounds ?

15. What is the total number of pies in a rupees and b annas ?

16. What is the cost of x maunds of sugar at y rupees per maund ? How many maunds for a rupees ?

17. If x rupees be equally divided amongst y boys, how much does each boy get ?

18. If I walk a miles an hour, how far do I walk :

(i) in 3 hours? (ii) in half-an-hour? (iii) x hours?

19. Express a square feet in square inches.

20. Two boys write an essay; one writes x lines and a words in a line; the other writes y lines and b words in a line; find the total number of words written between them.

21. Express in symbols the following :—

(1) The sum of x and y divided by z.

(2) Five times the square of z, multiplied by x minus y.

(3) Six times the cube of a, diminished by the fourth power of x, and the whole multiplied by the square of c.

(4) The square of a added to the cube of b divided by the excess of the sum of a and b over d.

(5) The square of the sum of a and b multiplied by the difference of c and d.

22. What is the coefficient of a^2 in $4a^2b$? of a^5 in $3a^5b^2c$? of x^3 in $\frac{3}{4}abx^3$? of y in ax^2y? of x^2 in a^2x^2?

23. What does the expression $a(b+c)$ denote?

24. What does $3a$ stand for? And what does a^3 mean?

25. Of how many dimensions does the expression (i) $7a^3b^4c$ consist? (ii) $3a^4b^3c^3$ consist? (iii) a^2xy^2 consist?

II. NEGATIVE QUANTITIES.

40. Any quantity to which a + sign is prefixed is called a **positive** quantity, and any quantity to which a − sign is prefixed is called a **negative** quantity.

41. The expression $3-5$ has no arithmetical meaning *i. e.*, we cannot subtract 5 from 3. In Algebra, however, such an expression has a meaning quite intelligible.

This idea may be made clearer by considering the following few examples.

(i) If a man whose income is *Rs.*100, spends *Rs.*70, he *saves* *Rs.*30. But, on the other hand, if his income be *Rs.*70 and spends *Rs.*100, he incurs a *debt* of *Rs.*30.

Thus, *Rs.*100 − *Rs.* 70 = + *Rs.*30,

and *Rs.* 70 − *Rs.*100 = − *Rs.*30.

(ii) If a man gains *Rs.*50 and then loses *Rs.*30, his total *gain* is *Rs.*20. But, on the other hand, if he gains *Rs.*30 and then loses *Rs.*50, the result of his trading is a *loss* of *Rs.*20.

Thus, $Rs.50 - Rs.30 = +Rs.20$,

and $Rs.30 - Rs.50 = -Rs.20$.

Moreover, if he loses Rs. 50 and then gains Rs. 30, the result of his trading is still a *loss* of Rs. 20.

Thus, $-$Rs. $50 +$ Rs. $30 = -$Rs. 20.

(iii) If a man sells first 5 horses and then again 3 horses, he has 8 horses *less* than he had at first.

Thus, -5 horses -3 horses $= -8$ horses.

(iv) If a man starting from a given place walks 100 yds. along a road, and then walks 30 yds. backwards, he will be 70 yds. from his starting place. But if he first walks 30 yds. and then 100 yds. backwards, he will still be 70 yds. from his starting place, but *on the opposite side of it.*

Thus, 100 yds. $-$ 30 yds. $= +70$ yds.,

and 30 yds. $-$ 100 yds. $= -70$ yds.

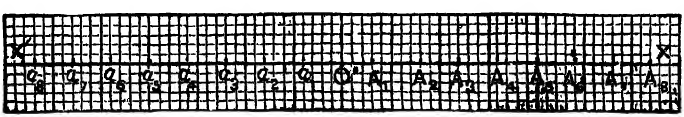
Hence, we see that $+70$ and -70 are the exact *opposite* of one another. In considering a man's money $+Rs.30$ means an ***increase***, whilst $-Rs.30$ means an equal *decrease*; $+70$ yds. and -70 yds. means 70 yds. in *opposite directions*, and so on.

(v) Again, in periods of time, if a date *after* A. D. be denoted by a number with a $+$ sign prefixed, a date *before* B. C. will be denoted by a number with a $-$ sign prefixed.

Hence, using symbols, we have

$$8a - 5a = +3a\,; \quad 5a - 8a = -3a\,;$$
$$-8a - 5a = -13a\,; \quad -8a + 5a = -3a.$$

42. Graphical Illustrations. Take a straight line **XOX′** of unlimited length, and consider all distances measured ***to the right*** as positive, whilst all distances measured in the ***opposite direction***, from right to left, as negative.



Take $OA_1 = A_1A_2 = A_2A_3 = A_3A_4 = \ldots\ldots\ldots x$ along **OX**,

and $Oa_1 = a_1a_2 = a_2a_3 = a_3a_4 = \ldots\ldots\ldots\ldots x$ along **OX′**.

Taking O as the starting point in each case,

OA_5 denotes $+5x$, whilst Oa_5 denotes $-5x$, and so on.

Also A_4A_7 denotes $+3x$, whilst A_7A_4 denotes $-3x$.

Again, since OA_5 denotes $5x$ (5 spaces *to the right*),

and A_5A_3 denotes $-2x$ (2 spaces *to the left*);

$\therefore\ 5x-2x=OA_3=3x$, (3 spaces *to the right*).

Again, since Oa_3 denotes $-3x$ (3 spaces *to the left*),

and a_3A_4 denotes $+7x$ (7 spaces *to the right*);

$\therefore\ -3x+7x=OA_4=4x$, (4 spaces *to the right*).

Again, since Oa_6 denotes $-6x$ (6 spaces *to the left*),

and a_6a_2 denotes $+4x$ (4 spaces *to the right*);

$\therefore\ -6x+4x=Oa_2=-2x$ (2 spaces *to the left*).

Again, since Oa_3 denotes $-3x$ (3 spaces *to the left*),

and a_3a_8 denotes $-5x$ (5 spaces *to the left*):

$\therefore\ -3x-5x=Oa_8=-8x$ (8 spaces *to the left*).

Exercise VII.

1. If *Rs.*5 loss is the unit, what is a gain of *Rs.*60?

2. If *Rs.*5 gain is the unit, what is the loss of *Rs.*60, and a gain of *Rs* 60? and a gain of *Rs.*40?

3. If a debt of *Rs.*50 be represented by 10, what will an income of *Rs.*600 represent?

4. A boy lost 40 marbles and afterwards won 25; how many did he (i) win (ii) lose altogether?

5. A man walks East for 20 miles, then 35 miles due West and then East for 30 miles. How far (i) East, (ii) West is he from the starting place?

6. A man said that he missed the 5 o'clock train by -15 minutes. When did the train start?

7. If a maund weight is too heavy by -5 seers, what is its actual weight?

8. If a man is -10 years younger than another whose age is 40 years, how old is he? How old, if -10 years older?

9. If a foot measure is -3 inches too long, what is its length?

10. A man has 15 cows in a field; he drove away -25 cows; how many has he now in the field?

Graphical Examples.

Prove the following graphically, using squared paper :—

1. $5-3=2$.
2. $8-3=5$.
3. $6-4=2$.
4. $-7+3=-4$.
5. $3-8=-5$.
6. $-2-3=-5$.
7. $-3x+5x=2x$.
8. $-3x-5x=-8x$.
9. $-5x+2x=-3x$.
10. $-7a+3a=-4a$.
11. $5a-7a=-2a$.
12. $-3a+5a=2a$.

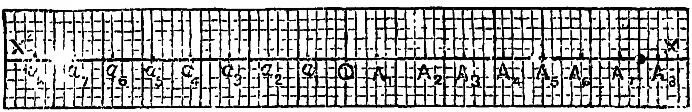
III. CHANGE OF THE ORDER OF TERMS.

43. When several Arithmetical quantities are connected together by the signs + and −, the value of the result is the same in whatever order the terms are taken. (See Arith. Art. 57). This rule also holds good for Algebraical quantities.

Thus, $a+b=b+a$; $a-b=-b+a$;

$$a+b-c=a-c+b=b+a-c=b-c+a=-c+a+b=-c+b+a.$$

44. Graphical Illustrations. Using the diagram of Art. 42, and adopting the same convention as regards signs, &c.



$6x-7x+4x$ brings us first from O to A_6 (6 spaces to the right), then from A_6 to a_1 (7 spaces to the left), then from a_1 to A_3 (4 spaces to the right) ;

$\therefore$ $6x-7x+4x=OA_3=3x$ (3 spaces to the right).

Similarly, $6x+4x-7x$ brings us first from O to A_6 (6 spaces), then from A_6 to A_{10} (4 spaces), then from A_{10} to A_3 (7 spaces to the left), (*i. e.*) to the same point as before ;

$\therefore$ $6x+4x-7x=OA_3=3x$ (3 spaces to the right).

Hence, $6x-7x+4x=6x+4x-7x$.

Again, $-x-3x+6x$ brings us first from O to a_1 (1 space to the left), then from a_1 to a_4 (3 spaces to the left), then from a_4 to A_2 (6 spaces to the right) ;

$\therefore$ $-x-3x+6x=OA_2=2x$, (2 spaces to the right).

Similarly, $6x - 3x - x$ brings us first from **O** to $\mathbf{A}_6$ (6 spaces), then from $\mathbf{A}_6$ to $\mathbf{A}_3$ (3 spaces to the left), then from $\mathbf{A}_3$ to $\mathbf{A}_2$ (1 space to the left);

$\therefore\ 6x - 3x - x = \mathrm{OA}_2 = 2x$ (2 spaces to the right).

Hence, $-x - 3x + 6x = 6x - 3x - x$.

45. Any algebraical expression consisting of **like** terms may be simplified by **collecting terms.**

Ex. Find the simplest form of

$$6a + 3b - a + 3c - b - 2a + 5c - 5a - 6c + 4b.$$

The Exp. $= 6a - a - 2a - 5a + 3b - b + 4b + 3c + 5c - 6c$

(collecting like terms)

$= 6a - 8a + 7b - b + 8c - 6c = -2a + 6b + 2c.$

Exercise VIII.

Find the simple forms of the following expressions :—

1. $12 - 8 + 7 - 5 + 3.$ **2.** $-8 + 4 - 9 + 6 - 10 + 2.$

3. $3a - 6a + 4a - 5a - 3a.$ **4.** $6x + 3x - 9x - 7x + 5x.$

5. $7a - 2b - 3c - 4a + 5b + 4c + 2a - 3b - 5c.$

6. $5a^2 + 3ab - 2b^2 - ab + 9b^2 - 2ab - 7b^2 + 4ab - b^2.$

7. $3a^3 - 2a^2 + 5a + a^3 + a + 9a^2 - 4a^3 - 6a + 2a^2 - 7a.$

8. $3x^2y - 4x^3 - 4xy^2 - 6x^2 + 2xy^2 - 3x^2y - 5x^2 - 3x^3 + 6 + 2xy^2 + 7x^3.$

Graphical Examples.

Prove the following graphically, using squared paper :—

1. $5 + 3 - 4 = 4.$ **2.** $3 + 6 - 7 = 2.$ **3.** $-1 - 2 - 4 = -7.$

4. $-5 + 3 - 4 = -6.$ **5.** $-6 - 7 + 3 = -10.$ **6.** $-7 + 3 + 4 = 0.$

7. $-2 + 3 - 4 + 5 - 6 + 4 = 0.$ **8.** $1 - 2 + 3 - 4 + 5 - 6 = -3.$

9. $3x + 4x - 9x = -2x.$ **10.** $-3x - 4x + 7x = 0.$

11. $-7x + 5x + x = -x.$ **12.** $-a - 3a - 5a = -9a.$

13. $3a + 4a - 6a = a.$ **14.** $-7a + 4a - 3a = -6a.$

15. $-9x + 8x + 6x - 7x = -3x.$ **16.** $-8a + 6a - 3a + 5a = 0.$

IV. SUBSTITUTIONS.

46. Hitherto we have considered the numerical value of every expression to be a *positive* quantity, but the numerical value of an algebraical expression may be a *negative* quantity.

Ex. 1. If $a=2$, $b=3$, $c=4$, find the numerical value of
$2a^2b+3b^2c-a^2bc-4bc^2+abc$.

The Exp. $=2.2.2.3+3.3.3.4-2.2.3.4-4.3.4.4+2.3.4$
$=24+108-48-192+24=156-240=-84$.

Ex. 2. Find the value of $3x^2-5xy+\frac{y^2z}{2x}$, when $x=1, y=2, z=3$,

The Exp. $=3.1.1-5.1.2+\frac{2.2.3}{2.1}=3-10+6=9-10=-1$.

Ex. 3. Find the values of $2x^2-5x-10$, when x has the values 0, 1, 2, 3, 4, 5, 6. Tabulate the work.

When

$x=$	0	1	2	3	4	5	6
$2x^2=$	0	2	8	18	32	50	72
$-5x=$	0	-5	-10	-15	-20	-25	-30
$-10=$	-10	-10	-10	-10	-10	-10	-10
$2x^2-5x-10=$	-10	-13	-12	-7	2	15	32

$\therefore$ -10, -13, -12, -7, 2, 15, 32 are the required values.

Exercise IX.

1. Find the value of $3b-4a-6c+7x+2y$, and of $3ab-8bc+2xy$, when $a=6$, $b=5$, $c=4$, $x=3$, $y=2$.

2. Find the value of $-3ab-abc+4bc-3ac$, and of $4a^2b-9b^2c+25abc$ when $a=6$, $b=5$, $c=4$.

3. Find the value of $(a^3-3a^2b-3ab^2+b^3)(b^3-3b^2)-(c^2+2c)\times(abc-a^2b^2)$, when $a=1$, $b=4$, $c=6$.

4. If $a=2$, $b=3$, $c=4$, find the values of

(1) $\{7a-(a+b+c)\}-\{6b+a(2a-b+2c)\}$.

(2) $a-(b+c)-\{b-(c-a)\}-\{3a-(2b+a-c)\}$.

(3) $\left\{4b+2c-\frac{a+2b}{3}-(c-a)(a+c)\right\}\times(2c-3b)$.

5. Find the value of $6ax+2by-cz-(2ax-3by+4cz)-cz-ax$, when $a=0$, $b=1$, $c=2$, $x=8$, $y=3$, $z=4$.

6. When $a=4$, $b=3$, $c=2$, $d=0$, find the values of
(1) $3a^2-4bc-5ad+3a^2bcd$. (2) $2b^2c+9a-ab+cd$.

7. If $a=1$, $b=2$, $c=3$, $d=0$, find the value of
$(ac-bd)\sqrt{(a^2bc+b^2cd+c^2ad-2)}$.

8. Find the values of x^2-5x+6, when x has the values 0, 1, 2, 3, 4, 5. Tabulate the work.

9. Find the values of $6x^2-11x-10$, when x has the values 0, 1, 1·5, 2, 3, 5·5, 8. Tabulate the work.

10. Prove that $x^2-7x+12=0$, when $x=3$ or 4.

11. Prove that $3x^3-26x^2+71x-60=0$, when $x=1\frac{2}{3}$, 3 or 4.

12. Given $V=4$, $t=10$ and $g=32$, find the value of
(1) $S=Vt+\frac{1}{2}gt^2$. (2) $S=V^2/2g$. (3) $S=Vt-\frac{1}{2}gt^2$.

CHAPTER III.

THE FUNDAMENTAL OPERATIONS.

I. ADDITION.

47. **Addition** is the process of collecting several quantities into a **single** term. This single term is called their **sum.**

48. **The Addition of like quantities with like signs** will be obtained by addding the numerical coefficients, prefixing the common sign and annexing the common letter or letters.

Ex. 1. Add together $4a$, $5a$.

Here, we have to add 4 like things to 5 like things of the same kind, and the total is 9 of such things ;

$$\therefore 4a+5a=\underline{9a}.$$

Similarly, $4a^2b+5a^2b+2a^2b+a^2b=\underline{12a^2b}$.

Ex. 2. Add together $-2x$, $-5x$, $-x$, $-3x$.

Here, we have to take away successively 2, 5, 1, 3 like things of the same kind, and the result is the same as taking away $(2+5+1+3)$ or 11 such things on the whole.

Therefore the sum of $-2x$, $-5x$, $-x$, $-3x=\underline{-11x}$.

Similarly, $(-6x^2y)+(-3x^2y)+(-x^2y)+(-9x^2y)=\underline{-19x^2y}$.

49. The Addition of like quantities with unlike signs will be found by adding separately the positive and negative coefficients; taking the difference of these two sums, prefixing the sign of the greater and annexing the common letter or letters.

Ex. 1. Add together $6a$ and $-2a$.

Here, we have to take away 2 like things from 6 like things of the same kind. Now as the difference of 6 and 2 is 4, and the greater is positive,

$\therefore\ 6a-2a=\underline{4a}.$

Ex. 2. Find the sum of $-12bc$, $4bc$, $-8bc$, $5bc$, $3bc$.

The sum of the coefficients of the positive terms $=4+5+3=12$.

....negative........$=12+8=20$.

The difference of these two sums is $(20-12)$ or 8, and the sign of the greater is negative;

$\therefore$ the required sum $=\underline{-8bc}.$

Exercise X.

Add together :—

1. $5a$, $7a$, $3a$, $6a$, a, $2a$.
2. $4x$, $9x$, $5x$, $6x$, x, $2x$.
3. $5ab$, $6ab$, ab, $8ab$, $3ab$.
4. $3a^2b$, $4a^2b$, $6a^2b$, $9a^2b$, a^2b.
5. $-2x$, $-7x$, $-5x$, $-9x$, $-x$.
6. $-3a$, $-7a$, $-12a$, $-9a$, $-4a$.
7. $4ab$, $-5ab$. $-3ab$, $6ab$, ab.
8. $4x$, $-5x$, $-8x$, $6x$, $3x$.
9. $5a$, $-3a$, $6a$, $9a$, $-7a$.
10. ab, $-2ab$, $5ab$, $-6ab$.
11. $-3abc$, $-2abc$, $5abc$, $10abc$.
12. a^2b^2, $-5a^2b^2$, $-6a^2b^2$, $4a^2b^2$.
13. $-4a^2b^4$, $-5a^2b^4$, $10a^2b^4$, $6a^2b^4$.
14. $11ab^2$, $-2ab^2$, $-5ab^2$, $10ab^2$.
15. $3abcd$, $-4abcd$, $5abcd$, $-10abcd$, $6abcd$.

Find the value of

16. $5a^2b^2-2a^2b^2+4a^2b^2-8a^2b^2-6a^2b^2$.
17. $-2a+10a+a-4a-5a$.
18. $2ax-5ax-7ax+10ax+4ax$.
19. $3a^2+4a^2-8a^2+5a^2-7a^2$.
20. $7yz-2yz-5yz-6yz+3yz+9yz-6yz$.

---

50. The Addition of unlike quantities can be found by writing the quantities one after the other and prefixing the proper sign to each.

Ex. Add together $5a^2b$, $3b^2$, $-2ab^2$, $-3b$, $2a$.

The sum $=\underline{5a^2b-2ab^2+2a-3b+3b^2}.$

Exercise XI.

Find the sum of the following quantities :—

1. $a, -3b, 5c, 6d, -7x.$ 2. $5xy, -3x^2y, -6xy^2, 9x^2y^2.$
3. $-3ab, 5a^2b, 4b^3, -5b^2.$ 4. $2a, -3a^2, 2a^3, -5a^4.$
5. $4a^3b^2, -3ab^4, -7a^2b^3, -7b^5, -5a^5, 29a^4b.$

---

51. **The Addition of quantities partly like and partly unlike** may be found by writing the like quantities so that they may fall conveniently in the same column, and adding each column separately by the preceding Rules and writing down the others with their proper signs.

Ex. 1. Find the sum of $2a-3b+c$, $2b-3c+a$, $-3a+2c+b$ and $5a-4c+3b$.

$$\begin{array}{r}
2a-3b+c\\
a+2b-3c\\
-3a+b+2c\\
5a+3b-4c\\
\hline
5a+3b-4c
\end{array}$$

Here, re-arranging the expressions so that like terms may fall in the same vertical columns, and then adding up each column separately, we have the sum $=\underline{5a+3b-4c}$.

Ex. 2. Add together $3a^2+2ab^2$, $4a^2-3ab^2$, $-8a^2+4ab^2$, $5a^2-6ab^2$ and $7a^2+3ab^2$.

$$\begin{array}{r}
3a^2+2ab^2\\
4a^2-3ab^2\\
-8a^2+4ab^2\\
5a^2-6ab^2\\
7a^2+3ab^2\\
\hline
11a^2
\end{array}$$

Here, the algebraical sum of the terms in the first column is $11a^2$, and that of the terms in the second column is zero.

Hence the sum $=\underline{11a^2}$.

Ex. 3. Find the sum of $2a+c+d$, $-b+a+e$, $c-d$, $-3a-e-f$ and $-2c+2d-2e$.

$$\begin{array}{rrrrrr}
2a & & +c & +d & & \\
a & -b & & & +e & \\
 & & c & -d & & \\
-3a & & & & -e & -f\\
 & & -2c & +2d & -2e & \\
\hline
 & -b & & +2d & -2e & -f
\end{array}$$

Here, the terms are arranged in alphabetical order and then each column is added separately.

The sum $=\underline{-b+2d-2e-f}$.

Ex. 4. Add together $a^3-2b^3+3ab^2$, $5a^2b-ab^2-3a^3$, $8a^3+5b^3$, $9a^2b-2a^3+ab^2$ and $-3b^3-12a^2b-6a^3$.

$$\begin{array}{rrrr}
a^3 & & +3ab^2 & -2b^3\\
-3a^3 & +5a^2b & -ab^2 & \\
8a^3 & & & +5b^3\\
-2a^3 & +9a^2b & +ab^2 & \\
-6a^3 & -12a^2b & & -3b^3\\
\hline
-2a^3 & +2a^2b & +3ab^2 &
\end{array}$$

Here, the terms are arranged according to the *descending* powers of a and *ascending* powers of b.

The sum $=\underline{-2a^3+2a^2b+3ab^2}$.

52. The Rule laid down in Art. 51 is only an illustration of the principle given in Art. 45.

Ex. Add together $x-2y+3z$, $3y-4z-2x$, $3x-5z-5y$, $x+2y$.

The sum $=x-2y+3z+3y-4z-2x+3x-5z-5y+x+2y$

$=x-2x+3x+x-2y+3y-5y+2y+3z-4z-5z$ (re-arranging terms)

$=3x-2y-6z$ (collecting like terms).

Exercise XII.

Add together :—

1. $3x+y$, $2x+2y$, $4x+3y$, $5x+2y$, $x+y$, $4x+6y$.

2. $-4a+b$, $-2a+3b$, $-5a+2b$, $-6a+3b$, $-7a+b$, $-a+2b$.

3. $a-b$, $b-c$, $c-a$.

4. $a+b-c$, $b+c-a$, $c+a-b$.

5. $2a+3b-4c$, $-3a+4b-c$, $4a+7b+7c$, $a-b-4c$, $-5a+2b-6c$.

6. $7a-3b+4c-2d+7$, $-8a+4b-6c+2d-11$, $13a+3b-5c+4d-4$, $2a-b+c+11$ and $a+2d-3$.

7. $ax-4by+3cz$, $13ax-9by+7cz$, $-5ax+7by-14cz$, $2ax-by+cz$, and $-11ax+13by-4cz$.

8. $2a^2+ab+3b^2$, $3a^2-4ab+2b^2$, $3a^2+3ab-b^2$, $12a^2-14ab-7$[illegible] and $3a^2-12ab+17b^2$.

9. $2x-3y+4z-4$, $x+2y-3z$, $-3x+2y-5z+7$, $4x-y+2z-3$, $9x-10y+11z-12$ and $x+y+z$.

10. $8ab+3cd-5c^2$, $7ab-2cd+6c^2$, $9ab-4cd-10c^2$, $7ab-2cd+6c^2$, $6ab+7cd-7c^2$ and $7ab-3cd+4c^2$.

11. $20x^3+20x^2y-3xy^2+14y^3$, $-17x^3+14x^2y-12xy^2-3y^3$, $14x^3+17x^2y+15xy^2-5y^3$, $-12x^3-13x^2y-14xy^2-5y^3$ and $12x^2y+3y^3$.

12. $x+xy-3x^2y^2$, $3x+3xy-4x^2y^2$, $6x+7xy+10x^2y^2$, $3x+2xy+6x^2y^2$, $7x-6xy+6x^2y^2$, $3x-6xy-3x^2y^2$, $-4x-3xy-x^2y^2$, $5x+7xy+x^2y^2$ and $-2x-10xy-9x^2y^2$.

13. $8ax-7by+3y^2$, $ax^2+2by-7y^2$, $9ax+4y^2-2$, $7+3by-2y^2+4ax$, $6ax^2-y^2$ and $2by+3y^2-5$.

14. $2x^2-3xy-4y^2$, $3xz+2y^2-z^2$, $x^2-2yz+5z^2$, $3xy-6xz-3x^2$, $3xz-2z^2+5yz$ and $4y^2-3yz+2x^2$.

15. $x^3-3ax^2+3a^2x-a^3$, $4x^3-5ax^2+6a^2x-15a^3$, $3x^3+4ax^2+2a^2x+6a^3$, $-17x^3+19ax^2-15a^2x+8a^3$, $-13ax^2-27a^2x+18a^3$ and $9x^3-12a^2x+4ax^2-16a^3$.

16. $4a^2b^2-5a^3b^3+6abcd,\ -5a^2b^2+6abcd,\ 6c^2d^2-5a^2b^2-6a^3b^3,\ ax^2-6a^3b^3+5,\ -6+ax^2-6a^2b^2+2abcd$ and $-6c^2d^2-2ax^2+17a^3b^3+1$.

17. $a^3-2ab^2-ac^2+a^2b+2a^2c+2abc,\ -a^2b+b^3-2bc^2+2ab^2+2abc+b^2c,\ -2a^2c-b^2c+c^3+2abc+ac^2+2bc^2$ and $-a^3-b^3-c^3$.

18. $11x^3+14x^2y-7xy^2+z^3,\ 3y^2z+x^3-2x^2y-2z^3+3xy^2-7y^3,\ 2y^3-11z^3-3xyz+4yz^2+4x^3,\ 12x^3-4yz^2+4xyz+4xy^2+3y^3$ and $12z^3-12x^2y-xyz+2y^3-3y^2z$.

---

53. When compound expressions, with brackets, are to be added, it is more convenient to retain the brackets.

Ex Find the sum of $6(a+b)$ and $3(a+b)$.

Here, taking $a+b$ as a single quantity x, we have

$$6x+3x=9x,$$

$$\therefore\ 6(a+b)+3(a+b)=9(a+b).$$

54. When the numerical coefficients are fractions, they must be treated by the Rules of Fractions in Arithmetic.

Ex Add together $\frac{1}{5}a^2+\frac{1}{6}ab-\frac{1}{4}b^2,\ \frac{1}{5}b^2-\frac{1}{6}a^2+\frac{1}{3}ab,\ \frac{1}{2}ab-\frac{1}{4}b^2+\frac{1}{3}a^2$.

$$\begin{array}{r|l}
\frac{1}{5}a^2+\frac{1}{6}ab-\frac{1}{4}b^2 & \text{Here, } \frac{1}{5}-\frac{1}{6}+\frac{1}{3}=\frac{6}{30}-\frac{5}{30}+\frac{10}{30}=\frac{16}{30}-\frac{5}{30}=\frac{11}{30}. \\
-\frac{1}{6}a^2+\frac{1}{3}ab+\frac{1}{5}b^2 & \quad \frac{1}{6}+\frac{1}{3}+\frac{1}{2}=\frac{1}{6}+\frac{2}{6}+\frac{3}{6}=\frac{6}{6}=1. \\
\frac{1}{3}a^2+\frac{1}{2}ab-\frac{1}{4}b^2 & -\frac{1}{4}+\frac{1}{3}-\frac{1}{4}=-\frac{3}{12}+\frac{4}{12}-\frac{3}{12}= \\
\hline
\frac{11}{30}a^2+ab-\frac{1}{6}b^2 & \text{Hence sum}=\frac{11}{30}a^2+ab-\frac{1}{6}b^2.
\end{array}$$

Exercise XIII.

Add together :—

1. $4(a-b),\ 2(a-b),\ -7(a-b)$.

2. $-2(a-b)x^2,\ 3(a-b)x^2,\ 4(a-b)x^2$.

3. $4(x^2+y^2)+2ab(x^2-y^2)-3,\ -2(x^2+y^2)-5ab(x^2-y^2)+4,\ 3(x^2+y^2)-2ab(x^2-y^2)-5,\ 5(x^2+y^2)+7ab(x^2-y^2)+4,$ $-3(x^2+y^2)-2ab(x^2-y^2)-5$ and $2(x^2+y^2)-5ab(x^2-y^2)+5$.

4. $6a-2(x-y)a^2+3a^3,\ 7a+3(x-y)a^2+4a^3,\ -6a-6(x-y)a^2-6a^3,$ $7a+13(x-y)a^2-10a^3,\ -10a+4(x-y)a^2+10a^3$ and $7a-12(x-y)a^2-a^3$.

5. $\frac{1}{2}x^2-\frac{1}{3}x^2+\frac{4}{9}x^2$.

6. $\frac{3}{4}xy+\frac{5}{6}xy-3xy-\frac{1}{4}xy$.

7. $6a^2b+\frac{1}{2}a^3b^2-\frac{1}{3}a^4b^4,\quad -7a^2b-\frac{1}{4}a^3b^2+\frac{1}{2}a^4b^4,\quad 2a^2b+\frac{1}{8}a^3b^2-\frac{1}{8}a^4b^4,$ $-9a^2b+\frac{1}{10}a^3b^2-\frac{2}{5}a^4b^4$ and $-\frac{19}{40}a^3b^2+\frac{43}{120}a^4b^4$.

8 $\frac{1}{4}a+\frac{1}{8}b+\frac{1}{2}a^2b^2+1$, $7-\frac{2}{3}a+\frac{1}{2}b+\frac{1}{4}a^2b^2$, $\frac{3}{4}a^2b^2-9-\frac{3}{4}a-\frac{5}{8}b$, $b+2a+1$ $-2a^2b^2$ and $\frac{1}{4}a-b+3$

9 $\frac{1}{10}x+\frac{1}{3}x^2+\frac{4}{7}x^3+\frac{4}{1}x^4$, $\frac{5}{7}x^4+\frac{1}{8}x^3+\frac{1}{4}x^2+\frac{2}{5}x$, $-\frac{1}{5}x^3+\frac{1}{10}x^2+\frac{3}{10}x-\frac{4}{5}x^4$ and $-\frac{4}{5}x-\frac{5}{7}x^4+\frac{1}{4}x^2-\frac{1}{8}x^3$

10 $9a^2b^2+3(a^2-b^2)+\frac{5}{8}a-ab$, $\frac{1}{3}a^2b^2+\frac{3}{4}(a^2-b^2)+\frac{1}{4}a-ab$, $4a^2b^2-\frac{5}{6}(a^2-b^2)+\frac{1}{4}a+2ab$ and $-5a^2b^2-\frac{7}{12}(a^2-b^2)-\frac{9}{8}a+ab$

II. SUBTRACTION.

55 **Subtraction** being the reverse of Addition, it is evident, that to subtract algebraical quantities, *change the signs of the quantities to be subtracted* and then proceed as in Addition

Thus, if we subtract b from a, the result will be $a-b$, but if we take $b-c$ from a the result will be *greater* than by c than the former, since the quantity now to be subtracted is *less* by c than in the former case, hence the result will be $a-b+c$, which is the value of $a-(b-c)$, so that the quantities b, $-c$ when subtracted, become $-b$, $+c$, respectively

56 We will further confirm the principle of the last Article as follows —

(i) Since $a=a-b+b$, if we subtract $+b$ from a, the result is $a-b$, the same as if we add $-b$ to it,

(ii) Since $a=a+b-b$, if we subtract $-b$ from a, the result is $a+b$, the same as if we add $+b$ to it

Thus, if a person possesses a Rupees and owes b Rupees, his money in hand may be expressed by $+a$ Rupees and his debt by $-b$ Rupees, so that he may be said to possess $+a$ and $-b$ Rupees, or in one sum, $a-b$ Rupees Now if we *subtract* or annul his debt, that is take away his negative property, $-b$ Rupees, he will possess the whole positive property, $+a$ Rupees, the same as if we *give* him $+b$ Rupees, to pay his debt with

Hence, the following results are obvious —

$$6a-2a=4a\ ,\ \ -6a-2a=-8a\ ,\ \ 6a-(-2a)=8a\ ,$$
$$-6a-(-2a)=-4a\ ,\ \ 2a-6a=-4a$$

Ex 1 Subtract $a+b$ from $a-b$

The result $=a-b-(a+b)=a-b-a-b=-2b$

Ex 2. What must be added to $2a+b$ to make $2a$?

The result $=2a-(2a+b)=2a-2a-b=-b$

Exercise XIV.

Subtract :—

1. $4a$ from $9a$.	**2.** $-4ab$ from $-8ab$.	**3.** $-3ab$ from $10ab$.
4. $-4xy$ from $3xy$.	**5.** $-8b$ from $12b$.	**6.** $6a^2b$ from $-a^2b$.
7. $-4a$ from 0.	**8.** $2a+5b$ from 0.	**9.** $-7ax^2$ from $11ax^2$.
10. a from $-a$.	**11.** x from 0.	**12.** $2a-b$ from $3a-2b$.
13. $-2a$ from $-3b$.	**14.** abc from $2bcd$.	**15.** $-3a^2$ from $-3b^2$.
16. $b+c$ from a.	**17.** $-a$ from $3ax$.	**18.** $-2a$ from $-3ax$.
19. $a+b$ from $x-y$.		**20.** $3a-b+c$ from $3a+b$.

What must be added to

21. $2x+3y$ to make $2x$?	**22.** $a^2-b^2-c^2$ to make $3b^2+c^2$?
23. [illegible]${}^2+ax+b$ to make $5x^2-ax$?	**24.** $a+2b+c$ to make a?
25. [illegible] $7x+6$ to make $7x-6$?	

57. In dealing with compound expressions containing unlike terms we may conveniently apply the following Rule.

Rule. *Write like quantities under like quantities, change the sign of all the quantities to be subtracted and then proceed as in Addition.*

Ex. 1. Subtract $3x+4y-5z$ from $5x+7y-8z$.

$$\begin{array}{r} 5x+7y-8z \\ \underline{-3x-4y+5z} \\ \underline{2x+3y-3z} \end{array}$$

(by *addition*).

The like terms are written in the same vertical column (the signs of all the terms in the lower line being changed) and each column is treated separately.

Note. It is not necessary that the signs should be *actually* changed; the change may be made *mentally*.

Ex. 2. From $5x^2-3xy+4y^2$ take $-4x^2-3xy+7y^2$.

$$\begin{array}{r} 5x^2-3xy+4y^2 \\ \underline{-4x^2-3xy+7y^2} \\ \underline{9x^2 \qquad\quad -3y^2} \end{array}$$

Here, $5x^2-(-4x^2)=5x^2+4x^2=9x^2$.
$-3xy-(-3xy)=-3xy+3xy=0$.
$4y^2-(+7y^2)=4y^2-7y^2=-3y^2$.

Ex. 3. Subtract $-7a^2+3b^2-2c^2$ from $-3a^2+4ab-5b^2$.

$$\begin{array}{r} -3a^2+4ab-5b^2 \qquad \\ \underline{+7a^2 \qquad\quad +3b^2-2c^2} \\ 4a^2+4ab-8b^2+2c^2 \end{array}$$

Here, $-3a^2+7a^2=4a^2$; $4ab-0=4ab$; $-5b^2-3b^2=-8b^2$; $0+2c^2=2c^2$.

Exercise XV.

From

1. $2a-2b+c$ take $a+b-2c$. **2.** $7a^2-8a-1$ take $5a^2-6a+3$.

3. $2x^2-3xy+y^2$ take $4x^2+4xy-2y^2$.

4. $5ax-7by+cz$ take $ax+2by-cz$.

5. $7x^2-2x+4$ take $2x^2+3x-1$. **6.** $2a^2-3ab+b^2$ take $4a^2-4ab+3b^2$.

7. $4a^2-5b^2+11c^2$ take $3a^2-2b^2-8c^2+1$.

8. $4x^3-2x^2y+4xy^2+3y^3$ take $3x^3+4x^2y-7xy^2+5y^3$.

9. $-5a^3+7a^2b-3ab^2+6b^3$ take $7a^3-4a^2b-3ab^2+3b^3$.

10. $5a^3-7a^2b+6ab^2-b^3+5$ take $3a^3+4a^2b+3-8ab^2+3b^3$.

11. $8a^2-2a+6b^2-5ab+5c^2-3bc+2$ take $a^2+a+2b^2+2ab+3c^2+3bc+2$; $5a^2b-3b^2+2xy$ take $-3a^2b-5b^2+xy$.

12. $2x^3-4x^2y-3y^2+6-2x^2-3xy^2-14y^3$ take $3x^3+2x^2y-y^2-3xy^2+x^2-10y^3$; $3x^2-2xy+4y^2$ take $5x^2+7xy$.

13. $5x^2+6xy-4y^2-12xz-7yz-5y^2$ take $2x^2-3y^2+4xz-5z^2+6yz-7xy$; $7x^2+5xz-2z^2$ take $3z^2-7xz+5x^2$.

14. $4a^4-3a^3-2ab-7a+7$ take $a^4-2a^3-2ab+7a-7$.

15. $5a+3c-4b-7d-e$ take $4a+7b-5d+6c-5e$.

16. $6p^2-9q^2+12pq$ take $6p^2+8q^2-10pq$ and $5p^2-3pq+2q^2$.

17. $p^2q^3+2x^2y^3-4xy^4$ take $x^5+4p^2q^3-3x^2y^3-4xy^4$ and $5x^2y^3-x^5$.

18. $3x^2+2xy-y^2$ take $-x^2-3xy+3y^2$ and $3x^2+4xy-5y^2$.

19. $1-2x+3x^2$ take $7x^3-4x^2+3x+1$ and $-4x^3+3x-2x^2$.

20. $a^4-2a^3b+3a^2b^2-4ab^3+5b^4$ take $2ab^3-3a^2b^2+4a^3b-5a^4$, and $3a^4-2a^3b+6a^2b^2-2ab^3+3b^4$.

21. What must be added to $p^2-q^2+2pq-q^3$ to make $p^2-4pq+2q^3$?

22. What expression must be taken from $2a^2-6a^2b+4a^2b^2-2$ to leave $a^2-7a^2b-4a^2b^2$?

23. From $4a+5b-5c$ take the sum of $a-b+c$, $2a+2b-3c$, $-a-b-c$ and $-3a+4b+4c$.

24. Subtract $5x^2-2x+6$ from unity, and $2x-3x^2-5$ from zero, and add the results.

25. If $V=5a+4b-6c$, $X=-3a-9b+7c$, $Y=20a+7b-5c$, $Z=13a-5b+9c$, calculate the value of $V-(X+Y)+Z$. (M. M. 1883.)

58. In the subtraction of compound expressions, with brackets, retain the brackets, and of quantities with fractional coefficients, deal with the fractional coefficients as in Arithmetic.

Ex. From $(a-b)x^2+3(c-d)y^2$ take $2(a-b)x^2-4(c-d)y^2$ and from $\frac{1}{3}xy-\frac{1}{4}ab+\frac{1}{5}c^2$ take $\frac{1}{2}xy+\frac{1}{3}ab-\frac{1}{4}c^2$.

(i)
$$\begin{array}{r} (a-b)x^2+3(c-d)y^2 \\ 2(a-b)x^2-4(c-d)y^2 \\ \hline -(a-b)x^2+7(c-d)y^2 \end{array}$$

(ii)
$$\begin{array}{r} \frac{1}{3}xy-\frac{1}{4}ab+\frac{1}{5}c^2 \\ \frac{1}{2}xy+\frac{1}{3}ab-\frac{1}{4}c^2 \\ \hline -\frac{1}{6}ab-\frac{7}{12}ab+\frac{9}{20}c^2. \end{array}$$

Exercise XVI.

1. From $4a(x-y)+5(x^2-y^2)$ take $5(x^2-y^2)+2a(x-y)$.
2. Take $a^2(a-b)-(x+y)b^2+5(a-b)(x-y)$ from $a^2(a-b)+3(a-b)(x-y)+(x+y)b^2$.
3. Subtract $4(a+b)^2-2x(a^2+b^2)+x^2(a+b)$ from $2(a+b)^2-x(a^2+b^2)+3x^2(a+b)$.
4. Subtract $\frac{3}{4}abc-\frac{17}{18}a^2b+\frac{9}{10}a^3$ from $\frac{1}{8}abc+\frac{1}{6}a^2b-\frac{3}{5}a^3$.
5. From $\frac{1}{5}a^2+\frac{3}{4}a^2b+\frac{1}{5}a^3b^2$ take $\frac{1}{6}a^2+\frac{2}{3}a^2b+\frac{3}{4}a^3b^2$.
6. Subtract $7a^2b^2(a-b)+6x^2y^2(a^2+b^2)-2ab(a^3-b^3)$ from $11a^2b^2(a-b)-10x^2y^2(a^2+b^2)+7ab(a^3-b^3)$.
7. Take $\frac{1}{8}x^2y^2+\frac{1}{5}x^2yz+\frac{1}{6}xy^2z+\frac{1}{3}$ from $\frac{3}{4}x^2y^2+\frac{9}{10}x^2yz+\frac{2}{3}xy^2z+\frac{7}{8}$.
8. From $\frac{1}{6}a^2b^2-\frac{2}{5}abc+\frac{7}{8}x^2y^2z^2+\frac{3}{4}(a-b)$ take $\frac{1}{8}a^2b^2+\frac{3}{5}abc-\frac{3}{4}x^2y^2z^2-\frac{1}{2}(a-b)$.
9. Subtract $3a-\frac{2}{3}b+\frac{3}{4}c$ from $2a+\frac{1}{3}b-\frac{1}{4}c$. (C. E. 1875).

III. BRACKETS.

59. We have already seen (Art. 28) that when two or more quantities are to be treated as a single quantity, they are frequently enclosed in **brackets**.

60. Removal of brackets. Since $a+(b+c)$ means that the sum of b and c is to be added to a, we have, by the rule of addition,

$$a+(b+c)=a+b+c.$$

Similarly, $a+(b-c)$ means that to a we are to add b, diminished by c.

Thus, $a+(b-c)=a+b-c$.

In like manner, $a+b-c+(d-e+f)=a+b-c+d-e+f$.

Hence, the following Rule.

Rule. *When an expression within brackets is preceded by the sign* $(+)$, *simply remove the brackets.*

61. Since $a-(b+c)$ means that the sum of b and c is to be taken from a, the result will be the same whether b and c are taken away separately or in one sum. Thus, by the rule of subtraction, we have

$$a-(b+c)=a-b-c.$$

Again, $a-(b-c)$ means that we are to take away the excess of b over c from a. If from a we take away b, we get $a-b$; but by so doing we shall have taken away c too much, and must therefore add c to $a-b$ (Art. 54).

Thus, $a-(b-c)=a-b+c$.

In like manner, $a-b-(c+d-e)=a-b-c-d+e$.

Hence the following Rule :—

Rule. *When an expression within brackets is preceded by the sign* $(-)$, *the brackets may be removed, provided the signs of all the quantities within the brackets be changed.*

Ex. 1. Prove, by removing the brackets, that

$$(a+b)-(3a-2b)-(2a-3c)+3c-(3a+3b-4c)=-7a+10c.$$

The given expression $=a+b-3a+2b-2a+3c+3c-3a-3b+4c$

$$=a-8a+3b-3b+10c=-7a+10c.$$

62. A straight line, called a **vinculum** drawn over several quantities is equivalent to a bracket. (Art. 29).

Thus, $a-\overline{3b+2c}$ is the same as $a-(3b+2c)$.

63. In the case of a fraction with a numerator of more than one term, the line separating its numerator and denominator is also a species of vinculum —, drawn underneath and may be removed by the preceding Rules.

Thus, $5+\dfrac{3x-4}{7}=5+\frac{1}{7}(3x-4)=5+\dfrac{3x}{7}-\dfrac{4}{7}$.

Also $5-\dfrac{3x-4}{7}=5-\frac{1}{7}(3x-4)=5-\dfrac{3x}{7}+\dfrac{4}{7}$.

Ex. 2. Simplify the expression

$$\frac{6x-8}{2}+\frac{10x-5}{5}-\frac{14x-21}{7}.$$

The given expression $=\dfrac{6x}{2}-\dfrac{8}{2}+\dfrac{10x}{5}-\dfrac{5}{5}-\dfrac{14x}{7}+\dfrac{21}{7}$

$$=3x-4+2x-1-2x+3=5x-2x-5+3=3x-2.$$

Exercise XVII.

Remove the brackets from the following, and then reduce the resulting expressions to their simplest forms :—

1. $2a+(3a-4b)$.   2. $3x-(2x-4y)$.   3. $x-(-x+y)$.

4. $7-3x-(2x-7)$.   5. $(3a+4b)-(5a-7b)$.   6. $3a-(a+2b-3c)$.

7. $x+(2y-z)-\overline{3x-3y+z}$.   8. $2x-\overline{4x+3y}-\overline{3y-x}$.

9. $a-x-(2x-a)-(2-2a)+(3-2x)-(1-x)$.

10. $(a^3-2a^2c+3ac^2)-(a^2c-2a^3+2ac^2)+(a^3-ac^2-a^2c)$.

11. $(2x^2-2y^2-z^2)-(3y^2+2x^2-z^2)-(3z^2-2y^2-x^2)$.

12. $(x^3+ax^2+a^2x)-(y^3-by^2+b^2y)+(z^3+cz^2+c^2z)-(x^3-y^3+z^3)$
$+(ax^2+by^2+cz^2)-(a^2x-b^2y+c^2z)$.

13. $(10x^2-2y^2+15yz)-(8yz-7y^2+x^2)-(6y^2+7yz-9x^2)$.

14. $(3a-b+7c)-(2a+3b)-(5b-4c)+(-a+3c)$.

15. $(7a^3-8a^2b+3b^3)-(17a^3-2a^2b+15b^3)-(-15a^3-13b^3+5a^2b)$.

Prove the following by removing brackets :—

16. $\dfrac{5x-15}{5}+\dfrac{2x-4}{2}-\dfrac{22-33x}{11}=5x-7$.

17. $\dfrac{6-9x}{3}-\dfrac{7-21x}{7}+\dfrac{25x-20}{5}=5x-3$.

18. $\dfrac{6x+8}{12}+\dfrac{27x-54}{9}-\dfrac{12+42x}{6}=-x-$

19. $(4a-2b+5c)-\overline{2a-3b+7c}+\overline{9c+3b-2a}=4b+7c$.

20. $\overline{9a-b}+-\overline{2a+3b}-\overline{6a+5b}-\overline{a-3b}=0$.

---

64. When there are brackets within brackets, first remove the innermost bracket, then the next, and so on.

Ex. 1. Remove the brackets from

$$\{3a-b-(3c-d)\}-\{2a-(b+2c)+d\}.$$

The given expression $=\{3a-b-3c+d\}-\{2a-b-2c+d\}$
$=3a-b-3c+d-2a+b+2c-d=a-c$.

65. When there is a **coefficient** before a bracket, all the quantities within the bracket must be multiplied by that *coefficient*.

Ex. 2. Simplify $4(1+2a)-2[3a+2\{2a-(4a-1)\}]$.

The given expression $=4+8a-2[3a+2\{2a-4a+1\}]$
$=4+8a-2[3a+4a-8a+2]$
$=4+8a-6a-8a+16a-4=10a$.

Exercise XVIII.

Express, removing brackets, in their simplest forms :—

1. $2a-\{b-(a-2b)\}$.
2. $a-\{2b-(2b+3c)-a\}$.
3. $7a-\{b+(2a+b)-(a-b)\}$.
4. $3x-\{y-(2x-y)-(x+y)\}$.
5. $a^2-(b^2-c^2)-\{b^2-(c^2-a^2)\}+\{c^2-(b^2-a^2)\}$.
6. $\{2a^2-(3ab-b^2)\}-\{a^2-(4ab+b^2)\}+\{2b^2-(a^2-ab)\}$.
7. $\{x^3+y^3-(3x^2y+3xy^2)\}-\{(x^3-3x^2y)-(3xy^2-y^3)\}$.
8. $\{2x-(3y-z)\}-\{y+(2x-z)\}+3z-(x-2y)\}-\{2x-(y-z)\}$.
9. $2a-\{3a+4(b-a)-2b\}$.
10. $2a-3(b-c)-2\{a-2(b-c)\}$.
11. $2x-3(y-z)+\{x-2(y-z)\}-2\{x-3(x-y)\}$.
12. $a-[5b-\{a-(3c-3b)+2c-(a-2b-c)\}]$.
13. $7a-2[3a-2b+\{(a+b)-(a-b)\}]$.
14. $\{2a-(3b+c-2d)\}-\{(2a-3b)+(c-2d)\}+\{2a-(3b+c)-2d\}$
$-\{(2a-3b+c)-2d\}$.
15. $4(3b-a)-3[7a-3\{(2a-b)-2(b-a)\}]$.
16. $\{m-n-(3x-2y)\}-[3m+2n-\{x-y+(m+2n)-(2y-x)\}]$.
(M. M. 1890.)
17. $x-[a-\{2a-(3a-\overline{4a-x})\}]$. (M. M. 1889).
18. $\{3a-(b-2c)\}-\{2b-(c-a)\}+\{2b-[4a-a-\overline{a-2b}]\}$.
19. $3a-2[3a-2\{3a-2(3a-2a+b)+b\}+b]$.
20. $5x-3[2x+9y-2\{3x-2(x-\overline{5x-2y})+3y\}]$.
21. $3x-[7y-\{2x-(6z-\overline{4z-3y}-y)+4x-(x-5\overline{y+z})\}]$.
22. $5x+[3x-\{7y-\overline{2x+3y}-21y\}-6y]-[6y+\{7x-(3y+4x)+9y\}$
$+5x]$, and then find the value of the resulting expression, when $x=1, y=2$.

66. Formation of brackets. It is often necessary not only to break up, or **resolve**, quantities contained in brackets, but also to **form** such quantities, that is, to take up in a bracket any given terms of an expression. Now, in doing this, it should be noticed that, whatever term we choose to set as *first* term within the bracket, the sign of that term will have to be placed *before* the bracket, and this sign will of course affect all the terms we may place within the bracket.

Thus, $+a-b-c$, collected in a bracket with $+a$ as *first* term, will be $+(a-b-c)$; but, with $-b$ as *first* term, $-(b-a+c)$, and with $-c$ as *first* term, $-(c-a+b)$.

So also we might use an *inner* bracket, and write the quantity thus :—

$$+\{(a-b)-c\}, \text{ or } +\{a-(b+c)\}, \text{ or } -\{(b-a)+c\}, \text{ or } -\{b-(a-c)\}, \text{\&c.}$$

Hence, the following Rules :—

Rule (i) *Quantities can be enclosed within a bracket, preceded by the sign* (+), *without changing their signs.*

Rule (ii) *Quantities can be enclosed within a bracket, preceded by the sign* (−), *provided the signs of all the quantities within the bracket be changed.*

Ex. $a-2b+4c-2d-e-5=a-[2b-4c+2d+e+5]$
$$=a-[2b-\{4c-2d-e-5\}]$$
$$=a-[2b-\{4c-(2d+e+5)\}]$$
$$=a-[2b-\{4c-(2d+\overline{e+5})\}].$$

67. When any terms of a quantity contain some common **factor**, a bracket is often employed to collect the *other* factors, considered as its literal coefficients, into one quantity, which is set before or after the common *factor.*

Thus, we have seen already that $3x+5x-6x=2x$, that is, $=(3+5-6)x$.

In like manner, $ax+bx-x=(a+b-1)x$.

$$2a-4ax+6ay=2a(1-2x+3y).$$

$$(a+2b)x^2-(2b-c)x^2-(2c-a)x^2=\{(a+2b)-(2b-c)-(2c-a)\}x^2$$
$$=(2a-c)x^2.$$

68. *Conversely*, when a bracket comes in this way before or after a single term as **factor**, it may be resolved after multiplying each term of the quantity within it by the common *factor.*

Thus, $a(b-x)-(a-y)b=(ab-ax)-(ab-by)$
$$=ab-ax-ab+by=by-ax=-(ax-by).$$

Exercise XIX.

Place the following in a bracket preceded (i) by a positive sign, (ii) by a negative sign :—

1. $-3a+5b-4c$. **2.** $2-a+b-c$. **3.** $2a+3b-4c-5$.

Express by brackets, keeping the terms in the order given, and taking them (i) in sets of two, (ii) in sets of three :—

4. $3x-2y+5z+a+3b-2c$. **5.** $2a-3b+4c-2d-e+5$.

6. $a^5+2a^4-3a^3+5a^2-3a-1$. 7. $4a^2+5b^2-3c^2-2x^2-3y^2+2z^2$.

8. $-5a+2c-3d-2z-y+3x$ 9. $-3x^2-2y^2-5z^2-a+2b-3c$.

10-15. Express Examples 4-9 by brackets, in sets of three, with the second and third of each set enclosed in an inner bracket

In the following expressions bracket like powers of x

16. $ax^3-bx^2-cx-bx^3+cx^2-dx+cx^3-dx^2-ex$

17. $4x+5bx^3+3bx-2x^2+ax^3-ax$.

18. $-3x^3+5ax^2-3cx+8ax^2+6x-3x^3$

19. $3x^3-2c^2x^3-4abx^6+3ax^2+3x^2-3abx^3$

20. $4ax+6x^2+a^2x^3-3bx^2-2x-5bx^3$.

Add together —

21. $ax-by$, $x+y$, and $(a-1)x-(b+1)y$

22. $(a+c)x^2-3(a-b)xy+(b-c)y^2$, $(b-c)x^2+2(a+b)xy+(a-b)y^2$, and $(a-5b)xy-(a-c)y^2+(a+b)x^2$

23. $(a-2p)x^3-2x^2+(2c-3r)x$, $(a+2p)x^3+(q-b)x^2-x$, $-(p-a)x^3-(b+q)x^2-(c-1)x$, and $-x^3+3bx^2-(c-2r)x$

24. $(a+b)x+(b+c)y$ and $(a-b)x-(b-c)y$, and subtract the latter from the former

25. From ax^3-bx^2+x take $-px^3-qx^2+rx$.

IV MULTIPLICATION.

69 Multiplication consists in finding the sum of a number repeated any number of times

Thus, $a\times b$ means a repeated b times

$$=a+a+a+\ \ldots\ \text{continued } b \text{ terms}$$

70 To prove that $a\times b=b\times a$, when a and b are positive integers.

As a numerical example, let a and b stand for 4 and 5 respectively.

Then $4\times 5=4$ repeated 5 times

$$=(1+1+1+1)+(1+1+1+1)+\ldots \text{ repeated 5 times}$$

$$\begin{aligned}=\ &1+1+1+1\\ &+1+1+1+1\\ &+1+1+1+1\\ &+1+1+1+1\\ &+1+1+1+1\end{aligned}$$

$=(1+1+1+1+1)+(1+1+1+1+1)+\ldots\ldots$ repeated 4 times
(adding vertical columns)
$=5+5+\ldots\ldots\ldots$ repeated 4 times
$=5\times 4$

Hence, algebraically, $a\times b=b\times a$

71 The definition of Multiplication given in Art 69 does not hold good when the quantities to be multiplied together are not positive whole numbers It has to be modified. (See Arith. Art 269)

Thus, to multiply 5 by $\frac{3}{7}$, we do to 5 that operation which when done to unity gives $\frac{3}{7}$, that is, we divide 5 into 7 equal parts and take 3 such parts

So algebraically, to multiply a by b in this sense, is to do to a that operation which has been done to unity to get b

72 In this extended sense of multiplication the statement $ab=ba$ can be extended so as to include every case in which a and b stand for any quantities

Hence, $abc=a\times b\times c=(a\times b)\times c=b\times a\times c=bac,$
$=b\times(a\times c)=b\times c\times a=bca$, and so on

Thus, as we have already seen (Art 10) that factors can be multiplied in any order without affecting the value of the result, but it is usual to write the letters in alphabetical order

73 **Rule of Indices** Since $a^3=a\times a\times a$, and $a^5=a\times a\times a\times a\times a$, (by definition)

$\therefore a^3\times a^5=a\times a\times a\times a\times a\times a\times a\times a$ (8 factors)
$=a^8$ (by definition) $=a^{3+5}$

Similarly, $a^2\times a^3\times a^4=a^5\times a^4=a^9$, and so on

Hence the following Rule —

Rule *The product of powers of the same quantity is obtained by adding the indices of the powers*

74 If m be a *positive* integer, a and b having any values whatever, then

$(a+b)m=(a+b)+(a+b)+(a+b)+\ldots$ repeated m times
$=a+a+a+\ldots$ repeated m times
$+b+b+b+\ldots$ repeated m times
$=a$ taken m times $+b$ taken m times
$=am+bm$

Hence, when m is a *positive* integer, we have

$$(a+b)m=am+bm \ldots\ldots\ldots\ldots\ldots (1)$$

The above law being true for any *positive* value of m, it must also be true for any *negative* value. For, suppose $m=-x$, where x is any positive quantity.

Then $(a+b).(-x)=-\{(a+b)x\}=-(ax+bx)$, by (1)
$=-ax-bx \;=a(-x)+b.(-x)$;

Hence, for all values of a, b and m, we have

$$(a+b)m=am+bm \quad \ldots \quad \ldots\ldots \quad (2)$$

In like manner,

$$(a-b)m=(a-b)+(a-b)+(a-b)+ \ldots \text{ repeated } m \text{ times}$$
$$=a \text{ repeated } m \text{ times} - b \text{ repeated } m \text{ times}$$
$$=am-bm.$$

75. Rule of Signs. Let it be required to multiply $a-b$ by $c-d$.

Here, $(a-b)(c-d)=(a-b)x$, (writing x for $c-d$)
$=ax-bx$, Art. 74
$=a(c-d)-b(c-d)$
$=(ac-ad)-(bc-bd)$
$=ac-ad-bc+bd.$

If on the right-hand side of this result, each term be considered separately, we find that

$$(+a)\times(+c)=+ac, \quad (+a)\times(-d)=-ad,$$
$$(-b)\times(+c)=-bc, \quad (-b)\times(-d)=+bd.$$

The results enable us to state the following Rule :—

Rule. *Like signs multiplied together give plus* $(+)$.
Unlike signs multiplied together give minus $(-)$.
Or shortly, *Like signs produce* $+$ *and unlike* $-$.

76 The convention of signs for direction (Art. 42) enables us to illustrate more clearly the *Rule of Signs*.

(i) To multiply $+3$ by $+4$: this is arithmetically intelligible.

$(+3)\times(+4)=+3$ repeated 4 times,
$=(+3)+(+3)+(+3)+(+3)=+12.$

So, in algebra, $(+a)\times(+b)=+ab.$

(ii) To multiply -3 by $+4$; this is arithmetically intelligible.

$(-3)\times(+4)=-3$ repeated 4 times,

$=(-3)+(-3)+(-3)+(-3)=-12$

So, in algebra, $(-a)\times(+b)=-ab$.

(iii) To multiply $+3$ by -4, this is meaningless in arithmetic.

Its algebraic interpretation is the following:—

$(+3)\times(-4)=+3$ repeated 4 times, *but in the opposite direction*,

$=+12$, *in the opposite direction*,

$=-12$

So, in algebra, $(+a)\times(-b)=-ab$

(iv) To multiply -3 by -4, this is arithmetically meaningless.

Its algebraic interpretation is the following:—

$(-3)\times(-4)=-3$ repeated 4 times, *but in the opposite direction*,

$=-12$, *in the opposite direction*,

$=+12$

So, in algebra, $(-a)\times(-b)=+ab$

77. To illustrate the Rule of Signs, we shall add here a few examples in Substitution, where some of the symbols denote negative quantities. It may easily be shewn by repeated applications of the Rule of Signs that any *odd* power of a negative quantity is *negative*, and any *even* power of a negative quantity is *positive*.

Ex. 1 If $a=-1$, $b=3$, $c=-2$, find the value of $-6a^2bc^3$.

Here, $-6a^2bc^3=-6\times(-1)^2\times3\times(-2)^3$
$=-6\times1\times3\times-8$
$=\underline{144}$.

We write down at once, $(-1)^2=+1$ and $(-2)^3=-8$.

Ex. 2 If $a=0$, $b=-1$, $c=2$, find the value of

$$(a-b)^2+(b-c)^2+(c-a)^2.$$

The given expression $=(1)^2+(-3)^2+(2)^2=1+9+4=\underline{14}$.

Ex. 3. Find the value of $(x^2+y^2)(x-y)^2+y(y-z)^3+(z-x)^4$, when $x=4$, $y=1$, $z=3$.

The given expression $=(16+1)(3)^2+1(-2)^3+(-1)^4$
$=17\times9+1\times-8+1=153-8+1=\underline{146}$.

Exercise XX.

If $a=-2$, $b=-3$, $c=1$, find the value of

1. $3ac$. 2. $-5bc$. 3. $4a^2b$. 4. $-5a^3b^2c$.

5. a^3b^3. 6. $(-ab)^3$. 7. $-a^4bc^3$. 8. $-3ab^3c^4$.

9. $a^3+b^3+c^3$. 10. $ab+bc+ca$ 11. $a^4+b^4+c^4$.

12. $a^2+b^2+c^2-bc-ca-ab$. 13. $a^2b^2-2ac^2+3b^2c$.

If $a=3$, $b=-2$, $c=0$, $d=2$, find the value of

14. $b(a+c)+c(a+b)+a(c-b)$ 15. b^c+a^d. 16. $3b^a-d^a$.

17. Find the value of $(a-b)^2+(b-c)^2+(a-b)(b-c)+5c^3$, when $a=1$, $b=-2$, $c=\frac{1}{2}$

18. Find the value of $(5a-3b)(a-b)-b\{3a-c(4a-b)-b^2(a+c)\}$, when $a=0$, $b=-1$, $c=\frac{1}{2}$.

19. Find the value of

(1) $(ac-bd)\sqrt{(a^2bc-b^2cd+c^2ad-2)}$, when $a=-1$, $b=2$, $c=3$, $d=0$

(2) $3abc-2bcd\sqrt[3]{(a^3bc-bc^2d+3)}$, when $a=0$, $b=1$, $c=-2$, $d=3$.

(3) x^2y^2, when $x=1·5$, $y=-·4$

(4) $a^2x^3y^4$, when $x=-2$, $y=3$, $a=-·25$.

(5) $3(a+2x)^2-2(a+2x)(a-2x)+(a-2x)^2$, when $x=1$, $a=-2$.

20. Find the value of $\frac{3x^3-2x^2y-3xy^2+2y^3}{4x^3-3x^2y-4xy^2+3y^3}$, when $x=2$, $y=-1$.

---

78. To multiply two simple algebraical quantities.

Rule. *Multiply together respectively the numerical coefficients and letters and give to each letter in the product an index equal to the sum of the indices that letter has in the separate factors; and then, if both the factors have the same sign, prefix to this product the sign +, if different signs, the sign −.*

Ex. 1. $3ac\times2b=+3\times2\times a\times b\times c=\underline{6abc}$.

Ex. 2. $-5a^2b^3\times4b^5=-5\times4\times a^2\times b^{3+5}=\underline{-20a^2b^8}$.

Ex. 3. $-7a^2bc\times-3a^3b^2c^4=+7\times3\times a^{2+3}\times b^{1+2}\times c^{1+4}=\underline{21a^5b^3c^5}$.

79. When several quantities are multiplied together, their product is called **the continued product**.

80. To find the continued product of several simple quantities, instead of multiplying them together successively by the above Rule,

it will be shorter to multiply them at once together, and then prefix to this product the sign + or −, according as the number of *negative* factors is **even** or **odd**.

Ex. Find the continued product of $2a^2b$, $-3a^3b^2$, $-5ab^4$.

$2a^2b \times -3a^3b^2 = -6a^5b^3$;
$-6a^5b^3 \times -5ab^4 = +30a^6b^7$.
Thus the complete product is $30a^6b^7$.

The work, however, may be done shortly, thus :—

$$2a^2b \times -3a^3b^2 \times -5ab^4$$
$$= +2 \times 3 \times 5a^{2+3+1} \times b^{1+2+4} = 30a^6b^7.$$

Exercise XXI.

Multiply :—

1. $7a$ by $4b$.
2. $-2a$ by $3c$.
3. $-3a$ by $-5b$.
4. $5ac$ by $2bd$.
5. $3a^2x^2$ by $4a^2$.
6. $3ab^2$ by $-a^2b$
7. $6ab^2y^3$ by $-2b^3y^3$.
8. $-3a^2b^2c^2$ by $5a^3b$.
9. $9a^3b^4$ by $-2b^2c^2$.
10. $-13x^6y^6z^6$ by $-2x^3y^2z^2$.
11. ax^2y^3 by bxy.
12. x^2y by $-xy^2$.
13. $-abc$ by $-ac$.
14. $-acx$ by $-2axy$.
15. mx^3 by $-nx^3$.

Multiply together

16. $3a^2b, -2ab^3, -2a^2$.
17. $4a^2, -3b, -3c^3, -ad$
18. $-2a^4b, -3a^2b, -6a$.
19. $3abc, 5a^2b, -4ax^2y$.
20. $6x^3y^3, -4xy, -9x^4y^2$.
21. $2a^3b^3, -3a^2b^4, 7ab^6$.

Write down the values of

22. $(-4xy)^2$.
23. $(-9a^2)^2$.
24. $(12x^4y^4)^2$.
25. $(6a^2b^2)^3$.
26. $(-9a^4)^3$.
27. $(2x^2y^3z^4)^2$.
28. $(3abc)^4$.
29. $(-3abc)^5$.
30. $(-2a^3b)^3$.
31. $(-3ab^2)^4$.
32. $(-a^2)^7$.
33. $(-a)^{11}$.
34. $(-ab)^2 \times (-ab)^3 \times (-ab)^4$.
35. $4(bc)^2 \times 3(ac)^3 \times 5abc$.

81. We have $(a-b+c)m = (x+c)m$, (writing x for $a-b$)

$$= xm + cm,$$
$$= (a-b)m + cm,$$
$$= am - bm + cm.$$

Hence, to multiply a compound expression by a single quantity we have the following Rule :—

Rule. *Multiply each term of the multiplicand separately by the simple quantity beginning at the left.*

Ex. Multiply $3x^2-2xy+4y^2$ by $2a^2x$ and $-2a^2b^2+5ab^3-7b^4$ by $-4ab$.

The process is generally conducted thus :—

(i)
$$\begin{array}{l} 3x^2-\ 2xy\ +4y^2 \\ 2a^2x \\ \hline 6a^2x^3-4a^2x^2y+8a^2xy^2 \end{array}$$

(ii)
$$\begin{array}{l} -2a^2b^2\ +5ab^3\ -7b^4 \\ -4ab \\ \hline 8a^3b^3-20a^2b^4+28ab^5 \end{array}$$

Exercise XXII.

Multiply

1. a^2+3ab by 4.
2. $x+3yz$ by ax.
3. $4x+3y$ by 6.
4. ax^2+3y^2z by $-2b$.
5. $a^2b^2-b^3c^3$ by ab.
6. $3ax+2b$ by $-3a$.
7. $-4x^2y-2y^2z$ by $-3xyz$.
8. $-3a^2b^3-4c^2d^2$ by $-5a^2d^2$
9. $-3ab^2+2a^2b+7a^4$ by $-3a^2b$.
10. $-4+6ab+4a^2b^2$ by $-a^3$.
11. x^2-xy+y^2 by x.
12. a^2-ax+x^2 by $-ax$.
13. x^2-ax+b by $-abx$.
14. $x^3-3x^2y+3xy^2-y^3$ by $-xy$.
15. $a^3+4a^2b-3ab^2-b^3$ by $-3ab$.
16. $a^2bc^2-3bc+2cd$ by $5a^2b^2c^2$.

Find the continued product of

17. $-3ab$, $4ac$, $-2b^2$, $2a^2b$.
18. $-x^2, -2x^2$, $-2xy$, y^3, $-4x^3y$.
19. $-xyz$, $-3y^2$, $2x^2$, $4z^2$, -5, $-2x^2y^2z^2$.
20. x^2y^2, $-xyz$, a^3b^4, $-7x^3y^2$, $-2a^3x^3$, $-3a^2$.

Simplify the following :—

21. $6bc^2(2b^2-3bc-4c^2)-8c^3(2b^2-3bc-4c^2)$.
22. $2xy(3xy+4y^2)-3y^2(3xy+4y^2)$.
23. $x^2(2x^2-ax+a^2)-ax(2x^2-ax+a^2)-a^2(2x^2-ax+a^2)$.
24. $a^2(a^4+2a^2b^2+4b^4)-2b^2(a^4+2a^2b^2+4b^4)$.
25. $4a(-a^3+2a^2b-b^3)+8b(-a^3+2a^2b-b^3)$.

---

82. Since $(a+b)m=am+bm$, Art. 74

We have $(a+b)(c+d)=a(c+d)+b(c+d)$, (writing $c+d$ for m)

$$=ac+ad+bc+bd.$$

Similarly, it may be shewn that

$$(a+b)(c-d)=ac-ad+bc-bd\ ;$$
$$(a-b)(c+d)=ac+ad-bc-bd\ ;$$
$$(a-b)(c-d)=ac-ad-bc+bd.$$

83. Again, since $(a-b+c)m=am-bm+cm$, Art. 81

We have $(a-b+c)(x-y)=a(x-y)-b(x-y)+c(x-y)$

(writing $x-y$ for m)

$$=(ax-ay)-(bx-by)+(cx-cy)$$
$$=ax-ay-bx+by+cx-cy.$$

Hence, to multiply one compound expression by another, we have the following Rule.

Rule. *Multiply each term of the multiplicand by each term of the multiplier and add the several products together for the complete product.*

Ex. 1. Multiply $2x-5$ by $3x+2$.

$$(2x-5)(3x+2)=(2x-5)3x+(2x-5)2$$
$$=6x^2-15x+4x-10$$
$$=6x^2-11x-10. \quad \textit{Ans.}$$

The operation is generally arranged thus :—

Like terms are placed in the same column.

$$\begin{array}{rl}
2x-5 & \\
3x+2 & \\
\hline
6x^2-15x & =\text{product by } 3x \\
+4x-10 & =\text{product by } 2. \\
\hline
6x^2-11x-10 & =\text{whole product (by addition).}
\end{array}$$

Ex. 2. Multiply $3a^3-2a^2b-ab^2$ by $7ab-5b^2$.

$$\begin{array}{r}
3a^3-2a^2b-ab^2 \\
7ab-5b^2 \\
\hline
21a^4b-14a^3b^2-7a^2b^3 \\
-15a^3b^2+10a^2b^3+5ab^4 \\
\hline
21a^4b-29a^3b^2+3a^2b^3+5ab^4
\end{array}$$

Ex. 3. Multiply $2x^2-5x+6$ by $3x^2-4x-3$.

$$\begin{array}{rl}
2x^2-5x+6 & \\
3x^2-4x-3 & \\
\hline
6x^4-15x^3+18x^2 & =\text{product by } 3x^2 \\
-8x^3+20x^2-24x & =\text{product by } -4x \\
-\;\;6x^2+15x-18 & =\text{product by } -3 \\
\hline
6x^4-23x^3+32x^2-\;\;9x-18 & =\text{whole product.}
\end{array}$$

84. A re-arrangement of the terms will be found convenient when the expressions are not arranged according to the powers (ascending or descending) of some common letter.

Ex. 4. Find the product of $a^2-3ab-4b^2$ and $5b^2+2a^2-ab$.

$$\begin{array}{l} a^2-3ab-4b^2 \\ \underline{2a^2-ab+5b^2} \\ 2a^4-6a^3b-8a^2b^2 \\ \qquad -a^3b+3a^2b^2+4ab^3 \\ \qquad\qquad \underline{+5a^2b^2-15ab^3-20b^4} \\ 2a^4-7a^3b \qquad\qquad -11ab^3-20b^4 \end{array}$$

Exercise XXIII.

Multiply

1. $x-6$ by $x+13$. **2.** $x+5$ by $x+3$. **3.** $x+5$ by $x-3$

4. $x-5$ by $x+3$. **5.** $2a-b$ by $a+2b$. **6.** $5a+2b$ by $4a+3b$.

7. $2a+b$ by $a+3b$. **8.** $2a-b$ by $c-3d$. **9.** $3x+2y$ by $2x+3y$.

10. $3ab+4b^2$ by $2ab-3b^2$. **11** x^2-xy+y^2 by $x+y$.

12. x^2+3x-2 by $x+3$. **13.** x^2-4x+3 by $x-2$.

14. $x^3+2x^2y+4xy^2+8y^3$ by $x-2y$. **15.** $2x^3+4x^2+8x+16$ by $3x-6$.

16. $6x^3-4ax^2-3a^2x+2a^3$ by $3x+2a$.

17. $5x^4-2x^3y-3x^2y^2+2xy^3+3y^4$ by $2x-3y$.

18. $27x^3+9x^2y+3xy^2+y^3$ by $3x-y$.

19. $a^4-2a^3b+4a^2b^2-8ab^3+16b^4$ by $a+2b$.

20. $a^4+a^2x^2+x^4$ by a^2-x^2. **21.** $a+2b-3c$ by $a-2b+3c$.

22. a^2+2a-1 by a^2-a+1 and by a^2-3a-1.

23. $x^2+2ax+3a^2$ by $x^2-2ax+a^2$.

24. $x^4-ax^3+a^3x-a^4$ by x^2+ax+a^2.

25. $2x^4-3x^3-3x^2+3x-1$ by x^2-3x-3.

26. $a^4-2a^3b+3a^2b^2-2ab^3+b^4$ by $a^2+2ab+b^2$.

27. $x^2-ax+2a^2$ by $x^2+3ax+4a^2$. **28.** $3x^3-2x-5$ by $2x^3-x^2+3$.

29. $9a^2-3ab+b^2-6a-2b+4$ by $3a+b+2$.

30. $a^2+b^2+c^2+ab-ac+bc$ by $a-b+c$.

31. $a^2+4b^2+9c^2+2ab+3ac-6bc$ by $a-2b-3c$.

32. $5x^3-10x^2+12x-8$ by $5x^2+10x+8$.

33. $a^3-2a^2b+2ab^2-b^3$ by $3a^2-2ab+b^2$.

34. $9a^2+4b^2+c^2-2bc-3ac-6ab$ by $3a+2b+c$.

35. $3a^4+5a^2b^2-6ab^3-4a^3b+7b^4$ by $2a-3b$.

36. $2a^2-6-4a+a^3$ by $2a-3+a^2$.

37. $a^3+11a-4a^2-24$ by a^2+5+4a.

38. a^3+2a^2+2a+1 by $a^3-1+2a-2a^2$.

39. x^3-2x^2+3x-4 by $4x^3+3x^2+2x+1$.

40. $1-2x+3x^2-4x^3$ by $1+2x+3x^2+4x^3$.

41. $ab+ac+bd+cd$ by $ab-ac-bd+cd$.

42. $a^2-2ab+b^2+c^2$ by $a^2+2ab+b^2-c^2$.

43. $1-x+x^2-x^3$ by $1+x+x^2+x^3$. (C. E. 1859).

Find the coefficient of x^4 and of x^3 in the following products :—

44. $(x^3+6x^2+8x-8)(x^2-2x+4)$.

45. $(3x^3-7x^2-8x-9)(5x^3+11x^2-7x+8)$.

46. $(81x^4+27x^3y+9x^2y^2+3xy^3+y^4)(3x-y)$.

85. The following Examples with their Solutions, illustrating the use of brackets in Multiplication, should be carefully noticed.

Ex. 1. Multiply $x+a$ by $x+b$ and $x^2-(a+b)x+ab$ by $x+c$.

(i)
$$\begin{array}{rl} & x+a \\ & x+b \\ \hline & x^2+ax \\ & \quad +bx+ab \\ \hline \textit{Ans.} & x^2+(a+b)x+ab \end{array}$$

(ii)
$$\begin{array}{l} x^2-(a+b)x+ab \\ x+c \\ \hline x^3-(a+b)x^2+abx \\ \quad + \quad cx^2-(ac+bc)x+abc \\ \hline x^3-(a+b-c)x^2+(ab-ac-bc)x+abc \end{array}$$

Ex. 2. Multiply x^3-ax^2+bx-c by x^2+mx+n.

$$\begin{array}{l} x^3-ax^2+bx-c \\ x^2+mx+n \\ \hline x^5-ax^4+bx^3\quad -cx^2 \\ \quad +mx^4-amx^3+bmx^2-cmx \\ \quad\quad +nx^3\quad -anx^2+bnx-cn \end{array}$$

Ans. $x^5-(a-m)x^4+(b-am+n)x^3-(c-bm+an)x^2-(cm-bn)x-cn$

Exercise XXIV.

Multiply

1. $x-a$ by $x-b$. 2. $x-a$ by $x+b$. 3. $x+a$ by $x-b$.
4. x^2-ax+b by $x-c$, and by x^2+ax-c.
5. $a+mx-nx^2$ by $a-2mx+nx^2$, and by $a^2+2nx-mx^2$.
6. $x^2+(a+b)x+ab$ by $x+c$, and by $x^2-(a+b)x+ab$.
7. $x^3+(m+n)x^2+2mnx+1$ by $(m+n)x-1$.
8. $x^2-(a-p)x+a^2-ap+q$ by $x-a$.
9. $1-ax+bx^2-cx^3$ by $1+x-x^2$. 10. ax^2-bx+c by x^2-x+1.
11. Find the coefficient of x^4 in the product of
$x^4-ax^3+bx^2-cx+d$ by x^2+px+q. (C.E. 1885).
12. Find the coefficient of x^3 in the following product :—
$(ax^4+bx^3+cx+d)(ax^2-bx+c)$.

Find the continued product of

13. $ax-by$, $ax+cy$, and $ax-dy$.
14. $2x-m$, $2x+n$, $x+2m$ and $x-2n$.
15. x^2+ax-b^2, x^2+bx-a^2 and $x-(a+b)$.

86. When the coefficients are fractional, they should be dealt with by the Rules of Fractions in Arithmetic.

Ex. 1. $\frac{5}{8}a^3b^2 \times -\frac{8}{3}a^2bx = -\frac{5}{8}\times\frac{8}{3}a^{3+2}b^{2+1}x = -\underline{\frac{5}{3}a^5b^3x}.$

Ex. 2. $24a(\frac{1}{3}a^2+\frac{3}{4}b^2-\frac{5}{8}bc) = 24a\times\frac{1}{3}a^2+24a\times\frac{3}{4}b^2-24a\times\frac{5}{8}bc.$
$$=\underline{8a^3+18ab^2-15abc}.$$

Ex. 3. $(\frac{1}{6}x-\frac{4}{3}y-z)\times-\frac{3}{5}xy^2z = -\frac{3}{5}xy^2z\times\frac{1}{6}x+\frac{3}{5}xy^2z\times\frac{4}{3}y$
$$+\frac{3}{5}xy^2z\times z$$
$$= -\underline{\frac{1}{10}x^2y^2z+\frac{4}{5}xy^3z+\frac{3}{5}xy^2z^2}.$$

Ex 4. Multiply $\frac{1}{2}a^2+\frac{1}{4}ab-\frac{2}{3}b^2$ by $\frac{1}{4}a+\frac{3}{5}b$.

$$\begin{array}{l}
\frac{1}{2}a^2+\frac{1}{4}ab-\frac{2}{3}b^2 \\
\frac{1}{4}a+\frac{3}{5}b \\
\hline
\frac{1}{8}a^3+\frac{1}{16}a^2b-\frac{1}{6}ab^2 \\
\qquad +\frac{3}{10}a^2b+\frac{3}{20}ab^2-\frac{2}{5}b^3 \\
\hline
\frac{1}{8}a^3+\frac{29}{80}a^2b-\frac{1}{60}ab^2-\frac{2}{5}b^3
\end{array}$$

Exercise XXV.

Multiply

1. $-\frac{3}{9}a^2b^3$ by $-\frac{9}{13}ab^4$.
2. $\frac{6}{7}x^2y^2z^3$ by $\frac{7}{8}x^3z$.
3. $\frac{5}{8}x^3$ by $-\frac{8}{3}ax^2$.
4. $-\frac{8}{11}a^2b^2$ by $\frac{33}{5}bc^4$.
5. $\frac{7}{9}a^3b^4$ by $-\frac{9}{35}ab^2c^3$.
6. $-\frac{1}{9}x^2y$ by $-\frac{2}{9}y^2$.
7. $3x^2y^3+2xy^3-\frac{1}{2}z^3$ by $-\frac{3}{5}xy$.
8. $\frac{1}{4}a^2+\frac{2}{3}b^2-\frac{3}{4}c^2$ by $12x$.
9. $\frac{4}{9}a^4x^4-\frac{1}{2}a^3bx^3y+\frac{9}{16}a^2b^2x^2y^2-\frac{81}{128}ab^3xy^3$ by $-\frac{3}{4}by$.
10. $\frac{1}{5}a^4-\frac{1}{4}a^3b+\frac{1}{3}a^2b^2+1$ by $\frac{10}{3}a^3b^2$, and by $-2a^4$.

Find the value of

11. $\frac{3}{4}a^3b^2\times-\frac{4}{5}ab^4\times-\frac{5}{6}a^2b$.
12. $\frac{10}{3}a^2b^2\times-\frac{5}{4}a^3b\times 2a^4$.
13. $\frac{20}{25}x^2\times-\frac{3}{5}x^2y\times-\frac{3}{2}xy^2\times\frac{1}{2}xy\times-\frac{1}{5}x^3y^3$.

Multiply

14. $a^3-\frac{1}{2}a^2b+\frac{1}{3}b^3$ by $2a^3-3b^2$.
15. $\frac{1}{2}x^2+\frac{1}{4}xy-\frac{2}{5}y^2$ by $\frac{1}{4}x+\frac{1}{2}y$.
16. $\frac{2}{3}a^3-\frac{4}{5}a^2b+\frac{1}{2}b^3$ by $\frac{3}{5}a-2b$.
17. $\frac{3}{2}a^2-a+\frac{2}{3}$ by $3a^2+2a+\frac{4}{3}$.
18. $\frac{3}{4}x^2+\frac{3}{2}y^2-\frac{3}{2}xy$ by $\frac{2}{3}xy+\frac{2}{3}y^2+\frac{1}{3}x^2$.
19. $3x^4-2x^3+\frac{3}{2}x^2-3$ by $\frac{1}{2}x^4+\frac{1}{3}x^3-\frac{1}{4}x^2-\frac{1}{2}$.
20. $1-\frac{1}{2}a+\frac{1}{3}a^2+\frac{1}{4}a^3$ by $1+\frac{1}{2}a-\frac{1}{3}a^2+\frac{1}{4}a^3$.

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87. Continued product. A judicious choice of arrangement of the quantities to be multiplied together always lessens the trouble of multiplication.

Ex. Find the continued product of $a-b$, a^2+b^2, $a+b$ and a^4+b^4.

Here, instead of multiplying the quantities in the order as they stand, we first multiply $a-b$ by $a+b$, and then the product by a^2+b^2 and then by a^4+b^4. Thus,

(i)
$$\begin{array}{lr} a-b & \\ a+b & \\ \hline a^2-ab & \\ \quad +ab & -b^2 \\ \hline a^2 \qquad & -b^2 \end{array}$$

(ii)
$$\begin{array}{lr} a^2-b^2 & \\ a^2+b^2 & \\ \hline a^4-a^2b^2 & \\ \quad +a^2b^2 & -b^4 \\ \hline a^4 \qquad & -b^4 \end{array}$$

(iii)
$$\begin{array}{lr} a^4-b^4 & \\ a^4+b^4 & \\ \hline a^8-a^4b^4 & \\ \quad +a^4b^4 & -b^8 \\ \hline a^8 \qquad & -b^8 \end{array}\ \textit{Ans.}$$

Exercise XXVI.

Multiply together :—

1. $a-x$, a^2-x^2, and a^3-x^3.
2. $2x^2+ax+a^2$, $2x-a$, and $x+a$.
3. $x+1$, $x-1$, $x+3$, and $x-3$.
4. $x+a$, $x+b$, and $x+c$.

5. a^2-a+1, a^2+a+1, a^4-a^2+1, and a^8-a^4+1.
6. $a-2b$, $a-b$, $a+b$ and $a+2b$.
7. a^2+ab+b^2, a^2-ab+b^2 and $a^4-a^2b^2+b^4$.
8. $a+1$, $a+2$, $a+3$ and $a+4$.

Find the continued product of

9. $x+a$, $x-a$, $x+2a$, and $x-2a$.
0. $mx+2ny$, $mx-2ny$, $mx-3ny$ and $mx+3ny$.
1. x^2-2y^2, $x^2-2xy+2y^2$, x^2+2y^2, and $x^2+2xy+2y^2$. (B.M. 1885.)

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88. Detached Coefficients. In such cases the coefficients nly are written down, and the powers of the symbols are understood ust as in Arithmetic the powers of 10 are understood in expressing number by digits in the ordinary system of Notation. If any ower be missing, 0 must be inserted as in Arithmetic.

Ex. 1. Multiply x^3-2x^2+4x+5 by $x-3$.

$$\begin{array}{rrrrr}
\multicolumn{5}{l}{x^3-2x^2+4x+5} \\
\multicolumn{5}{l}{x-3} \\
1 & -2 & +4 & +5 & \\
 & -3 & +6 & -12 & -15 \\
\hline
\multicolumn{5}{l}{x^4-5x^3+10x^2-7x-15}
\end{array}$$

inserting the requisite powers of x in the last line.

Ex. 2. Multiply $3x^4-5x^2+6$ by $2x^2-3x+4$.

$$\begin{array}{rrrrrrr}
\multicolumn{7}{l}{3x^4+0x^3-5x^2+0x+6} \\
\multicolumn{7}{l}{2x^2-3x+4} \\
\hline
6 & +0 & -10 & +0 & +12 & & \\
 & -9 & -0 & +15 & -0 & -18 & \\
 & & +12 & +0 & -20 & +0 & +24 \\
\hline
\multicolumn{7}{l}{6x^6-9x^5+2x^4+15x^3-8x^2-18x+24}
\end{array}$$

89. *Analogy between the Arithmetical and Algebraical methods multiplication.*

Multiply 425 by 23.

$$\begin{array}{r}
425 \\
23 \\
\hline
850 \\
1275 \\
\hline
9775
\end{array}$$

The above is an abbreviated form of the following :—

$$
\begin{array}{l}
4.10^2+2.10\ +5 \\
\underline{2.10\ +3\qquad\qquad\qquad} \\
8.10^3+4.\ 10^2+10.10 \\
\underline{\qquad\quad 12.10^2+6.\ 10+15} \\
8.10^3+16.10^2+16.10+15
\end{array}
$$

$$=8.10^3+(1.10^3+6.10^2)+(1.10^2+6.10)+(1.10+5)$$

$$=9.10^3+7.10^2+7.10+5=9775.$$

If we now multiply $4x^2+2x+5$ by $2x+3$, the analogy between the two methods is at once evident. Thus,

$$
\begin{array}{l}
4x^2+2x\ +5 \\
\underline{2x\ +3\qquad\qquad} \\
8x^3+\ 4x^2+10x \\
\underline{\quad +12x^2+\ 6x+15} \\
8x^3+16x^2+16x+15
\end{array}
$$

Exercise XXVII.

Multiply (by the method of *detached coefficients*) :—

1. $3x^2-2x+7$ by $2x-7$.

2. $3x^2+4x+5$ by $4x-5$.

3. $2x^2-3x-6$ by x^2-x+2.

4. x^2-2x-5 by x^2+2x+3.

5. $9a^2-6ab+4b^2$ by $3a+2b$.

6. $x^2+2xy+4y^2$ by $x-2y$.

7. $4x^3-5x^2+3x-1$ by x^2+2x-4, and by $4x^2-3x+2$.

8. $5x^4-x^3+4x^2-2x+3$ by x^2-2x+3.

V. IMPORTANT RESULTS IN MULTIPLICATION.

90. The student should notice certain results in so as to be able to apply them when similar cases oc down at once the corresponding products.

91. The Square of a Binomial.

$$
(1)\quad
\begin{array}{l}
a\ +b \\
\underline{a\ +b\qquad} \\
a^2+\ ab \\
\underline{\quad +\ ab+b^2} \\
\underline{\underline{a^2+2ab+b^2}}
\end{array}
\qquad\qquad
(2)\quad
\begin{array}{l}
a\ -b \\
\underline{a\ -b\qquad} \\
a^2-\ ab \\
\underline{\quad -\ ab+b^2} \\
\underline{\underline{a^2-2ab+b^2}}
\end{array}
$$

Thus, (1) $(a+b)^2=a^2+2ab+b^2$. (2) $(a-b)^2=a^2-2ab+b^2$.

From the above products we learn that :—

(1) *The square of the* sum *of any two quantities is equal to the sum of their squares* plus *twice their product.*

(2) *The square of the* difference *of any two quantities is equal to the sum of their squares* minus *twice their product.*

Ex. 1. $(x+2)^2 = x^2 + 2.x.2 + 2^2 = x^2 + 4x + 4.$

Ex. 2. $(3a+2b)^2 = (3a)^2 + 2.3a.2b + (2b)^2 = 9a^2 + 12ab + 4b^2.$

Ex. 3. $(2x-y)^2 = (2x)^2 - 2.2x.y + y^2 = 4x^2 - 4xy + y^2.$

Ex. 4. $(2ax-3by)^2 = (2ax)^2 - 2.2ax.3by + (3by)^2$
$= 4a^2x^2 - 12abxy + 9b^2y^2.$

92. The formulae $(a+b)^2 = a^2 + 2ab + b^2$ and $(a-b)^2 = a^2 - 2ab + b^2$ may advantageously be applied in arithmetical work.

$312^2 = (300+12)^2 = 90000 + 7200 + 144 = 97344.$

$198^2 = (200-2)^2 = 40000 - 800 + 4 = 39204.$

$101{\cdot}3^2 = (100 + 1{\cdot}3)^2 = 10000 + 260 + 1{\cdot}69 = 10261{\cdot}69.$

Exercise XXVIII.

Write down the squares of :—

1. $x+2y.$	**2.** $3x-y.$	**3.** $5a+3b.$	**4.** $3a-5b.$
5. $a^2+b^2.$	**6.** $a^2-b^2.$	**7.** $4ab+3.$	**8.** $2x^2+3.$
9. $5ab+7.$	**10.** $ab-3cd.$	**11.** $2x^2+1.$	**12.** $3x-4y.$
13. $2a^2+3.$	**14.** $3+2x.$	**15.** $2x-3y.$	**16.** $a^2-3ax.$
17. $bx^2-cxy.$	**18.** $2ab+5c.$	**19.** $1+2abc.$	**20.** $4ab^2-3b^2c.$

Without going through the operation of multiplication, find the square of :—

21. 99.	**22.** 85.	**23.** 78.	**24.** 105.	**25.** 1004.
26. 999.	**27.** 1005.	**28.** 500·3.	**29.** 7·996.	**30.** 99·97.

93. The Square of a Multinomial. Art. 91 enables us, by using brackets, to find the square of an expression consisting of any number of terms.

Ex. 1. $(a+b+c)^2 = \{a+(b+c)\}^2$, (taking $b+c$ as one term)

$$\left.\begin{aligned} &= a^2 + 2a(b+c) + (b+c)^2 \\ &= a^2 + 2ab + 2ac + b^2 + 2bc + c^2 \end{aligned}\right\} \text{(Art. 91)}$$

$$= \boldsymbol{a^2 + b^2 + c^2 + 2ab + 2ac + 2bc}$$

$$= \boldsymbol{a^2 + b^2 + c^2 + 2a(b+c) + 2bc}.$$

Ex. 2. $(a-b-c)^2=\{a-(b+c)\}$, (taking $b+c$ as one term)

$$\left.\begin{array}{l} =a^2-2a(b+c)+(b+c)^2 \\ =a^2-2ab-2ac+b^2+2bc+c^2 \end{array}\right\} \text{(Art. 91)}$$
$$=a^2+b^2+c^2-2ab-2ac+2bc$$
$$=a^2+b^2+c^2-2a(b+c)+2bc.$$

From the above results we learn that :—

The square of a multinomial consisting of any number of terms is equal to the sum of the squares of each term of the multinomial plus *twice the product of each term and the sum of all the terms that follow it, remembering* the Rule of Signs.

Ex. 1. $(2x^2+3x+1)^2=(2x^2)^2+(3x)^2+1^2+2\,2x^2.3x+2.2x^2.1+2.3x.1$
$$=4x^4+9x^2+1+12x^3+4x^2+6x$$
$$=4x^4+12x^3+13x^2+6x+1.$$

Ex. 2. $(3x^2-2x+4)^2=(3x^2)^2+(2x)^2+4^2-2.3x^2(2x-4)-2.2x.4$
$$=9x^4+4x^2+16-12x^3+24x^2-16x$$
$$=9x^4-12x^3+28x^2-16x+16.$$

Exercise XXIX.

Write down the squares of :—

1. $a-b+c.$	2. $x^2+3x+1.$	3. $2x^2+3x-4.$
4. $4x^2-2x-5.$	5. $a^3+b^3-c^3.$	6. $a^2+ab-2b^2.$
7. $2x^2-3x-4.$	8. $2+3x-4x^2.$	9. $a+b-c-d.$
10. $a^3-\frac{1}{2}a^2-a-1.$	11. $1-3x+3x^2-x^3.$	12. $a^2-b^2-c^2+d^2.$

94. The Product of the Sum and Difference of Two Quantities.

$$\begin{array}{l} a+b \\ \underline{a-b} \\ a^2+ab \\ \quad\;\underline{-ab-b^2} \\ \underline{\underline{a^2 \qquad\; -b^2}} \end{array}$$

Thus, $(\boldsymbol{a}+\boldsymbol{b})(\boldsymbol{a}-\boldsymbol{b})=\boldsymbol{a}^2-\boldsymbol{b}^2.$

From the above product, we learn that :—

The product of the sum *and* difference *of any two quantities equal to the* difference *of their squares.*

Ex. 1. $(3a+2b)(3a-2b)=(3a)^2-(2b)^2=9a^2-4b^2.$

Ex. 2. $(2ax+3by)(2ax-3by)=(2ax)^2-(3by)^2=4a^2x^2-9b^2y^2.$

95 The above formula $(a+b)(a-b)=a^2-b^2$ may advantageously be employed in arithmetical work

Ex 1 $83\times77=(80+3)(80-3)=6400-9=6391.$

Ex 2 $9.3\times10.7=(10-.7)(10+.7)=100-.49=99.51.$

Exercise XXX

Write down the following products

1	$(a+1)(a-1)$	**2**	$(x-3)(x+3)$	**3**	$(a+x)(a-x)$.
4	$(2a+1)(2a-1)$	**5**	$(3ax+b)(3ax-b)$	**6**	$(3x+5)(3x-5)$.
7	$(3a+5b)(3a-5b)$	**8**	$(a+7b)(a-7b)$	**9**	$(2p+q)(2p-q)$.
10	$(5x-4a)(5x+4a)$	**11**	$(a^2+3b^2)(a^2-3b^2)$	**12**	$(2a^3+x)(2a^3-x)$.
13	$(4-a^3)(4+a^3)$	**14**	$(12-7x)(12+7x)$	**15**	$(8-5x)(8+5x)$
16	$(-a-7b)(-a+7b)$	**17**	$(a^2+b^2)(a^2-b^2)$	**18**	$(x^9-a^8)(x^9+a^8)$.
19	$(1-ax^2)(1+ax^2)$	**20**	$(3a+b)(3a-b)$	**21**	$(px^3+q)(px^3-q)$.

22 $(1-x)(1+x)(1+x^2)$ **23** $(x+3)(x-3)(x^2+9)$

24 $(a+5)(a-5)(a^2+25)$ **25** $(3+6x)(3-6x)(9+36x^2)$

26 $(a+b)(a-b)(a^2+b^2)$ **27** $(x+8)(x-8)(x^2+64)$

28 $(3x+2a)(3x-2a)(9x^2+4a^2)$ **29** $(a^2+4c^2)(a^2-4c^2)(a^4+16c^4)$

Without performing the actual multiplication, find the value of

30	999×1001	**31**	98×102	**32**	205×195	**33**	11.5×10.5
34	1.95×2.05	**35**	512×488	**36**	20.06×19.94	**37**	305×295
38	121×119	**39**	208×192	**40**	9000.5×8999.5		

96 Extended Application of the Product of the Sum and Difference Art 94 enables us, by using brackets, to find the product of the sum and difference of two expressions other than binomials

Ex 1 $(a+b+c)(a+b-c)=\{(a+b)+c\}\{(a+b)-c\},$

(taking $a+b$ as one term)

$$=(a+b)^2-c^2$$

$$=a^2+2ab+b^2-c^2 \quad \text{(Art 91)}$$

Ex 2 $(a^2+ab+b^2)(a^2-ab+b^2)=\{(a^2+b^2)+ab\}\{(a^2+b^2)-ab\}$

$$=(a^2+b^2)^2-(ab)^2$$

$$=a^4+2a^2b^2+b^4-a^2b^2$$

$$=a^4+a^2b^2+b^4$$

Ex. 3. $(a^2+ax-x^2)(a^2-ax-x^2)=\{(a^2-x^2)+ax\}\{(a^2-x^2)-ax\}$

$$=(a^2-x^2)^2-(ax)^2$$
$$=a^4-2a^2x^2+x^4-a^2x^2$$
$$=a^4-3a^2x^2+x^4.$$

Ex. 4. $(a+2b-3c-d)(a-2b+3c-d)$

$$=\{(a-d)+(2b-3c)\}\{(a-d)-(2b-3c)\}$$
$$=(a-d)^2-(2b-3c)^2$$
$$=(a^2-2ad+d^2)-(4b^2-12bc+9c^2) \quad \text{(Art 91.)}$$
$$=a^2-2ad+d^2-4b^2+12bc-9c^2.$$

Ex. 5. $(a+b+c)(a+b-c)(a-b+c)(b+c-a)$, (C. E. 1866-67).

$$=\{(a+b)+c\}\{(a+b)-c\}\{c+(a-b)\}\{c-(a-b)\}$$
$$=\{(a+b)^2-c^2\}\{c^2-(a-b)^2\}$$
$$=(a^2+2ab+b^2-c^2)(c^2-a^2+2ab-b^2) \quad \text{(Art. 91.)}$$
$$=\{2ab+(a^2+b^2-c^2)\}\{2ab-(a^2+b^2-c^2)\}$$
$$=(2ab)^2-(a^2+b^2-c^2)^2$$
$$=4a^2b^2-(a^4+b^4+c^4+2a^2b^2-2a^2c^2-2b^2c^2) \quad \text{(Art. 91.)}$$
$$=2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4.$$

Exercise XXXI.

Write down the following products :—

1. $(3a-2b+c)(3a+2b-c)$.
2. $(x^2+x+2)(x^2+x-2)$.
3. $(a+2b-3c)(a-2b+3c)$.
4. $(a^2+2ab+2b^2)(a^2-2ab+2b^2)$.
5. $(a^2-ax+x^2)(a^2-ax-x^2)$.
6. $(a^2+ax-x^2)(a^2-ax+x^2)$.
7. $(a^2-ax+x^2)(x^2-a^2+ax)$.
8. $(1-2a+3b)(1+2a-3b)$.
9. $(2a+3b-5)(2a+3b+5)$.
10. $(a^3-a^2b-2b^3)(a^3+a^2b-2b^3)$.
11. $(2a^2-3ab+b^2)(2a^2+3ab+b^2)$.
12. $(x^2+2xy+2y^2)(x^2-2xy+2y^2)$.

Multiply

13. $a+b+c$ by $a+b-c$, by $a-b+c$, and by $a-b-c$.
14. $a-b+c$ by $a-b-c$, by $b+c-a$, and by $c-b-a$.
15. $2a+b-3c$ by $2a-b+3c$, and by $b+3c-2a$.
16. $2a-b-3c$ by $2a+b+3c$, and by $b-3c-2a$.
17. $a+b+c+d$ by $a-b+c-d$, by $a-b-c+d$, by $a+b-c-d$, by $b+c-d-a$, and by $a-b-c-d$.
18. $a-2b+3c+d$ by $a+2b-3c+d$, by $2b-a+3c+d$, by $a+2b+3c-d$, and by $a-2b+3c-d$.

Find the value of :

19. $(a^2+b^2+c^2-2bc)(a^2-b^2-c^2+2bc)$.

20. $(1+2x+3x^2+4x^3)(1-2x+3x^2-4x^3)$.

21. $(ab+cd+ac+bd)(ab+cd-ac-bd)$.

22. $(a^3+2a^2b+2ab^2+b^3)(a^3-2a^2b+2ab^2-b^3)$.

23. $(a^2x^2+ax+1)(a^2x^2-ax+1)(a^4x^4-a^2x^2+1)$.

24. $(x^2+xy+y^2)(x^2-xy+y^2)(x^4-x^2y^2+y^4)(x^8-x^4y^4+y^8)$.

25. $(x+3)^2(x^2-6x+9)(x^4+18x^2+81)$.

97. The Product of Two Quantities having a Common Term.

(1)
$$\begin{array}{lll} x & +a & \\ x & +b & \\ \hline x^2 & +ax & \\ & +bx & +ab \\ \hline x^2 & +(a+b)x & +ab \end{array}$$

(2)
$$\begin{array}{lll} x & -a & \\ x & -b & \\ \hline x^2 & -ax & \\ & -bx & +ab \\ \hline x^2 & -(a+b)x & +ab \end{array}$$

(3)
$$\begin{array}{lll} x & +a & \\ x & -b & \\ \hline x^2 & +ax & \\ & -bx & -ab \\ \hline x^2 & +(a-b)x & -ab \end{array}$$

(4)
$$\begin{array}{lll} x & -a & \\ x & +b & \\ \hline x^2 & -ax & \\ & +bx & -ab \\ \hline x^2 & -(a-b)x & -ab \end{array}$$

From the product of the above expressions, we deduce the following **Rules** :—

(1) *The product consists of* three *terms.*

(2) *The first term is the* square *of the common term.*

(3) *The second term is the common term multiplied by the* sum *of the second terms of the binomial factors.*

(4) *The third term is the* product *of the second terms of the binomial factors.*

Ex. 1. $(x+5)(x+2)=x^2+(5+2)x+5\cdot 2$
$$=x^2+7x+10.$$

Ex. 2. $(x-5)(x-2)=x^2+(-5-2)x+(-5)\cdot(-2)$
$$=x^2-7x+10.$$

Ex. 3. $(x+5)(x-2)=x^2+(5-2)x+(+5)\cdot(-2)$
$$=x^2+3x-10.$$

Ex. 4. $(x-5)(x+2)=x^2+(-5+2)x+(-5)\cdot(+2)$
$$=x^2-3x-10.$$

Ex. 5. $(x+2)(x-2)(x-3)(x+3)=(x^2-4)(x^2-9)$, (Art 94.)

$$=x^4-(9+4)x^2+36$$
$$=x^4-13x^2+36.$$

Exercise XXXII.

Write down the following products :—

1. $(x+1)(x+3)$. **2.** $(x+4)(x-6)$. **3.** $(x-4)(x-6)$.

4. $(ab-3)(ab+2)$. **5.** $(2ax-3b)(2ax-b)$. **6.** $(x^2+4)(x^2-1)$.

7. $(x^2-3y^2)(x^2-y^2)$. **8.** $(5x-2a)(5x+3a)$. **9.** $(5+x)(3+x)$.

10. $(ab-3)(ab-7)$. **11.** $(x^2+2y)(x^2-3y)$. **12.** $(7x+3y)(7x-y)$.

13. $(3x+2a)(3x-a)$. **14.** $(4a^2+3)(4a^2-5)$. **15.** $(3-2x)(7+2x)$.

Find, *by inspection*, the coefficient of x in the following products :—

16. $(x+3)(x+7)$. **17.** $(x-15)(x+2)$. **18.** $(x+7)(x-2)$.

19. $(x+y)(x-z)$. **20.** $(2x+3)(2x-8)$. **21.** $(5x-4)(5x-9)$.

22. Find the coefficient of a in $(x+2a)(x-5a)$.

23. Find the coefficient of b in $(x+3b)(x-2)$.

24. Find the product of $x+2$, $x-3$, $x+4$ and $x-5$.

---

98. The Product of Three Quantities having each a Common Term.

$$\begin{array}{llll}
x & +a & & \\
x & +b & & \\
\hline
x^2 & +ax & & \\
 & +bx & +ab & \\
\hline
x^2 & +(a+b)x & +ab &
\end{array}
\qquad
\begin{array}{lllll}
x^2 & +(a+b)x & +ab & & \\
x & +c & & & \\
\hline
x^3 & +(a+b)x^2 & +abx & & \\
+ & cx^2 & +(ac+bc)x & +abc & \\
\hline
x^3 & +(a+b+c)x^2 & +(ab+ac+bc)x & +abc &
\end{array}$$

From the above product we deduce, the following **Rules** :—

(1) *The product consists of four terms.*

(2) *The first term is the* cube *of the common term.*

(3) *The second term is the* square *of the common term multiplied by the* sum *of the second terms of the binomial factors.*

(4) *The third term is the common term multiplied by the* sum *of the* products *of every two of the second terms of the binomial factors.*

(5) *The fourth term is the* product *of the second terms of the binomial factors.*

Ex. 1. $(x+3)(x-4)(x+2)=x^3+(3-4+2)x^2+\{(3\times -4)+(3\times 2)$
$+(-4\times 2)\}x+(3\times -4\times 2)$
$=x^3+x^2-14x-24.$

Ex. 2. Find the coefficient of x in the product

$$(x+2)(x-4)(x+6).$$

The coeff. of x is the coefficient of the third term

$$=(2\times -4)+(2\times 6)+(-4\times 6)=-8+12-24=-20.$$

Exercise XXXIII.

Write down the following products :—

1. $(x+1)(x+2)(x+3)$.
2. $(x-1)(x-2)(x-3)$.
3. $(a+2)(a+3)(a-4)$.
4. $(a-6)(a-3)(a-7)$.
5. $(x-4)(x+5)(x-6)$.
6. $(x+4)(x-3)(x+5)$.
7. $(a+2b)(a+6b)(a-3b)$.
8. $(1-x)(1+3x)(1-5x)$.
9. $(x+2y)(x-3y)(x-4y)$.
10. $(a^2+b^2)(a^2-2b^2)(a^2+5b^2)$.

Find the coefficient of x and of x^2 in the following products :-

11. $(x+3)(x-5)(x-6)$.
12. $(x+2y)(x-4y)(x+5y)$.
13. $(x-y)(x-4y)(x-3y)$.
14. $(x+2)(x+4)(x-6)$.

---

99. The Product $(ax\pm b)(cx\pm d)$.

By actual multiplication, we obtain

(1) $(ax+b)(cx+d)=acx^2+(bc+ad)x+bd.$

(2) $(ax-b)(cx-d)=acx^2-(bc+ad)x+bd.$

(3) $(ax+b)(cx-d)=acx^2+(bc-ad)x-bd.$

(4) $(ax-b)(cx+d)=acx^2-(bc-ad)x-bd.$

From the above products, we learn that :—

(1) *The product consists of three terms.*

(2) *In the first term, the coefficient of x^2 is the product of the coefficients of x in the first terms of the given binomials.*

(3) *In the second term, the coefficient of x is the sum of the products of the coefficient of x in one and the second term of the other.*

(4) *The third term is the product of the second terms of the given binomials.*

Ex. 1. $(2x+3)(4x+5)=2.4x^2+(3.4+2.5)x+3.5$
$=8x^2+(12+10)x+15=8x^2+22x+15.$

Ex. 2. $(3x-2)(2x-3)=3.2x^2-(2.2+3.3)x+2.3=6x^2-13x+6.$

Ex. 3. $(4x-1)(x+3)=4.1x^2-(1.1-4.3)x-1.3=4x^2+11x-3.$

Ex. 4 $(5x+8)(6x-7)=5.6x^2+(8.6-5.7)x-8.7$
$=30x^2+(48-35)x-56=30x^2+13x-56.$

Exercise XXXIV.

Write down the following products :—

1. $(3x+2)(4x+3)$. **2.** $(3x-4)(2x-5)$ **3.** $(8x+1)(3x-4)$.
4. $(8x+7)(2x-1)$. **5** $(2x-5y)(7x+3y)$. **6.** $(4x-5)(2x-7)$.
7. $(6x-7)(x-2)$. **8.** $(4x+3)(2x-5)$. **9.** $(2x+1)(3x-8)$.
10. $(7x+9y)(4x-5y)$. **11** $(3x-4)(4x+5)$. **12.** $(7x-4)(2x-3)$
13. $(7x-3)(2x+5)$. **14.** $(13x-1)(2x-3)$. **15.** $(5-x)(1+2x)$.
16. $(4x-5)(6x-5)$. **17.** $(3x-4)(4x+3)$. **18** $(4x-3)(3x-4)$

Find the coefficient of x in the following products :—

19. $(5x-9y)(2x+y)$. **20.** $(13x-1)(2x-3)$. **21.** $(2x-y)(x+2y)$
22. $(4x+3)(3x-5)$. **23.** $(3x+5)(x-6)$. **24.** $(2x-3)(x+7)$.

100. The Cube of a Binomial.

$$(1)\quad \begin{array}{l} a+b \\ \underline{a+b} \\ a^2+\ ab \\ \underline{\quad+\ ab+b^2} \\ a^2+2ab+b^2 \\ \underline{\ a+b} \\ a^3+2a^2b+ab^2 \\ \underline{\quad+\ a^2b+2ab^2+b^3} \\ \underline{a^3+3a^2b+3ab^2+b^3} \end{array} \qquad (2)\quad \begin{array}{l} a-b \\ \underline{a-b} \\ a^2-\ ab \\ \underline{\quad-\ ab+b^2} \\ a^2-2ab+b^2 \\ \underline{\ a-b} \\ a^3-2a^2b+\ ab^2 \\ \underline{\quad-\ a^2b+2ab^2-b^3} \\ \underline{a^3-3a^2b+3ab^2-b^3} \end{array}$$

Thus, (1) $(a+b)^3=a^3+3a^2b+3ab^2+b^3$
$=a^3+3ab(a+b)+b^3.$

(2) $(a-b)^3=a^3-3a^2b+3ab^2-b^3$
$=a^3-3ab(a-b)-b^3.$

From the above products, we learn that :—

(1) *The cube of the* sum *of a binomial is equal to the sum of their cubes* plus *three times their product multiplied by their sum.*

(2) *The cube of the* difference *of a binomial is equal to the difference of their cubes* minus *three times their product multiplied by their difference.*

Ex. 1. $(x+4)^3=x^3+3.x.4(x+4)+4^3=x^3+12x^2+48x+64.$

Ex. 2. $(ax-2)^3=(ax)^3-3.ax.2(ax-2)-2^3$
$=a^3x^3-6a^2x^2+12ax-8.$

101. The Product $(a\pm b)(a^2\mp ab+b^2)$.

(1)
$$\begin{array}{l} a^2-ab+b^2 \\ a+b \\ \hline a^3-a^2b+ab^2 \\ \quad +a^2b-ab^2+b^3 \\ \hline a^3 \qquad\qquad +b^3 \end{array}$$

(2)
$$\begin{array}{l} a^2+ab+b^2 \\ a-b \\ \hline a^3+a^2b+ab^2 \\ \quad -a^2b-ab^2-b^3 \\ \hline a^3 \qquad\qquad -b^3 \end{array}$$

Thus, (1) $(a+b)(a^2-ab+b^2)=a^3+b^3.$
(2) $(a-b)(a^2+ab+b^2)=a^3-b^3.$

From the above products, we learn that :—

(1) *The product consists of only two terms.*

(2) *Each term is the cube of the first and second terms of the given binomial.*

Ex. 1. $(2x+3)(4x^2-6x+9)=(2x)^3+(3)^3=8x^3+27.$

Ex. 2. $(3x-4)(9x^2+12x+16)=(3x)^3-(4)^3=27x^3-64.$

Exercise XXXV.

Write down the cubes of the following :—

1. $x-3.$ **2.** $2a+5.$ **3.** $2+ax.$ **4.** $a^2+4b^2.$

5. $x^2-2y^2.$ **6.** $2x-3.$ **7.** $3a^2+2b.$ **8.** $2a^3-3b^2.$

Find the following products :—

9. $(x+3)(x^2-3x+9).$ **10.** $(2a+3b)(4a^2-6ab+9b^2).$

11. $(1+ab)(1-ab+a^2b^2).$ **12.** $(x^2-y)(x^4+x^2y+y^2).$

13. $(2a+b)(4a^2-2ab+b^2).$ **14.** $(2xy-1)(4x^2y^2+2xy+1).$

15. $(4a-5b)(16a^2+20ab+25b^2).$ **16.** $(9a+2x)(81a^2-18ax+4x^2).$

17. $(2x-3y)(4x^2+6xy+9y^2).$ **18.** $(6a-b)(36a^2+6ab+b^2).$

19. $(a+b)(a-b)(a^2+ab+b^2)(a^2-ab+b^2).$

$(x-2)(x+2)(x^2+2x+4)(x^2-2x+4).$

Formulae for Multiplication.

102. The following results of Multiplication already proved in this Section should be committed to memory :—

1. $(a+b)^2 = a^2 + 2ab + b^2$.
2. $(a-b)^2 = a^2 - 2ab + b^2$.
3. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$.
4. $(a+b)(a-b) = a^2 - b^2$.
5. $(a^2+ab+b^2)(a^2-ab+b^2) = a^4 + a^2b^2 + b^4$.
6. $(x+a)(x+b) = x^2 + (a+b)x + ab$.
7. $(x-a)(x-b) = x^2 - (a+b)x + ab$.
8. $(x+a)(x-b) = x^2 + (a-b)x - ab$.
9. $(x-a)(x+b) = x^2 - (a-b)x - ab$.
10. $(x+a)(x+b)(x+c) = x^3 + (a+b+c)x^2 + (ab+ac+bc)x + abc$.
11. $(ax+b)(cx+d) = acx^2 + (bc+ad)x + bd$.
12. $(ax-b)(cx-d) = acx^2 - (bc+ad)x + bd$.
13. $(ax+b)(cx-d) = acx^2 + (bc-ad)x - bd$.
14. $(ax-b)(cx+d) = acx^2 - (bc-ad)x - bd$.
15. $(a+b)^3 = a^3 + 3ab(a+b) + b^3$.
16. $(a-b)^3 = a^3 - 3ab(a-b) - b^3$.
17. $(a+b)(a^2-ab+b^2) = a^3 + b^3$.
18. $(a-b)(a^2+ab+b^2) = a^3 - b^3$.

VI. DIVISION.

103. Division is the inverse of Multiplication. It consists in finding the quantity (called the **quotient**), by which another quantity (called the **divisor**) must be multiplied so as to produce the product (called the **dividend**).

Thus, **dividend = quotient × divisor.**

104. Rule of Signs. Since

$$(+a)\times(+b) = +ab, \qquad \frac{+ab}{+a} = +b.$$

$$(-a)\times(+b) = -ab, \qquad \frac{-ab}{-a} = +b.$$

$$(+a)\times(-b) = -ab, \qquad \frac{-ab}{+a} = -b.$$

$$(-a)\times(-b) = +ab, \qquad \frac{+ab}{-a} = -b.$$

Hence, in division as well as in multiplication,
Like signs produce + and unlike signs produce −.

105. Rule of Indices.

Since $a^6 = a \times a \times a \times a \times a$, and $a^3 = a \times a \times a$ } by definition

$$\therefore\ a^6 \div a^3 = \frac{a \times a \times a \times a \times a}{a \times a \times a} = a \times a = a^2 = a^{6-3}.$$

Hence, *One power of a quantity is divided by another power of the same quantity, by subtracting the index of the divisor from that of the dividend.*

106. Since by the above Rule, $a^3 \div a^3 = a^{3-3} = a^0$,

but by actual division, $a^3 \div a^3 = \dfrac{a^3}{a^3} = 1$.

$\therefore$ we obtain the curious result, $a^0 = 1$.

107. To divide one simple algebraical quantity by another.

Rule. *Divide respectively the coefficient and letters of the dividend by those of the divisor, and give to each letter in the quotient an index obtained by subtracting the index of that letter in the divisor from that in the dividend; and then, if the two quantities have the same sign, prefix to the quotient thus obtained the sign +, if different, the sign −.*

Ex. 1. Divide $-33a^2b^2c^2$ by $-11ac^2$.

$$\text{The quotient} = \frac{-33}{-11} \cdot \frac{a^2}{a} \cdot b^2 \cdot \frac{c^2}{c^2} = +3a^{2-1}b^2c^{2-2}$$
$$= 3ab^2, \text{ for } c^0 = 1. \quad \text{(Art. 106).}$$

Ex. 2. Divide $24a^4b^2x^3$ by $-3a^3bx^2$.

$$\text{The quotient} = \frac{24}{-3} \cdot \frac{a^4}{a^3} \cdot \frac{b^2}{b} \cdot \frac{x^3}{x^2} = -8a^{4-3}b^{2-1}x^{3-2}$$
$$-8abx.$$

Exercise XXXVI.

Divide

1. $6a^2b$ by $-2a$.
2. $-10a^3bc$ by $5ac$.
3. $6p^3qr^3$ by $3p^2r^2$.
4. $-24a^3bx^3z$ by $3a^2x^2$.
5. $12a^4b^3yz^4$ by $4a^2b^2z^3$.
6. $10a^2bc^3x$ by $-5ac$.
7. $-21p^4q^2r^3$ by $3p^2qr^2$.
8. $63a^2b^3c^4$ by $-7ab^2c$.
9. ab^2c^3 by $-abc$.

10. $-165x^2y^5$ by $-33xy^3$. **11.** $-70abx^3y$ by $2ax^2y$.
12. $-12b^2c^6$ by $6b^2c^5$. **13.** $24abc^3$ by $-3c^2$.
14. $-8x^8$ by $-x^7$. **15.** $-45a^4b^8$ by $9a^2b^3$.

Simplify the following :—

16. $\dfrac{24a^2b^2}{-6a}$. **17.** $\dfrac{-56a^9b^8c^6}{7a^6b^8c^2}$. **18.** $\dfrac{-72a^3b^5c^7}{9abc^3}$. **19.** $\dfrac{x^2y^2}{xy}$.

20. $\dfrac{51ab^2x}{-3ab}$. **21.** $\dfrac{-27a^6b^7c^2}{-9a^4b^6c}$. **22.** $\dfrac{21c^4d^3xy^3}{-3d^2xy^2}$.

108. To divide a compound expression by a single factor.

Rule. *Divide each term of the dividend separately by that divisor and take the algebraic sum of the several partial quotients thus obtained.*

Ex. 1. $(6x^3y-3x^2y^3+9xy^2)\div 3xy=\dfrac{6x^3y}{3xy}-\dfrac{3x^2y^3}{3xy}+\dfrac{9xy^2}{3xy}$

$$=2x^2-xy^2+3y.$$

Ex. 2. $(8a^2b^3c^2-4ab^2c^2-24abc^3)\div -4abc^2$

$$=\frac{8a^2b^3c^2}{-4abc^2}-\frac{4ab^2c^2}{-4abc^2}-\frac{24abc^3}{-4abc^2}=-2ab^2+b+6c.$$

Exercise XXXVII.

Divide

1. $ax-ay$ by a. **2.** $ax-bx$ by $-x$. **3.** $6a+8b$ by -2.
4. $-18ab+6bc$ by $-2b$. **5.** $5a^2b-7a^2b^3$ by ab.
6. $x^5-x^4+x^3$ by $-x^2$. **7.** $6x^2y-4x^2z+6xyz$ by $2x$.
8. $5a^3b^3-35a^2b^2c^2+20abc^4$ by $-5ab$.
9. $a^3bx^3y-3a^2bx^2y+3ab^2xy^2-ab^3xy^3$ by $abxy$.
10. $12m^5n^4-9m^4n^3+6m^3n^5-3m^2n^4$ by $-3m^2n^3$.
11. $-30a^6b^6+10a^5b^5-6a^4b^4-2a^3b^3+4a^2b^5$ by $-2a^2b^3$.
12. $4a^4b^4cd-8ab^3c^4d^2+16b^2c^3d^2$ by $-2bcd$.
13. $6p^4q^3+9p^3q^2-3p^2q^3+pq^4$ by pq, and by $-pq^2$.
14. $-12x^6y^9+9x^5y^7-6x^4y^6-3x^3y^5+18x^2y^3$ by $3x^2y^3$.
15. $16a^3bcd-8a^2b^3c^2d+12ab^4c^3d-4abcd$ by $-4abcd$.
16. $27x^4y^5z^6+45x^5y^4z^5+54x^6y^3z^4-18x^7y^8z^2$ by $9x^3y^3z^2$.

109. To divide one compound expression by another.

Rule. (i) *Place the expressions, as in Division of Arithmetic, arranging the terms of each of them, so that the different powers of some one letter, common to both of them, may follow in order of the magnitude of their indices, (it matters not whether in* ascending or descending *order, only the* same *order in each of them)* ;

(ii) *Divide the first term of the dividend by that of the divisor, and set the result in the quotient* ;

(iii) *Multiply the whole divisor by the first term of the quotient, and subtract the product from the dividend* ;

(iv) *Bring down fresh terms (as may be required) from the dividend, and repeat these operations till all the terms from the dividend are brought down.*

Ex. 1. Divide $x^2-7x+12$ by $x-4$.

Arrange the work thus :—

$$\begin{array}{r|l|l} x-4 & x^2-7x+12 & x-3 \\ & x^2-4x & \\ \hline & -3x+12 & \\ & -3x+12 & \end{array}$$

$x^2 \div x = x$. Put down x as the first term in the quotient. Then $x(x-4) = x^2-4x$. Write x^2-4x under the dividend. By subtraction, we obtain $-3x+12$. Divide $-3x$ by x and so obtain -3, the second term of the quotient.

Thus, the complete quotient is $\underline{x-3}$.

The reason for the process is the following :—

Since, $x^2-7x+12=x^2-4x-3x+12=x(x-4)-3(x-4)$;

$$\therefore \frac{x^2-7x+12}{x-4}=\frac{x(x-4)-3(x-4)}{x-4}=\frac{x(x-4)}{x-4}-\frac{3(x-4)}{x-4}$$

$$=x-3, \text{ the sum of the partial quotients.}$$

Ex. 2. Divide $6x^3-17x^2y+16y^3$ by $3x-4y$.

$6x^3 \div 3x = 2x^2$;
$2x^2(3x-4y)=$
$-9x^2y \div 3x = -3xy$;
$-3xy(3x-4y)=$
$-12xy^2 \div 3x = -4y^2$;
$-4y^2(3x-4y)=$

$$\begin{array}{r|l|l} 3x-4y & 6x^3-17x^2y+16y^3 & 2x^2-3xy-4y^2. \\ & 6x^3-\ \ 8x^2y & \\ \hline & -9x^2y & \\ & -9x^2y+12xy^2 & \\ \hline & -12xy^2+16y^3 & \\ & -12xy^2+16y^3 & \end{array}$$

$\therefore$ the required quotient $= \underline{2x^2-3xy-4y^2}$.

Ex. 3. Divide $11a^3-82a^2+30a^4+48-12a$ by $2a+3a^2-4$.

First arrange each of the expressions in descending powers of a.

$$
\begin{array}{r|l|l}
3a^2+2a-4 & 30a^4+11a^3-82a^2-12a+48 & 10a^2-3a-12 \\
 & 30a^4+20a^3-40a^2 & \\
\hline
 & \quad -9a^3-42a^2-12a & \\
 & \quad -9a^3-\ 6a^2+12a & \\
\hline
 & \qquad -36a^2-24a+48 & \\
 & \qquad -36a^2-24a+48 & \\
\hline
\end{array}
$$

$\therefore$ the required quotient $=\underline{10a^2-3a-12}$.

Exercise XXXVIII.

Divide

1. x^2+6x+5 by $x+1$.
2. $6a^2-16ab+8b^2$ by $2a-4b$.
3. $m^3-6m^2+11m-6$ by $m-2$
4. $6x^2+13xy+6y^2$ by $2x+3y$.
5. $6a^2b^2-ab^3-12b^4$ by $3ab+4b^2$.
6. $x^2+23x+102$ by $x+17$.
7. $x^2-23x+120$ by $x-15$.
8. $8a^2\ \ 34ab+21b^2$ by $4a+3b$.
9. $8x^2+14xy-15y^2$ by $2x+5y$.
10. $6a^2+7ab-3b^2$ by $3a-b$.
11. $64a^3+125b^3$ by $4a+5b$.
12. $8a^3-27b^3$ by $2a-3b$.
13. $a^2b^2+3ab-154$ by $ab+14$.
14. x^6+24x^3+144 by x^3+12.
15. $64a^6+b^6$ by $4a^2+b^2$.
16. a^4+4b^4 by $a^2-2ab+2b^2$.
17. $x^4+x^2y^2+y^4$ by x^2-xy+y^2.
18. $49a^2-112ab+64b^2$ by $7a-8b$.
19. a^6+1 by a^2+1.
20. $4x^4y^4+1$ by $2x^2y^2-2xy+1$.
21. $3a^2+a+9a^3-1$ by $3a-1$.
22. a^6-6a+5 by a^2-2a+1.
23. $38x^4-65x^3+27$ by $2x^2-5x+3$.
24. $a^4-3a^3b-9a^2b^2+23ab^3-12b^4$ by $a^2-5ab+4b^2$.
25. $6x^4+5x^2y+7x^2-6y^2+17y-5$ by $3x^2-2y+5$.
26. $x^6-3x^4+4x^3+26x^2-92x+55$ by $x^2-3x+11$.
27. $x^5-5x^3+x^2+2x+3$ by x^2+x-3.
28. $x^4-32a^3x+24a^2x^2-8ax^3+16a^4$ by $4a^2-4ax+x^2$.
29. $4ab^3+51a^2b^2+10a^4-48a^3b-15b^4$ by $4ab-5a^2+3b^2$.
30. $5x^5-7x^4-9x^3-11x^2-38x+40$ by $-5x^2+17x-10$.
31. $a^4+2a^3b+ab^3+2b^4-3a^3c-3b^3c$ by $a+2b-3c$.
32. $x^4-4xy^3+3y^4$ by $x^2-2xy+y^2$.
33. m^4+4m+3 by m^2+2m+1.
34. $1+6x^5+5x^6$ by $1+2x+x^2$.
35. $a^6+2a^3b^3+b^6$ by $a^2+2ab+b^2$.
36. x^6-2x^3+1 by x^2-2x+1.
37. a^6+1 by a^3+2a^2+2a+1.
38. $16x^4+36x^2+81$ by $4x^2+6x+9$.
39. $a^5-4a^3b^2-8a^2b^3-17ab^4-12b^5$ by $a^2-2ab-3b^2$.

40. $4a^5 - 29a - 36 + 8a^2 - 7a^3 + 6a^4$ by $a^3 - 4 + 3a - 2a^2$.

41. $3a^2 + 8ab + 4b^2 + 10ac + 8bc + 3c^2$ by $a + 2b + 3c$.

42. $15x^4 - 32x^3 + 15 + 50x^2 - 28x$ by $3 - 4x + 5x^2$.

43. $4a^6 - 25a^2x^4 + 20ax^5 - 4x^6$ by $2a^3 - 5ax^2 + 2x^3$.

44. $8x^4 - 2ax^3 + 3a^2x^2 - 2a^3x + a^4$ by $2x^2 + ax + a^2$.

45. $2a^4 - 13a^3b + 31a^2b^2 - 38ab^3 + 24b^4$ by $2a^2 - 3ab + 4b^2$.

46. $1 - 52x^4y^4 - 51x^3y^3$ by $4x^2y^2 + 3xy - 1$.

47. $-x^6 + 21x^3y^3 - 24xy^5 + 8y^6$ by $-x^2 + 3xy - y^2$.

48. $9x^5 - x^3 - 12x^2 - 50$ by $3x^2 - 2x + 5$.

49. $28x^4 + 13x^2y^2 - xy^3 + 15y^4$ by $4x^2 + 4xy + 3y^2$. (C. E. 1861).

50. $x^4 + x^3 - 24x^2 - 35x + 57$ by $x^2 + 2x - 3$. (C. E. 1877).

51. $x^9 - 1$ by $x^2 + x + 1$. **52.** $a^8 - x^8$ by $a + x$. (C. E. 1863).

53. $x^8 + x^6y^2 + x^4y^4 + x^2y^6 + y^8$ by $x^4 - x^3y + x^2y^2 - xy^3 + y^4$. (C. E. 1870).

110. Harder Examples in Division. The following Examples with their Solutions and others involving the use of brackets in Division should be carefully noticed.

Ex. 1. Divide $9a^2 - 4b^2 - c^2 - 4bc$ by $3a - 2b - c$.

$$
\begin{array}{l}
3a - 2b - c \,\big)\, 9a^2 - 4b^2 - c^2 - 4bc \,\big(\, 3a + 2b + c = \textit{Quot.} \\
\qquad\qquad\quad \underline{9a^2 - 6ab - 3ac} \\
\qquad\qquad\quad 6ab + 3ac - 4b^2 - 4bc - c^2 \\
\qquad\qquad\quad \underline{6ab \qquad\quad - 4b^2 - 2bc} \\
\qquad\qquad\qquad\qquad 3ac \qquad - 2bc - c^2 \\
\qquad\qquad\qquad\qquad \underline{3ac \qquad - 2bc - c^2}
\end{array}
$$

Ex. 2. Divide $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.

Arranging dividend and divisor in descending powers of a,

$$
\begin{array}{l}
a + b + c \,\big)\, a^3 - 3abc + b^3 + c^3 \,\big(\, a^2 - ab - ac + b^2 - bc + c^2 = \textit{Quot.} \\
\qquad\qquad\quad \underline{a^3 + a^2b + a^2c} \\
\qquad\qquad\quad - a^2b - a^2c - 3abc \\
\qquad\qquad\quad \underline{- a^2b - ab^2 - \ abc} \\
\qquad\qquad\qquad\quad - a^2c + ab^2 - 2abc \\
\qquad\qquad\qquad\quad \underline{- a^2c \qquad - abc - ac^2} \\
\qquad\qquad\qquad\qquad\quad + ab^2 - abc + ac^2 + b^3 \\
\qquad\qquad\qquad\qquad\quad \underline{+ ab^2 \qquad\qquad\quad + b^3 + b^2c} \\
\qquad\qquad\qquad\qquad\qquad - abc + ac^2 - b^2c \\
\qquad\qquad\qquad\qquad\qquad \underline{- abc \qquad\quad - b^2c - bc^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad + bc^2 + c^3 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad + bc^2 + c^3
\end{array}
$$

In the above Example, where a is taken as the letter of reference, and its powers arranged in descending order, there is found in the first remainder the terms $-a^2b$, $-a^2c$

These terms must be set *first*, but since both involve a^2, there is nothing as far as a is concerned to shew which is to be set *first of the two*. In such cases we take another letter, as b, to be, as it were, next in authority to a, and so, (arranging in descending powers of b), we prefer $-a^2b$ to $-a^2c$

The above method is the most *easy* in such a case; but the following, in which the coefficients of the different powers of a are collected in brackets, is the most neat and compendious

$$
\begin{array}{l|l|l}
a+b+c & a^3-3abc+(b^3+c^3) & a^2-a(b+c)+(b^2-bc+c^2)=Quot. \\
 & \underline{a^3+a^2(b+c)} & \\
 & -a^2(b+c)-3abc & \\
 & \underline{-a^2(b+c)-a(b^2+2bc+c^2)} & \\
 & \qquad +a(b^2-bc+c^2)+(b^3+c^3) & \\
 & \qquad \underline{+a(b^2-bc+c^2)+(b^3+c^3)} &
\end{array}
$$

for $(b+c)(b^2-bc+c^2)=b^3+c^3$, Art 100

Ex 3 Divide $(a-b)^2c^2+(a-b)c^3-(c^2-a^2)b^2+(c-a)b^3$ by $(a-b)c^2-(c-a)b^2$ (C E 1883)

Arranging divisor and dividend in descending powers of c,

$$
\begin{array}{l|l|l}
(a-b)c^2-b^2c+ab^2 & (a-b)c^3+(a-b)^2c^2-b^2c^2+b^3c+a^2b^2-ab^3 & c+(a-b) \\
 & \underline{(a-b)c^3 \qquad\qquad -b^2c^2+ab^2c} & \\
 & \qquad (a-b)^2c^2 \qquad -(a-b)b^2c+a^2b^2-ab^3 & \\
 & \qquad \underline{(a-b)^2c^2 \qquad -(a-b)b^2c+a^2b^2-ab^3} &
\end{array}
$$

Hence the required quotient $=c+(a-b)$, or $a-b+c$

Exercise XXXIX.

Divide

1 $a^2+b^2-c^2+2ab$ by $a+b-c$. 2 $a^2-b^2-c^2-2bc$ by $a+b+c$

3 $x^3-(a+p)x^2+(q+ap)x-aq$ by $x-a$

4 $mx^3+(mb-na)x^2-(mc+nb)x+nc$ by $mx-n$

5 $a^2-b^2-c^2+d^2-2ad+2bc$ by $a-b+c-d$.

6 $a^2+ab+2ac-2b^2+7bc-3c^2$ by $a-b+3c$

7. $9a^2+4b^2-c^2+12ab$ by $3a+2b-c$ 8 $a^3-b^3+c^3+3abc$ by $a-b+c$.

9 $a^3+b^3-c^3+3abc$ by $a+b-c$. (C E 1887.)

10. $1+x^3-8y^3+6xy$ by $1+x-2y$. 11. $1+x^3+8y^3-6xy$ by $1+x+2y$.

12. $x^3-8y^3+27z^3+18xyz$ by $x-2y+3z$.

13. $x^4+y^4-z^4+2x^2y^2-2z^2-1$ by $x^2+y^2-z^2-1$.

14. $a^3-b^3-c^3-3abc$ by $a^2+b^2+c^2+ab+ac-bc$.

15. $x^3+(a+b+c)x^2+(bc+ca+ab)x+abc$ by $x^2+(b+c)x+bc$.

16. $a^6-3a^2b^2(a^2-b^2)-b^6$ by $a^3+3ab(a+b)+b^3$.

17. $x^4-(2a+1)x^2+2a^2x-a^4+a^2$ by $x^2-a^2+(x-a)$.

18. $(x+y+z)(xy+xz+yz)-xyz$ by $x+y$. (C. E. 1866.)

19. $x^3+y^3+3xy-1$ by $x+y-1$. (C. E. 1869).

20. $x^3+8y^3-27z^3+18xyz$ by $x-3z+2y$. (B. M. 1883.)

21. $a^3+8b^3+27c^3-18abc$ by $a^2+4b^2+9c^2-6bc-3ca-2ab$. (M.M. 1888.)

22. $a^3+b^3+(m+1)ab(a+b)$ by $a^2+mab+b^2$.

111. When the coefficients are fractional, they are treated by the Rules of Fractions in Arithmetic.

Ex. 1. $-\frac{3}{4}a^2b^3c^4 \div \frac{1}{4}ab^2c^2 = -\frac{3}{1}\times\frac{4}{1}a^{2-1}b^{3-2}c^{4-2} = -3abc^2.$

Ex. 2. $(\frac{4}{5}a^3x^3-\frac{1}{2}a^2x^2+\frac{4}{7}ax)\div -\frac{4}{5}ax$

$$= -\frac{4}{5}\times\frac{5}{4}a^{3-1}x^{3-1}+\frac{1}{2}\times\frac{5}{4}a^{2-1}x^{2-1}-\frac{4}{7}\times\frac{5}{4}a^{1-1}x^{1-1}$$

$$= -a^2x^2+\frac{5}{8}ax-\frac{5}{7}, \text{ for } a^0=1 \text{ and } x^0=1. \quad (\text{Art. } 106.)$$

Ex. 3. Divide $x^4-\frac{5}{4}x^3+\frac{11}{8}x^2-\frac{1}{2}x$ by $x^2-\frac{1}{2}x$.

$$
\begin{array}{r|l}
x^2-\frac{1}{2}x & x^4-\frac{5}{4}x^3+\frac{11}{8}x^2-\frac{1}{2}x \;\big(\; x^2-\frac{3}{4}x+1 = \textit{Quot.} \\
 & \underline{x^4-\frac{1}{2}x^3\qquad\qquad\qquad} \\
 & \quad -\frac{3}{4}x^3+\frac{11}{8}x^2 \\
 & \quad \underline{-\frac{3}{4}x^3+\frac{3}{8}x^2\qquad\quad} \\
 & \qquad\qquad\quad x^2-\frac{1}{2}x \\
 & \qquad\qquad\quad \underline{x^2-\frac{1}{2}x}
\end{array}
$$

Exercise XL.

Divide

1. $\frac{4}{5}a^2x^2$ by $-\frac{2}{3}ax$.
2. $\frac{3}{7}ab^2x^3$ by $-\frac{4}{5}abx^2$.
3. $-\frac{3}{10}a^4b^3$ by $-\frac{2}{5}a^2b$.
4. $\frac{4}{7}a^5x^3$ by $-\frac{7}{2}a^4x^2$.
5. $\frac{3}{4}a^2b^2c$ by $\frac{5}{2}a^2bc$.
6. $\frac{4}{9}x^3y^2z$ by $-\frac{4}{3}x^2z$.
7. $-\frac{3}{8}a^2b^2+\frac{2}{3}ab$ by $-\frac{3}{4}ab$.
8. $\frac{1}{4}a^2-\frac{3}{4}ab-\frac{4}{5}ac$ by $-\frac{1}{4}a$.
9. $-a^3b^3c^3+\frac{3}{5}a^2b^3c^3-\frac{4}{5}a^5b^2c^4$ by $-\frac{1}{5}a^2b^2c^2$, and by $-2a^2c^2$.

Divide

10. $\frac{1}{4}a^3+\frac{1}{72}ab^2+\frac{1}{12}b^3$ by $\frac{1}{2}a+\frac{1}{3}b$.

11. $7a^2x^2-\frac{1}{7}$ by $ax-\frac{1}{7}$.

12. $\frac{3}{2}x^3-\frac{31}{12}x^2y+\frac{7}{8}xy^2-\frac{9}{16}y^3$ by $\frac{1}{2}x-\frac{3}{4}y$.

13. $5a^4+\frac{7}{4}a^3y-\frac{107}{12}a^2y^2+\frac{5}{6}ay^3+\frac{7}{6}y^4$ by $\frac{5}{2}a^2+3ay-\frac{7}{3}y^2$.

14. $\frac{1}{64}x^3-\frac{1}{125}y^3$ by $\frac{1}{4}x-\frac{1}{5}y$.

15. $\frac{1}{8}a^3+\frac{1}{27}b^3$ by $\frac{1}{2}a+\frac{1}{3}b$.

16. $a^4+\frac{9}{4}a^3+\frac{21}{8}a^2+\frac{33}{16}a+\frac{5}{16}$ by $a^2+\frac{3}{2}a+\frac{1}{4}$.

17. $a^3-2a^2b+\frac{11}{9}ab^2-\frac{2}{9}b^3$ by $a^2-\frac{5}{3}ab+\frac{2}{3}b^2$.

18. $\frac{1}{4}x^4+y^4$ by $\frac{3}{4}x^2+\frac{3}{2}y^2-\frac{3}{2}xy$.

19. $\frac{9}{2}x^4+2x^2+\frac{8}{9}$ by $3x^2+2x+\frac{4}{3}$.

20. $2a^5-a^4b-3a^3b^2+\frac{13}{6}a^2b^3-b^5$ by $2a^2-3b^2$.

21. $x^4-\frac{4}{27}x+\frac{1}{27}$ by $x^2-\frac{2}{3}x+\frac{1}{9}$.

22. $\frac{x^3}{125}+\frac{y^3}{27}$ by $\frac{x}{5}+\frac{y}{3}$.

23. $\frac{x^4}{3}-\frac{11x^3}{12}+\frac{41x^2}{8}-\frac{23x}{4}+6$ by $\frac{2x^2}{3}-\frac{5x}{6}+1$. (P. E. 1892).

24. $\frac{3}{4}x^6-4x^4+\frac{57}{8}x^3-\frac{43}{4}x^2-\frac{33}{4}x+27$ by $\frac{1}{2}x^3-x+3$.

25. $2a^2-\frac{55ab}{12}+\frac{29ac}{9}+\frac{21b^2}{8}-\frac{15bc}{4}+\frac{c^2}{3}$ by $\frac{2a}{3}-\frac{3b}{4}+c$.

---

112. Sometimes, it happens, that there is a remainder left after the operation of division is finished. In such cases, to obtain the **complete** quotient we place in the quotient, as in Arithmetic, the remainder with its proper sign, over the divisor, in the form of a fraction. But generally, the quotient is required to a certain number of terms.

Ex. 1. Divide $2x^4+3xy^3-4x^2y^2+x^3y-5y^4$ by x^2-2y^2+xy.

Arranging in descending powers of x and ascending powers of y,

$$\begin{array}{r|l|l}
x^2+xy-2y^2 & 2x^4+x^3y-4x^2y^2+3xy^3-5y^4 & 2x^2-xy+y^2 \\
 & \underline{2x^4+2x^3y-4x^2y^2} & \\
 & -x^3y\qquad\qquad+3xy^3 & \\
 & \underline{-x^3y-x^2y^2+2xy^3} & \\
 & x^2y^2+xy^3-5y^4 & \\
 & \underline{x^2y^2+xy^3-2y^4} & \\
 & -3y^4 &
\end{array}$$

Thus, the *complete* quotient $=2x^2-xy+y^2-\dfrac{3y^4}{x^2+xy-2y^2}$.

In this and other cases, as in common Arithmetic, this fraction cannot be avoided, since the dividend is not exactly divisible by the divisor; but the student should be cautioned, that, unless

attention is paid to the *arrangement according to powers*, alluded to in Art. 109, and that, not only with the dividend and divisor at starting, but also throughout the sum, if care be not taken in all the remainders to preserve the order of the indices of the principal letter, or *letter of reference*, as it is called, there will *always* be a fractional term of this kind, instead of a clear and complete quotient.

To illustrate this, take the follwing example :

Ex. 2. Divide $x^3+3ax^2+3a^2x+a^3$ by $x+a$.

$$
\begin{array}{l|l}
x+a) & x^3+3ax^2+3a^2x+a^3 \;(\; x^2+3a^2+2ax \\
& \underline{x^3+\;\;ax^2} \\
& \qquad 3a^2x+2ax^2 \\
& \qquad \underline{3a^2x+\;\;3a^3} \\
& \qquad\qquad 2ax^2-2a^3 \\
& \qquad\qquad \underline{2ax^2+2a^2x} \\
& \qquad\qquad\quad -2a^2x-2a^3
\end{array}
$$

The process will never terminate, and the result will be

$$x^2+3a^2+2ax-\frac{2a^2x+2a^3}{x+a}$$

Ex. 3. Divide $1-x$ by $1-x+x^2$ to four terms.

$$
\begin{array}{l|l}
1-x+x^2) & 1-x \;(\; 1-x^2-x^3+x^5 \\
& \underline{1-x+x^2} \\
& \quad -x^2 \\
& \quad \underline{-x^2+x^3-x^4} \\
& \qquad -x^3+x^4 \\
& \qquad \underline{-x^3+x^4-x^5} \\
& \qquad\qquad x^5 \\
& \qquad\qquad \underline{x^5-x^6+x^7} \\
& \qquad\qquad\quad x^6-x^7
\end{array}
$$

Thus the quotient $=1-x^2-x^3+x^5$ and the rem. $=x^6-x^7$.

Exercise XLI.

Divide (each to four terms in the quotient)

1. a by $1+x$.
2. $1+2x$ by $1-3x$.
3. $1-2x$ by $1+3x$.
4. 1 by $1-2a+4a^2$.
5. 1 by $1-2x+x^2$.
6. $1-ax$ by $1+bx$.
7. Divide $a-1$ by $1-2a+2a^2$ to four terms.
8. Find the remainder, when x^3-px^2+qx-r is divided by $x-a$.

Prove the following by division :—

9. $\dfrac{x^2+11x+35}{x+5}=x+6+\dfrac{5}{x+5}$
10. $\dfrac{x^2-16x+60}{x-9}=x-7-\dfrac{3}{x-9}$.
11. $\dfrac{35a^2-3ab-100b^2}{5a-9b}=7a+12b+\dfrac{8b^2}{5a-9b}$.
12. $\dfrac{a^3-\frac{1}{27}b^3}{a+\frac{1}{3}b}=a^2-\frac{1}{3}ab+\frac{1}{9}b^2-\dfrac{\frac{2}{27}b^3}{a+\frac{1}{3}b}$.

113. Detached Coefficients. The work of Division may be shortened as in Multiplication by using the method of *detached coefficients.* (See Art. 88).

Ex. 1. Divide $30x^4+11x^3-82x^2-12x+48$ by $3x^2+2x-4$.

$$\begin{array}{r|l|l}
3+2-4 & 30+11-82-12+48 & 10x^2-3x-12 \\
 & \underline{30+20-40} & \\
 & \quad -9-42-12 & \\
 & \quad \underline{-9-\ 6+12} & \\
 & \qquad -36-24+48 & \\
 & \qquad \underline{-36-24+48} &
\end{array}$$

Ex. 2. Divide $9a^4-4a^2+4$ by $3a^2-4a+2$.

$$\begin{array}{r|l|l}
3-4+2 & 9+\ 0-4+0+4 & 3a^2+4a+2 \\
 & \underline{9-12+6} & \\
 & \quad 12-10+0 & \\
 & \quad \underline{12-16+8} & \\
 & \qquad 6-8+4 & \\
 & \qquad \underline{6-8+4} &
\end{array}$$

114. *Analogy between the Arithmetical and Algebraical methoas of division.*

Arithmetical method.

$$\begin{array}{r|l|l}
123 & 13899 & 113 \\
 & \underline{123} & \\
 & \ 159 & \\
 & \ \underline{123} & \\
 & \quad 369 & \\
 & \quad \underline{369} &
\end{array}$$

Algebraical method.

$$\begin{array}{r|l|l}
10^2+2.10+3 & 10^4+3.10^3+8.10^2+9.10+9 & 10^2+1.10+3 \\
 & \underline{10^4+2.10^3+3\ 10^2} & \\
 & \quad 1.10^3+5.10^2+9.10 & \\
 & \quad \underline{1.10^3+2.10^2+3.10} & \\
 & \qquad 3.10^2+6.10+9 & \\
 & \qquad \underline{3.10^2+6.10+9} &
\end{array}$$

$$\begin{array}{r|l|l}
x^2+2x+3 & x^4+3x^3+8x^2+9x+9 & x^2+x+3 \\
 & \underline{x^4+2x^3+3x^2} & \\
 & \quad x^3+5x^2+9x & \\
 & \quad \underline{x^3+2x^2+3x} & \\
 & \qquad 3x^2+6x+9 & \\
 & \qquad \underline{3x^2+6x+9} &
\end{array}$$

Exercise XLII.

Divide (by the method of *detached coefficients*) :—

1. $8x^3-12x^2-14x+21$ by $2x-3$. **2.** a^4+9a^2+81 by a^2+3a+9.

3. $3x^4-10x^3+12x^2-11x+6$ by $3x^2-x+3$.

4. x^3-27 by x^2+3x+9. **5.** $27x^3+1$ by $3x+1$. **6.** x^4-1 by $x+1$.

7. $12x^4+5x^3-33x^2-6x+20$ by $4x^2-x-5$.

8. $6x^2-2x-x^4-4x^3+x^5$ by x^3-4x+2.

9. $27a^3-54a^2b+36ab^2-8b^3$ by $9a^2-12ab+4b^2$.

10. $-141x^2-180x+5x^4-58x^5-32+92x^3+24x^6$ by $2x^2-3x-4$.

VII. IMPORTANT RESULTS IN DIVISION.

115. The following results in Division should be carefully noticed.

1. a^n-b^n is divisible by $a-b$, if n be **any** whole number.

Thus, (1) $\dfrac{a-b}{a-b}=1$. (2) $\dfrac{a^2-b^2}{a-b}=a+b$.

(3) $\dfrac{a^3-b^3}{a-b}=a^2+ab+b^2$. (4) $\dfrac{a^4-b^4}{a-b}=a^3+a^2b+ab^2+b^3$

And so on.

2. a^n+b^n is divisible by $a+b$, if n be any **odd** whole number.

Thus, (1) $\dfrac{a+b}{a+b}=1$. (2) $\dfrac{a^3+b^3}{a+b}=a^2-ab+b^2$.

(3) $\dfrac{a^5+b^5}{a+b}=a^4-a^3b+a^2b^2-ab^3+b^4$. And so on.

3. a^n-b^n is divisible by $a+b$, if n be any **even** whole number.

Thus, (1) $\dfrac{a^2-b^2}{a+b}=a-b$. (2) $\dfrac{a^4-b^4}{a+b}=a^3-a^2b+ab^2-b^3$.

(3) $\dfrac{a^6-b^6}{a+b}=a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5$. And so on.

4. a^n+b^n is **never** divisible by $a+b$ or by $a-b$, when n is an **even** whole number.

Thus, (1) $\dfrac{a^2+b^2}{a+b}=a-b+\dfrac{2b^2}{a+b}$. (2) $\dfrac{a^2+b^2}{a-b}=a+b+\dfrac{2b^2}{a-b}$.

(3) $\dfrac{a^4+b^4}{a+b}=a^3-a^2b+ab^2-b^3+\dfrac{2b^4}{a+b}$.

(4) $\dfrac{a^4+b^4}{a-b}=a^3+a^2b+ab^2+b^3+\dfrac{2b^4}{a-b}$. And so on.

116. From the above results, we notice the following general facts :—

(1) The number of terms in the quotient is always equal to the number expressing the power of a or of b in the dividend.

(2) The powers of a **decrease** continually by one.

(3) The powers of b **increase** continually by one.

(4) The signs of the quotient are all + when the divisor is $a-b$ but they are alternately + and − when the divisor is $a+b$

117. The above results may now be applied to many similar cases.

Ex. 1. $\dfrac{x^3+27y^3}{x+3y}=x^2-3xy+9y^2$.

Ex. 2. $\dfrac{8a^3x^3-1}{2ax-1}=4a^2x^2+2ax+1$.

Ex. 3. $\dfrac{(x+y)^3+z^3}{x+y+z}=(x+y)^2-(x+y)z+z^2$

$=x^2+2xy+y^2-xz-yz+z^2$

Exercise XLIII.

Write down the quotients of the following (*by inspection*) :—

1. a^2-x^2 by $a+x$. **2.** a^5-x^5 by $a-x$. **3.** a^6-x^6 by $a+x$.

4. $9x^2-1$ by $3x-1$. **5.** $25x^2-1$ by $5x+1$. **6.** $4x^2-9$ by $2x+3$.

7. $1+8x^3$ by $1+2x$. **8.** $27x^3-1$ by $3x-1$. **9.** $1-16x^4$ by $1+2x$.

10. x^4-81y^4 by $x-3y$. **11.** a^5+32b^5 by $a+2b$. **12.** $x^{18}-y^{12}$ by x^3+y

13. $\frac{1}{8}a^3+b^3$ by $\frac{1}{2}a+b$. **14.** $x^4y^4-z^4$ by $xy+z$.

15. $16m^4-n^4$ by $4m^2+n^2$. **16.** a^5-32x^5 by $a-2x$.

17. $64-a^6$ by $2+a$. **18.** a^3+343 by $a+7$.

19. $a^2-(b-c)^2$ by $a-b+c$. **20.** $x^3-(y-z)^3$ by $x-y+z$.

REVISION PAPERS I.

Paper I.

1. Find the value of $\dfrac{8a^3-27b^3}{4a^2+6ab+9b^2}+\dfrac{7(a^4-b^4)}{17(a^2-b^2)}$, when $a=5$, $b=3$.

2. Add together $1-(1-\overline{1-x})$, $2x-(3-5x)$ and $2-(-4+5x)$.

3. Take $2a-3\{a-(b-a)\}$ from $2b-3\{b-(a-b)\}$.

4. Simplify $\dfrac{a}{4}-\left[\dfrac{2c-3a}{4}-\left\{c-\dfrac{a-2c}{2}-\left(2c-\dfrac{3a+c}{2}\right)\right\}\right]$.

5. Use squared paper to illustrate the following :—
(i) $9-6=3$. (ii) $7-4-6=-3$. (iii) $5-8+3=0$.

6. Find the value of $5x^2+x-3$, when $x=-2, -1, 0, 1, 2$. Tabulate your work.

7. Express in binomials, and also in trinomials :—
$$ax-by-cz-bx+cy+az.$$

8. Multiply $2x-3y-4(x-2y)+5\{3x-2(x-y)\}$ by $4x-(y-x)-3\{2y-3(x+y)\}$.

9. Divide $a^4-a^2b^2-a^2c^2+b^2c^2$ by $a^2-ab+ac-bc$.

10. Simplify
$$3(a-2x)^2+2(a-2x)(a+2x)+(3x-a)(3x+a)-(2a-3x)^2.$$

Paper II.

1. Find the value of
$$\frac{3(x+y+z)(yz+zx+xy)-x^3-y^3-z^3}{x^2+y^2+z^2-yz-zx-xy}, \text{ when } x=\tfrac{1}{2},\ y=\tfrac{2}{3},\ z=\tfrac{3}{4}.$$

2. Subtract the sum of $a^3-x^2-\{a-x-(2x^2-4)\}$ and $a^3-x^3-\{(a+x)-(a^2-x^2-3)\}$ from $2x^2-x^3$.

3. Use squared paper to illustrate the following :—
(i) $6-2-8=-4$. (ii) $5a-9a+2a=-2a$.

4. Prove that $4a-3(9a-5a)=-8a$ by two different methods.

5. Multiply $4a^2+9b^2+c^2+3bc+2ac-6ab$ by $2a+3b-c$.

6. Divide $72x^6-200x^4-512$ by $12x^3+4x^2-16x+32$.

7. What must be added to the expression $8x^3 - 12x^2 - 13x + 24$ to make it exactly divisible by $2x - 3$?

8. Find the value of $2x^2 - 3x - 5$, when $x = -3, -2, -1, 0, 1, 2, 3$. Tabulate your work.

9. Simplify

$$18\left\{\frac{2x}{9} - \frac{1}{6}\left(\frac{2y}{3} + z\right)\right\} + 24\left(\frac{3x}{8} - \frac{2y - 3z}{12}\right) + 30\left\{\frac{7z}{15} - \frac{4}{7}(2x - y)\right\}.$$

10. Simplify

$$(x^2 - xy + y^2)(x^2 - 2xy + y^2)(x^2 + xy + y^2)(x^2 + 2xy + y^2).$$

(M. M. 1890).

Paper III.

1. Find the value of $\dfrac{\sqrt{(a^2 + b^2 + c^2)}}{a - 2b(a - 3c + 2)} + \dfrac{\sqrt[3]{(b - a)}}{3c + \frac{1}{2}}$,

when $a = 7$, $b = -1$ and $c = \frac{5}{2}$.

2. Subtract $b\{a - (b + c)\}$ from the sum of $a\{a - (c - b)\}$ and $c\{a - (b - c)\}$. (M. M. 1885).

3. From $(3a - 4b)(3a + 4b) + (3a - 4b)^2$ take $6(3a^2 - 2b^2) - 24ab$.

4. Find the value of

$$\frac{4y}{5}(y - x) - 35\left[\frac{3x - 4y}{5} - \frac{1}{10}\{3x - \tfrac{5}{7}(7x - 4y)\}\right],$$

when $x = -\frac{1}{2}$ and $y = 2$. (C. E. 1892).

5. Multiply $\cdot 3 + x^2$ by $\cdot 2 - x$, and find the value of the product when $x = \cdot 1$.

6. Add together $x^2 - (x - y + z)(x + y - z)$, $y^2 - (y - x + z)(y + x - z)$ and $z^2 - (z - x + y)(z + x - y)$. (C. E. 1864).

7. Find the continued product $(x - a)(x^2 + ax + a^2)(x^3 + a^3)$.

(C. E. 1882).

8. Simplify

$$\tfrac{1}{6}\{x(x + 1)(x + 2) + x(x - 1)(x - 2)\} + \tfrac{2}{3}(x - 1)x(x + 1).$$

9. Divide $x^4 + 5ax^3 + (25a - b - 29)x^2 - 5(4a + b - 4)x + 4b$ by $x^2 + 5x - 4$. (M. M. 1893).

10. A person walks due East $2a - 3b + c$ miles, he then walks due West $3a + b - 2c$ miles, he then walks East again $4a - 2b - 3c$ miles; find how far is he then from his starting point.

Paper IV.

1. Given $a=\sqrt{2}$, $b=\sqrt{3}$, $c=4$ and $d=0$, find the value of $\sqrt{\{(a^2+b^2+c^2)(b^2+c^2)(b^2+d^2)\}}$. (C. E. 1868).

2. Add $(2x+3y)(2x-3y)$, $(2x+3y)^2$, $(2x-3y)^2$ and $6xy-7(x^2+y^2)$.

3. Subtract $bcd^2-(a^2-b^2)bd$ from $(a^2+bc)d^2-(a^2-c^2)bd$.
(C. E. 1859).

4. Divide $x^3+ax^2-3a^2x-3a^3$ by $x-2a$ and what is the remainder?

5. If $X=3x-2a$ and $Y=2x-3a$, find the value of

$$(2X-Y)(3X-2Y).$$

6. The product of two expressions is $6a^4-a^3-13a^2+10a-2$; one of them is $2a^2-3a+1$, what is the other expression?

7. Divide $a-1$ by $1-2a+2a^2$ to four terms.

8. The dividend is $6a^3-7a^2b+8ab^2-9b^3$, the quotient is $2a-5b$, and there is a remainder $2b^2(9a+8b)$. Find the divisor.

9. Use squared paper to illustrate the following :—

(i) $3a+5a-6a=2a$. (ii) $2a-8a+3a=-3a$.

10. Simplify $3a-[a+b-2\{a+b+c-(a-b+c-d)\}+a]$.
(C. E. 1876).

Paper V.

1. If $a=4$, $b=3$, $c=2$, $d=1$, $e=0$, find the value of

$$3(b+d)\{6(a-d)^3+b(a-c)^3\}-(c+d)\{15(c-a)^3-(a+c)^2d\}$$
$$+(b+c)\{(b-3c)^3+(a-d)^3\}-(a+c)^2(b+c)^2de.$$ (M. M. 1882).

2. Subtract $b^2-\frac{3}{5}a^2-(\frac{5}{9}c^2-\frac{6}{7}d^2)$ from $a^2-\frac{1}{4}b^2-(3d^2-\frac{1}{6}c^2)$.

3. Multiply $a^2+2b^2+9c^2-3ab+6ac-9bc$ by $a+2b-3c$, and divide the result by $a-b+3c$. (M. M. 1880).

4. Simplify

$$(x-a)(x-b)(x-c)-[bc(x-a)+\{(a+b+c)x-a(b+c)\}x].$$
(A. E. 1889).

5. Arrange the expression $x(p+x)\{b^2+q^2-x(p-x)\}-(p^2+qx)\times(2x^2-qx+q^2)$ in powers of x; and divide it by $x^2+(p-q)x-p^2$. (M. M. 1889).

6. Multiply $x^2+(2a+3b)x+6ab$ by $x^2-(2a+3b)x+6ab$.

7. Find the coefficient of x in the product $(x-a)(x-2b)(x-3c)$.

8. Divide $(4x^3-3a^2x)^2+(4y^3-3a^2y)^2-a^6$ by $x^2+y^2-a^2$.
(B. M. 1884).

9. One factor of $27a^4+11a-10$ is $9a^2+3a-5$; find the other factor.

10. Show that $(x+2)(x+3)(x+4)(x+5)+1$ is a perfect square.
(A. E. 1894).

Paper VI.

1. Find the value of $8a^3+27b^3+c^6-18abc$, when $6a=1$, $9b+1=0$. and $2c=1$.

2. Add $a+3b+5c$, $4a-7b+11c$, $4a-5b-15c$, $a+18b+8c$ and multiply the result by the difference between $11a+7c$ and $10a+6c-b$.

3. If X stands for $2x-a$ and Y for $x+2a$, find the product of $2X+3Y$ and $X-Y$.

4. Divide $ax^3-(a^2+b)x^2+b^2$ by $ax-b$.

5. Find the coefficient of x in the quotient obtained by dividing $8x^4+xy^3-y^4$ by $x-\frac{1}{2}y$.

6. Simplify $24\{x-\frac{1}{2}(x-3)\}\{x-\frac{2}{3}(x+2)\}\{x-\frac{3}{4}(x-1\frac{1}{3})\}$, and subtract the result from $(x+2)(x-3)(x+4)$. (M. M. 1886).

7. Divide $14x^4+45x^3y+78x^2y^2+45xy^3+14y^4$ by $2x^2+5xy+7y^2$; and test your answer by making $x=1$ and $y=2$.

8. Which of the quantities x^3+y^3, x^4-y^4, x^5-y^5, x^6+y^6 is divisible by $x-y$, and which by $x+y$?

9. Find the continued product of
$$x^2+x+1,\ x^2-x+1 \text{ and } x^4-x^2+1. \quad \text{(M. M. 1887)}.$$

10. Prove that $(y-1)(y-3)(y-4)(y-6)+10$ is a positive quantity. (A. I. E. 1892).

Paper VII.

1. If $a=1$, $b=2$, $c=-\frac{1}{2}$, $d=0$, find the value of
$$\frac{a-b+c}{a-b-c}-\frac{ad-bc}{bd+ac}-\sqrt{\left(\frac{b^3}{a^3}-\frac{a^3}{c^3}\right)}. \quad \text{(C. E. 1866)}.$$

2. Find by inspection the following quotients :–

(i) $(64a^3b^3-27c^3)\div(4ab-3c)$. (ii) $(27a^3b^3+8c^3)\div(3ab+2c)$.

3. Write down the following results :—

(i) $(a^2-4)(a^2+5)$. (ii) 115×125. (iii) $(998)^2$.

4. Multiply together x^3-99x^2+x-29 and $x^5-17x^4+105x^3-19x^2+23x-41$, and arrange the product in descending powers of x. (C. E. 1895).

5. Divide $6a^3-17a^2b+7ab^2-5b^3$ by $2a-5b$. (C. E. 1866).

6. If X stands for $ax^2+7bx+7c$ and Y for $ax^2-9bx-9c$, find the value of $9X+7Y$.

7. Multiply $x^3-\frac{1}{2}x^2y-3y^3$ by $2x^2-\frac{1}{3}y^2$. (C. E. 1871).

8. Divide $\frac{9}{16}x^4-\frac{3}{4}x^3y+\frac{4}{3}xy^3+\frac{16}{9}y^4-\frac{7}{4}x^2y^2$ by $\frac{3}{2}x^2-xy-\frac{8}{3}y^2$.

9. Simplify $(5x+3y+1)(5x+3y-1)+(7x-5y)(7x+5y)$
$-(3x+4y)(4x-3y)+1$.

10. Divide $x(1+y^2)(1+z^2)+y(1+z^2)(1+x^2)+z(1+x^2)(1+y^2)$
$+4xyz$ by $1+xy+yz+zx$. (C. E. 1878).

Paper VIII.

1. Find the value of

$$\frac{a+b}{ab}(a^2+b^2-c^2)+\frac{b+c}{bc}(b^2+c^2-a^2)+\frac{c+a}{ca}(c^2+a^2-b^2),$$

when $a=3$, $b=4$, $c=-5$.

2. Simplify the expression :—

$$7(a-3b+c)-[4(2b+4c)(6c-3b)-3(a-4b)(a+3b)+\{(5a-4b+3c)\times4+\overline{a-47b+2c}\}+7].$$ (M. M. 1891).

3. When $x=11$, find the value of

$$\sqrt[3]{[(x+2)\sqrt{x-2}-2\{\sqrt[3]{11x^2-x+2\sqrt{x-2}}\}]}.$$ (M. M. 1880).

4. Write down the results of the following multiplications :—

(i) $(x^2+5)(x^2-5)$. (ii) $(3x-8)(3x-7)$. (iii) $(3x-5)(4x-9)$.

5. Multiply $x^8+y^8+x^7y+xy^7-x^5y^3-x^3y^5-x^4y^4$ by x^2+y^2-xy.

6. Divide $4x^4-4x^3+5x^2+8x-5$ by $2x^2-3x+5$, and divide the quotient by $2x-1$.

7. Divide $a_1a_2x^2+b_1b_2y^2+c_1c_2z^2+(a_1b_2+a_2b_1)xy+(a_1c_2+a_2c_1)xz+(b_1c_2+b_2c_1)yz$ by $a_2x+b_2y+c_2z$. (B. M. 1895).

8. Find what quantity not involving higher powers of x beyond the second should be added to $x^8-3x^7-5x^5+2x^4+5x^3+4x^2+1$ to make it exactly divisible by x^3+2x-1. (B. M., 1897).

9. Divide $2x^4-6ax^3+(4a^2+ab-2b^2)x^2+3ab^2x-a^2b^2$ by

$$x^2-(2a-b)x-ab. \quad \text{(B. M., 1901).}$$

10. A man walks 5 miles North, then 9 miles South, then again 7 miles North. How far is he then from his starting point? Illustrate with a diagram.

Paper IX.

1. Find the numerical value of

$$\frac{a-b}{a+b}+\sqrt{\left(2-\frac{a}{b}-\frac{b}{a}\right)}, \text{ when } a=\tfrac{1}{8},\ b=-\tfrac{1}{2}.$$

2. Perform the multiplications :—(M. M. 1897).

(i) $(3a^4-4a^3x-5x^4)(3a^4+4ax^3+5x^4)$. (ii) $(x+a)^5(x-a)^5$.

3. Divide $x^4-3(a+1)x^3+2(3a+1)x^2+3(a+1)(a^2-1)x+(a^2-1)^2$ by $x^2-(a+3)x-a^2+1$. (B. M. 1900).

4. Find the product of $3a+2b$ and $3a+2c-3b$, and test the result by making $a=1$, $b=c=3$. (C. E. 1870).

5. Divide $\frac{1}{2}-x$ by $\frac{1}{4}-x+x^2$ to five terms. (M. M. 1891).

6. Simplify $\{a-(b-c)\}^2+\{b-(c-a)\}^2+\{c-(a-b)\}^2$, and find its numerical value, when $a=1$, $b=3$, $c=5$. (M. M. 1858).

7. Multiply $(x^2+2ax+4a^2)^2$ by $(x-2a)^3$. (M. M. 1899).

8. Divide $6x^6-19x^5+6x^3-3x+2$ by $3x^2-2x+1$. (M. M. 1899).

9. Divide x^6+y^6 by a number which is greater than x by y. (P. E. 1889).

10. What must be added to $a^3-3a(a-1)-1$ to make it equal to $a^3+3a(a+1)+1$?

Paper X.

1. Find the value of $(a+b+c)^3-(a+b-c)^3+(a-b+c)^3$

$$+(-a+b+c)^3-24abc, \text{ when } 2a=2=-2b=c.$$

2. Simplify

$$24\left\{\frac{2x}{3}-\frac{1}{4}\left(\frac{5y}{3}+\frac{z}{6}\right)\right\}+16\left(\frac{3x}{8}-\frac{2y-3z}{1}\right)+20\left\{\frac{3z}{10}-\frac{3}{5}(2x-y)\right\}.$$

3. Multiply $-6a^4xy^2$ by $10ax^2y^3$ and divide the product by $15a^3xy^4$. Verify the result by substituting $a=-1$, $3x=2$ and $y=-3$, in the multiplicand, multiplier, divisor and quotient.

4. Remove the brackets from

$$(3a-5b)(a-c)+c\{2a-c(3a-b)-b^2(a-c)\};$$

also find its value when $a=0$, $b=1$, $c=-\frac{1}{2}$.

5. Divide $15x^6-17x^4-24x^3+138x^2-130x+63$ by $5x^3+6x^2-9x+7$, and verify by multiplication.

6. Divide $a^3+3a^2b+3ab^2+b^3+c^3$ by $a+b+c$.

7. Shew that $x(x-1)(x-2)(x-3)+1=(x^2-3x+1)^2$.

8. Find an algebraic expression such that when it is divided by a^2-ab+b^2, the quotient is $a^2-2ab+b^2$ and the remainder is $2a^2b^2$.

9. Divide $x^4+5ax^3-(n^2+n-7)a^2x^2-5na^3x-6a^4$ by

$$x^2-(n-2)ax-2a^2.$$

10. If X stands for $a(m+n)$, and Y for $b(m-n)$, find the values of

$$\frac{X}{a}+\frac{Y}{b} \text{ and } \frac{X}{a}-\frac{Y}{b}.$$

CHAPTER IV.

FACTORS AND EASY IDENTITIES.

I. RESOLUTION INTO FACTORS.

118. When an algebraical expression is produced by the multiplication of two or more quantities, each of the latter is called a **factor** of the expression.

119. Hence, to **resolve** an algebraic expression into its elementary factors is to find out those simple quantities which make up the expression by way of multiplication. The process of finding the factors is called **resolution into factors**.

120. In this Section, we should treat of such easy factors as will exemplify the **Converse Use of Formulae for Multiplication** given in Art. 102.

121. When all the terms of an expression have a common factor.

Rule. *Divide each term separately by the common factor and enclose the several quotients in a bracket, placing the common factor outside as a coefficient.*

Ex. 1. Resolve x^2-4ax into factors.

Here, x is common to both terms.

$$\therefore\ x^2-4ax=x(x-4a).$$

Ex. 2. $3a^2bc-4ab^2c+7abc^2=abc(3a-4b+7c)$.

Ex. 3. $5x^3y+15x^2y^2-10x^4y^2=5x^2y(x+3y-2x^2y)$.

Exercise XLIV.

Resolve the following expressions into factors :—

1. x^2+3ax. 2. a^4-a^3b. 3. x^4+5x^2. 4. x^6-5ax^2.
5. $5x^3-15x^2y$. 6. a^2-2ab. 7. $21-35x$. 8. $-2a^2+6a$.
9. $3x^2+9x^3-12x^4$. 10. $4a^2-16ab+24a$. 11. $8xy+16yz$.
12. $3a^3b^4-18a^4b^3+21a^5b^2$. 13. $6x^2yz+12xy^2z-18x^3y^3z$.
14. $42x^2y^2z^3+49x^3y^3z^2-63x^4y^4z$. 15. $14x^3y-7x^2y^2+56xy^3$.
16. $5a^4b+25a^3b^2-10a^2b^3+15ab^4$. 17. $36x^2yz-54xy^2z+48xyz^2$.
18. $70a^2b^2c-60a^4b^3c^2+50a^5b^4c^3-40a^6b^5c^4$.

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122. The resolution of expressions consisting of four terms which can be arranged in groups, each group having a compound factor common.

Rule. *Divide each group by the common factor and enclose the different quotients in a bracket, placing beside it the common factor, also within a bracket as a compound coefficient.*

Ex. 1. Resolve into factors $ac-ad+bc-bd$.

Observing that the first two terms contain a common factor a, and the last two terms have a common factor b, we enclose the first two terms in a bracket, and the last two in another. Thus,

$$\begin{aligned} ac-ad+bc-bd &= (ac-ad)+(bc-bd) \\ &= a(c-d)+b(c-d) \\ &= aX+bX, \text{ putting } X \text{ for } (c-d) \\ &= (a+b)X \\ &= (a+b)(c-d), \text{ substituting for } X \end{aligned}$$

Ex. 2. Resolve into factors $x^2-ax-bx+ab$.

$$\begin{aligned} x^2-ax-bx+ab &= (x^2-ax)-(bx-ab) \\ &= x(x-a)-b(x-a) \\ &= (x-a)(x-b). \end{aligned}$$

Ex. 3. Resolve into factors $10ac-35ad+6bc-21bd$.

$$10ac-35ad+6bc-21bd=(10ac-35ad)+(6bc-21bd)$$
$$=5a(2c-7d)+3b(2c-7d)$$
$$=(2c-7d)(5a+3b).$$

Exercise XLV.

Resolve into factors :—

1. $a^2+ab+ac+bc$.
2. $a^2-ab+ac-bc$.
3. $ac+bd+bc+ad$.
4. $a^2bc-2ab^2+ac^2-2bc$.
5. $ax+bx+ac+bc$.
6. $ax+bx-ac-bc$.
7. $a^2c^2-3acd+abc-3bd$.
8. $a^2c+b^2d+b^2c+a^2d$.
9. $2x^3-3x^2+4x-6$.
10. $11x^3-55x^2+7x-35$.
11. $20ab-12bc-35ad+21cd$.
12. $12a^2-18ab+8ac-12bc$.
13. $15ab+9ac-20bd-12cd$.
14. $x^2y^2+x^2+y^2+1$.
15. $x^3+5x^2+4x+20$.
16. $x^2-y^2-5x+5y$.
17. $18xy^2z-12x^2z-15y^2z+10xz$.
18. $axy+bcxy-5az-5bcz$.
19. $a^2x-3aby+2ax-6by$.
20. $2a^2+6ab-2ac-6bc$.

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123. The resolution of trinomials which are perfect squares.

(1) $\boldsymbol{a^2+2ab+b^2}$. (2) $\boldsymbol{a^2-2ab+b^2}$.

(1) $a^2+2ab+b^2=a^2+ab+ab+b^2$
$$=(a^2+ab)+(ab+b^2)=a(a+b)+b(a+b)$$
$$=(a+b)(a+b)=(a+b)^2.$$

(2) $a^2-2ab+b^2=a^2-ab-ab+b^2$
$$=(a^2-ab)-(ab-b^2)=a(a-b)-b(a-b)$$
$$=(a-b)(a-b)=(a-b)^2.$$

Ex. 1. Resolve into factors $x^2+10x+25$.

$$x^2+10x+25=x^2+2\times5\times x+5^2$$
$$=(x+5)^2.$$

Ex. 2. Factorise $16x^2-24ax+9a^2$.

$$16x^2-24ax+9a^2=(4x)^2-2(4x).(3a)+(3a)^2$$
$$=(4x-3a)^2.$$

Ex. 3. Find the value of $(2\cdot37)^2+$ [illegible] $59\times1\cdot8$.

Since $1\cdot659\times1\cdot8=2\cdot37\times$ [illegible] $=2\times2\cdot37\times\cdot63$

$\therefore$ the given expression [illegible] $+(\cdot63)^2+2\times2\cdot37\times\cdot63$

$=($[illegible]$+\cdot63)^2=3^2=9$. *Ans.*

Exercise XLVI.

Resolve into factors :—

1. $x^2+12x+36$. 2. $a^2-8a+16$. 3. $x^2+14x+49$
4. $4a^2-4a+1$. 5. $4x^2+20xy+25y^2$. 6. $4x^2+4x+1$
7. $9a^2-30ab+25b^2$ 8. a^4+14a^2+49 9. $9a^4+6a^2b^2+b^4$
10. $16a^2b^4-8ab^3c^2+b^2c^4$. 11. $4a^4b^2-28a^2b^3c+49b^4c^2$.
12. $4a^2b^2c^2+4abc+1$. 13. $4x^6y^2+20x^3yz^4+25z^8$.
14. a^4-20a^2+100. 15. $\frac{9}{25}a^2-\frac{4}{5}ab+\frac{4}{9}b^2$
16. $9a^2-6a(2b-3c)+(2b-3c)^2$. 17. $4x^2+(y-z)^2-4(xy-xz)$
18. $(ax-by)^2+4c(ax-by)+4c^2$. 19. $x^4+4(x-1)^2+4(x^3-x)$
20. $(2a+3b)^2+(x+2y)^2+2(2a+3b)(x+2y)$

Find the value of

21. $(\cdot37)^2+(\cdot63)^2+\cdot518\times\cdot9$. 22. $(1\cdot784)^2+(\cdot216)^2+1\cdot728\times\cdot446$.
23. $(1\cdot85)^2+(1\cdot8)^2-11\cdot1\times\cdot6$. 24. $(2\cdot16)^2+(1\cdot85)^2-\cdot54\times14\cdot8$.
25. $25x^2+70xy+49y^2$, when $x=20, y=-14$.

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124 The resolution of an expression in the form of the difference of two squares

$$a^2-b^2=a^2-ab+ab-b^2=(a^2-ab)+(ab-b^2)$$
$$=a(a-b)+b(a-b)=(a-b)(a+b).$$

The difference of the squares of two quantities is [illegible]*al to the product of their sum and difference.*

Ex. 1. Resolve into factors $25x^2-16$.

$25x^2-16=(5x)^2-(4)^2=(5x+4)($[illegible]

Ex. 2. Factorise $4x^2-y^2$.

$4x^2-y^2=(2x)^2-(y)^2=(2x+y)(2x-y)$.

Ex. 3. Find the value of $(385)^2-(285)^2$.

The given expression $=(385+285)(385-285)$

$=670\times100=67000$.

125. When the t[illegible]ve a common factor, it should first be taken outside a brac[illegible] expression can then be further factorised.

Ex. 1. $x^3 - a^2x = x(x^2$ [illegible] $(x+a)(x-a)$.

Ex. 2. $75x^2 - 48a^2 = 3(25x^2 - 16a^2) = 3(5x+4a)(5x-4a)$.

Ex. 3. $7a^2b^2 - 28a^2b^4 = 7a^2b^2(1-4b^2) = 7a^2b^2(1+2b)(1-2b)$.

Exercise XLVII.

Resolve into factors :—

1. $1-4x^2$. 2. a^2-9x^2. 3. $9m^2-4n^2$. 4. $25x^2-16$.
5. x^2-9. 6. $1-x^4$. 7. a^2-169. 8. a^2b^2-1.
9. $25-a^2b^2$. 10. $9x^2-16y^2$. 11. $81x^2-64$. 12. $36-x^8$.
13. a^4-25. 14. a^4b^2-100. 15. $49a^2-81b^2$. 16. $9x^2-49y^2$.
17. a^2-289b^2. 18. $121a^2-144b^2$. 19. $144a^2b^2-121c^4$. 20. a^6-b^4.
21. $4a^2-16$. 22. $25-a^4$. 23. $49x^4-1$. 24. $a^4b^{12}-9c^8$.
25. $25a^2x^2-4y^2$. (B. M. 1862). 26. $16x^4y^2-25x^2y^4$.
27. $a^2xy^3-x^5y$. 28. $2a^3b^2c-8ab^2c^3$. 29. $25x^6-a^2x^3$.
30. $a^6-9a^4b^6$. 31. $81x^4-64$. 32. $7x^6-63a^2x^4$.
33. $3x^4-300$. 34. $11-99a^2$. 35. $45x^2y^2-80$.
36. $141a^9b^7-564a^3b^3$. 37. $605a^2c-720b^2c$.
38. $7a^4-343b^4$. 39. $17-68a^2b^2$.

Find by factorization the values of :—

40. 97^2-87^2. 41. 235^2-35^2. 42. 625^2-375^2.
43. [illegible]25^2-25. 44. 349^2-49^2. 45. 97^2-94^2.
46. [illegible]. 47. 999^2-1. 48. 1796^2-1792^2.
49. [illegible]73^2. 50. 2853^2-2845^2.

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126. T[illegible]a[illegible]e method may conveniently be applied when either (or both) [illegible] squares is a compound quantity.

Ex. 1. $(a+$[illegible]X^2-9c^2, writing X for $a+2b$

$=(X+3c)(X-3c)$

$=(a+2b+3c)(a+2b-3c)$, restoring the value of X.

Ex. 2. $(x-y)^2-4a^2 = X^2-4a^2$, writing X for $x-y$.

$=(X+2a)(X-2a)$

$=(x-y+2a)(x-y-2a)$.

Ex. 3. $(2a-b)^2-(a-2b)^2=\{(2a-b)+(a-2b)\}\{(2a-b)-(a-2b)\}$
$=(2a-b+a-2b)(2a-b-a+2b)$
$=(3a-3b)(a+b)=3(a-b)(a+b)$.

127. The terms of a compound expression can often be arranged so as to form the difference of two squares.

Ex. 1. $a^2+2ab+b^2-c^2=(a+b)^2-c^2$ Art. 123.
$=(a+b+c)(a+b-c)$.

Ex. 2. $x^2+12yz-4y^2-9z^2=x^2-(4y^2+9z^2-12yz)$
(re-arranging the terms)
$=x^2-(2y-3z)^2$, Art. 123.
$=(x+2y-3z)(x-2y+3z)$

Ex. 3. $a^2+b^2-c^2-d^2-2ab+2cd$
$=(a^2+b^2-2ab)-(c^2+d^2-2cd)$ (re arranging the terms)
$=(a-b)^2-(c-d)^2$
$=(a-b+c-d)(a-b-c+d)$.

Exercise XLVIII.

Resolve into factors, and simplify where possible :—

1. $(a-b)^2-c^2$.
2. $a^2-(b-c)^2$.
3. $(a+b)^2-4b^2$.
4. $(a^2+b^2)^2-(c^2+d^2)^2$.
5. $(a+2)^2-(b-2)^2$.
6. $(a+1)^2-(b+1)^2$.
7. $(x+2y)^2-16a^2$.
8. $25x^2-(a+b)^2$.
9. $c^2-(a-b)^2$.
10. $(3x-2)^2-(x-3)^2$.
11. $(2a+b)^2-(2a-b)^2$.
12. $a^2-(4x-y)^2$.
13. $(4x+3y)^2-(3x+4y)^2$.
14. $(x^2+y^2)^2-$ [illegible]
15. $a^2b^2-(ab-1)^2$.
16. $(a^2+2b^2$ [illegible]
17. $9(x+y)^2-16(x-y)^2$.
18. $3(a+$ [illegible] $c+d)^2$.
19. $(2x+3y+a)^2-(x-y+a)^2$.
20. a^2-b^2 [illegible] $c-4c^2$.
21. $a^2-2ab+b^2-c^2$.
22. [illegible] b^2+c^2.
23. $4a^4-9a^2+6a-1$.
24. [illegible] $xy+9y^2-16z^2$.
25. $9a^2-12ab+4b^2-16x^2-8xy-y^2$.
26. [illegible] $b+a^2+b^2-c^2$.
27. $a^2-b^2-c^2+d^2+2ad+2bc$.
28. x^4-x^2-2x-1.
29. $a^2+b^2-c^2-d^2-2ab-2cd$.
30. $a^2-b^2+c^2+2ac$.
31. $a^4-b^4+2b^2c^2-2a^2d^2-c^4+d^4$.
32. $25-a^2-b^2+2ab$.
33. $(x^2-2x+3)^2-(x^2+2x-2)^2$.
34. $(a^2+b^2)^2-4a^2b^2$.
35. $9a^2-b^2-16d^2+8bd-6a+1$.

128. The resolution of multinomials which are perfect squares.

$$a^2+b^2+c^2+2bc+2ac+2ab=a^2+(b^2+c^2+2bc)+2a(b+c)$$
$$=a^2+(b+c)^2+2a(b+c), \text{ Art. } 123.$$
$$=\{a+(b+c)\}^2=(a+b+c)^2.$$

Ex. 1. $x^4-2x^3+3x^2-2x+1=x^4-(2x^3-2x^2)+(x^2-2x+1)$
$$=x^4-2x^2(x-1)+(x-1)^2, \text{ Art. } 123.$$
$$=\{x^2-(x-1)\}^2=(x^2-x+1)^2.$$

Ex. 2. $a^2+4b^2+9c^2-4ab-6ac+12bc$
$$=(a^2+4b^2-4ab)+9c^2-(6ac-12bc) \text{ (re-arranging the terms)}$$
$$=(a-2b)^2+9c^2-6c(a-2b)=(a-2b)^2+9c^2-2.3c(a-2b)$$
$$=\{(a-2b)-3c\}^2=(a-2b-3c)^2.$$

Exercise XLIX.

Resolve into factors :—

1. $4a^2+4ab+b^2+4ac+2bc+c^2$.
2. $x^4+2x^3-x^2-2x+1$.
3. $4x^4-12x^3-7x^2+24x+16$.
4. x^4+4x^3-8x+4.
5. $1+\frac{1}{4}b^2+\frac{1}{16}c^2-b+\frac{1}{2}c-\frac{1}{4}bc$.
6. $x^4+\frac{2}{3}x^3+\frac{11}{18}x^2+\frac{1}{6}x+\frac{1}{16}$.
7. $x^4-10x^3+39x^2-70x+49$.
8. $a^2+b^2+c^2+d^2-2ab-2ac+2ad+2bc-2bd-2cd$.

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129. The resolution of trinomials in which the coefficient of the highest term is unity.

(i) The form of trinomials is either

$$x^2+(a+b)x+ab \text{ or } x^2-(a+b)x+ab,$$

which $=x^2+px+q$ or $=x^2-px+q$, where $p=a+b$ and $q=ab$.

From the above facts, we deduce the following Rule :—

Rule. *Find two numbers which, multiplied together, give the third term of the given trinomial, and added together give the coefficient of the second term.*

Ex. 1. $x^2+10x+24=x^2+4x+6x+24=x(x+4)+6(x+4)$
$$=(x+4)(x+6). \text{ [Since } 24=4\times6\text{ ; } 10=4+6].$$

Ex. 2. $x^2-8x+12=x^2-2x-6x+12=x(x-2)-6(x-2)$
$$=(x-2)(x-6). \text{ [Since } 12=2\times6\text{ ; } 8=2+6].$$

Ex. 3. $x^2 - 11xy + 10y^2 = x^2 - xy - 10xy + 10y^2$
$$= x(x-y) - 10y(x-y)$$
$$= (x-y)(x-10y).$$
[Since $10y^2 = y \times 10y$; $11y = y + 10y$]

(ii) The form of trinomials is either
$$x^2 + (a-b)x - ab \text{ or } x^2 - (a-b)x - ab$$
which $= x^2 + px - q$ or $= x^2 - px - q$, where $p = a - b$ and $q = ab$.

From the above facts, we deduce the following Rule :—

Rule. *Find two numbers which, multiplied together, give the third term of the given trinomial and whose difference is the coefficient of the second term.*

Ex. 1. $x^2 + 2x - 63 = x^2 - 7x + 9x - 63 = x(x-7) + 9(x-7)$
$= (x-7)(x+9)$. [Since $63 = 7 \times 9$; $2 = 9 - 7$].

Ex. 2. $x^2 - 3x - 130 = x^2 + 10x - 13x - 130 = x(x+10) - 13(x+10)$
$= (x+10)(x-13)$.
[Since $130 = 10 \times 13$; $3 = 13 - 10$].

Ex. 3. $72a^2 + a^2b - a^2b^2 = -a^2(b^2 - b - 72) = -a^2\{b^2 - 9b + 8b - 72\}$
$$= -a^2\{b(b-9) + 8(b-9)\}$$
$$= -a^2(b-9)(b+8).$$
[Since $72 = 8 \times 9$; $9 - 8 = 1$]
$$= a^2(9-b)(8+b).$$

Exercise L.

Resolve into factors :—

1. $x^2 + 6x + 5$.
2. $x^2 + 9x + 20$.
3. x^2 [illegible] $5x + 6$.
4. $x^2 - 8x + 15$.
5. $x^2 + 8x + 7$.
6. $x^2 - 10x + 9$.
7. $x^2 + x - 6$.
8. $x^2 - x - 6$.
9. $x^2 - 2x - 3$.
10. $x^2 + 2x - 15$.
11. $x^2 + 7x - 8$.
12. $x^2 - 8x - 9$.
13. $x^2 + 7x + 12$.
14. $x^2 - 9x + 14$.
15. $x^2 - 5x - 14$.
16. $x^2 + x - 12$.
17. $1 - 3x + 2x^2$.
18. $x^2 + x - 110$.
19. $x^2 + 16x + 63$.
20. $x^2 - 23x + 132$.
21. $x^2 - 30x + 200$.
22. $x^2 - 16ax + 39a^2$.
23. $a^4 + 9a^2b^2 + 14b^4$.
24. $a^6 - 7a^3 + 12$.
25. $x^2y^2 + 14xyz + 33z^2$.
26. $a^2b^2 - 24abc + 143c^2$.
27. $42 - x - x^2$.
28. $66 + 5x - x^2$.
29. $5 - 4x - x^2$.

30. $x^2+11xy-26y^2$. **31.** a^6-4a^3-45. **32.** x^2-x-72.
33. $a^2b^2-3ab-4$. **34.** $x^2y^2+3xy-154$. **35.** $x^2-13x-48$.
36. $4a^2b^2-ab-1$. **37.** 2[illegible]$y^2-10xy+1$. **38.** $54x^2-15xy-y^2$.
39. $x^2+(2a-5b)x-10ab$. **40.** $x^2-(6a+5b)x+30ab$.
41. $x^2-26xy+169y^2$. **42.** $26x^2+11x-1$. **43.** $43x^2-42x-1$.
44. $1-7x+6x^2$. **45.** $x^2-(7a-3b)x-21ab$.
46. $a^2x^2-3a^3x+2a^4$. **47.** $a^3-a^2x-6ax^2$.
48. $a^6+a^4b^2-56a^2b^4$. **49.** $p^4m^2-5p^4m-84p^4$.
50. $x^2-(3m-5n)px-15mnp^2$.

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130. The resolution of trinomials in which the coefficient of the highest term is not unity.

(i) The form of the trinomial is either

$$acx^2+(bc+ad)x+bd \text{ or } acx^2-(bc+ad)x+bd,$$
$$\text{which} = px^2+qx+r \text{ or } = px^2-qx+r,$$
$$\text{where } p=ac,\ q=bc+ad, \text{ and } r=bd.$$

Hence the following method of solution :—

Ex. 1. $15x^2+11x+2=15x^2+5x+6x+2=5x(3x+1)+2(3x+1)$
$=(5x+2)(3x+1)$.
[Since $15\times 2=30=5\times 6$; $5+6=11$].

Ex. 2. $14x^2-41x+15=14x^2-6x-35x+15$
$=2x(7x-3)-5(7x-3)$
$=(7x-3)(2x-5)$
[Since $14\times 15=210=6\times 35$; $35+6=41$].

(ii) The form of the trinomial is either

$$acx^2+(bc-ad)x-bd \text{ or } acx^2-(bc-ad)x-bd,$$
$$\text{which} = px^2+qx-r \text{ or } = px^2-qx-r,$$
$$\text{where } p=ac,\ q=bc-ad \text{ and } r=bd.$$

Hence the following method of solution :—

Ex. 1. $6x^2+x-12=6x^2-8x+9x-12=2x(3x-4)+3(3x-4)$
$=(3x-4)(2x+3)$.
[Since $6\times 12=72=8\times 9$; $9-8=1$].

Ex. 2. $6x^2-7x-3=6x^2-9x+2x-3=3x(2x-3)+1(2x-3)$
$=(2x-3)(3x+1)$.
[Since $6\times 3=18=9\times 2$; $9-2=7$]

Note. The student may notice that, if the last term of the given trinomial be *positive*, then the last terms of the two factors will have the same sign as the middle term of the trinomial; but if *negative*, one of them will have the sign +, and the other −.

Exercise LI.

Resolve into factors :—

1. $4x^2+8x+3$.	**2.** $4x^2+13x+3$.	**3.** $3x^2-13x+14$.
4. $12x^2-7x+1$.	**5.** $8x^2+22x+12$.	**6.** $2a^2+7a+3$.
7. $4x^2+11x-3$.	**8.** $4x^2-4x-3$.	**9.** $3x^2+4x-4$.
10. $6x^2+5x-4$.	**11.** $12x^2-5x-2$.	**12.** $12x^2-14x+2$.
13. $12x^2-x-1$.	**14.** $3x^2-2x-5$.	**15.** $12a^4+a^2x^2-x^4$.
16. $3a^3b+a^2b^2-2ab^3$.	**17.** $2x^3y+5x^2y^2+2xy^3$.	**18.** $9x^2y^2-3xy^3-6y^4$.
19. $6a^4x^2+a^3x-a^2$.	**20.** $6b^2x^2-7bx^3-3x^4$.	**21.** $8+18a-5a^2$.
22. $28-31a-5a^2$.	**23.** $14x^2-29x+12$.	**24.** $13x^2+41x+6$.
25. $9x^2+6x-8$.	**26.** $4x^2+4x-63$.	**27.** $2a^2+9a-5$.
28. $3+23x-8x^2$.	**29.** $3x^2-13x-30$.	**30.** $18x^2-9x-2$.

131. Sometimes the following method is recommended.

Ex. 1. Find the factors of $3x^2-7x+2$.

$$\begin{aligned}3x^2-7x+2 &= \tfrac{1}{3}\{(3x)^2-7(3x)+6\}\\ &= \tfrac{1}{3}\{y^2-7y+6\}, \text{ (writing } y \text{ for } 3x)\\ &= \tfrac{1}{3}(y-1)(y-6)=\tfrac{1}{3}(3x-1)(3x-6), \quad \text{Art. 129.}\\ &= (3x-1)(x-2).\end{aligned}$$

Ex. 2. Factorize $12x^2-7x-12$.

$$\begin{aligned}12x^2-7x-12 &= \tfrac{1}{12}\{(12x)^2-7(12x)-144\}\\ &= \tfrac{1}{12}\{y^2-7y-144\}, \text{ (writing } y \text{ for } 12x)\\ &= \tfrac{1}{12}(y-16)(y+9), \quad \text{Art. 129.}\\ &= \left(\frac{12x-16}{4}\right)\left(\frac{12x+9}{3}\right)=(3x-4)(4x+3).\end{aligned}$$

Ex. 3. Factorize $10x^2-13xy-9y^2$.

$$\begin{aligned}10x^2-13xy-9y^2 &= \tfrac{1}{10}\{(10x)^2-13(10x)y-90y^2\}\\ &= \tfrac{1}{10}\{a^2-13ay-90y^2\}, \text{ (writing } a \text{ for } 10x)\\ &= \tfrac{1}{10}(a-18y)(a+5y), \quad \text{Art. 129.}\\ &= \tfrac{1}{10}(10x-18y)(10x+5y)=(5x-9y)(2x+y).\end{aligned}$$

Exercise LII.

Resolve into factors :—

1. $3x^2+14x+8$.　2. $3x^2-10x-8$.　3. $14x^2-29x-15$.
4. $4x^2-8ax+3a^2$.　5. $7x^2+4xy-3y^2$.　6. $2x^2+5x-3$.
7. $5x^2+17x+6$.　8. $24x^2-50x+25$.　9. $28-31x-5x^2$.
10. $6x^2-5ax-6a^2$.　11. $91x^2+186xy-85y^2$.
12. $99x^2-4xy-143y^2$.　13. $210a^4+103a^2b^2-153b^4$.

132. The resolution of an expression which is the sum or difference of two cubes.

(1) a^3+b^3.　(2) a^3-b^3.

(1) $a^3+b^3=(a^3+a^2b)-(a^2b+ab^2)+(ab^2+b^3)$

$$=a^2(a+b)-ab(a+b)+b^2(a+b)=(a+b)(a^2-ab+b^2).$$

(2) $a^3-b^3=(a^3-a^2b)+(a^2b-ab^2)+(ab^2-b^3)$

$$=a^2(a-b)+ab(a-b)+b^2(a-b)=(a-b)(a^2+ab+b^2).$$

Ex. 1. $x^3+8=x^3+2^3=(x+2)(x^2-2.x+2^2)=(x+2)(x^2-2x+4)$.

Ex. 2. $8a^3+27b^3=(2a)^3+(3b)^3$

$$=(2a+3b)\{(2a)^2-2a.3b+(3b)^2\}$$
$$=(2a+3b)(4a^2-6ab+9b^2).$$

Ex. 3. $125a^3-1=(5a)^3-1^3=(5a-1)\{(5a)^2+5a+1\}$

$$=(5a-1)(25a^2+5a+1).$$

Ex. 4. $8x^3-729y^6=(2x)^3-(9y^2)^3$

$$=(2x-9y^2)\{(2x)^2+2x.9y^2+(9y^2)^2\}$$
$$=(2x-9y^2)(4x^2+18xy^2+81y^4).$$

Exercise LIII.

Resolve into factors :—

1. x^3+y^3. (B. E. 1862.)　2. x^3-y^3. (B. E. 1862).　3. $1+x^3$.
4. $1-x^3$.　5. x^6+y^3.　6. x^6-y^3.　7. a^3-8.
8. $8x^3+1$.　9. a^3b^3+1.　10. a^3b^3-1.　11. a^3+64b^3.
12. $27a^3+1$.　13. a^3-64b^3.　14. x^6+1.　15. $1-27x^6$.
16. $8a^3-27b^3$.　17. $216a^3-b^3$.　18. $729x^3+8a^3$.　19. x^6+64.
20. $1+729a^3$.　21. $343a^3-1$.　22. $64x^3-125y^3$.　23. $2a^3+128$.

24. $125a^6+512x^3$. **25.** $a^3x^3+27x^6$. **26.** $81-3a^3$.
27. $a^3x^2y+27x^2y^4$. **28.** $8x^9+y^6$. **29.** $64-(a-b)^3$.
30. $216+(4a-5b)^3$. **31.** x^4-27x. **32.** $729(a+b)^3-8a^3$.

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133. The resolution of an expression which is a perfect cube.

(1) $\boldsymbol{a^3+3a^2b+3ab^2+b^3=a^3+3ab(a+b)+b^3=(a+b)^3}$.

(2) $\boldsymbol{a^3-3a^2b+3ab^2-b^3=a^3-3ab(a-b)-b^3=(a-b)^3}$.

(1) $a^3+3a^2b+3ab^2+b^3=(a^3+b^3)+3ab(a+b)$, (rearranging the terms)
$=(a+b)(a^2-ab+b^2)+3ab(a+b)$, Art. 132.
$=(a+b)\{(a^2-ab+b^2)+3ab\}$
$=(a+b)(a^2+2ab+b^2)$
$=(a+b)(a+b)^2=(a+b)^3$. Art. 123.

(2) $a^3-3a^2b+3ab^2-b^3=(a^3-b^3)-3ab(a-b)$, (rearranging the terms)
$=(a-b)(a^2+ab+b^2)-3ab(a-b)$, Art. 132.
$=(a-b)\{(a^2+ab+b^2)-3ab\}$
$=(a-b)(a^2-2ab+b^2)$
$=(a-b)(a-b)^2=(a-b)^3$. Art. 123.

Ex. 1. $a^3+12a^2+48a+64=(a^3+64)+(12a^2+48a)$
$=(a+4)(a^2-4a+16)+12a(a+4)$
$=(a+4)\{(a^2-4a+16)+12a\}$
$=(a+4)(a^2+8a+16)$
$=(a+4)(a+4)^2=(a+4)^3$.

Ex. 2. $8x^3-12x^2+6x-1=(8x^3-1)-(12x^2-6x)$
$=(2x-1)(4x^2+2x+1)-6x(2x-1)$
$=(2x-1)\{(4x^2+2x+1)-6x\}$
$=(2x-1)(4x^2-4x+1)$
$=(2x-1)(2x-1)^2=(2x-1)^3$.

Exercise LIV.

Resolve into factors :—

1. $x^3+6x^2+12x+8$. **2.** $a^3+6a^2x+12ax^2+8x^3$.
3. $8x^3-36x^2+54x-27$. **4.** $125a^3-150a^2+60a-8$.
5. $x^3-15x^2+75x-125$. **6.** $\frac{1}{8}a^3-\frac{1}{2}a^2b+\frac{2}{3}ab^2-\frac{8}{27}b^3$.
7. $64x^3-144x^2+108x-27$. **8.** $a^3-18a^2+108a-216$.

134. The resolution of trinomials of the following form :—

$$a^4+a^2b^2+b^4.$$ (Easy to understand:

$a^4+a^2b^2+b^4=(a^4+2a^2b^2+b^4)-a^2b^2$, (adding and subtracting a^2b^2)

$=(a^2+b^2)^2-(ab)^2$, Art. 123

$=(a^2+b^2+ab)(a^2+b^2-ab)$, Art. 126.

$=(a^2+ab+b^2)(a^2-ab+b^2)$.

Ex. 1. $x^4+4x^2+16=(x^4+8x^2+16)-4x^2$

$=(x^2+4)^2-(2x)^2=(x^2+4+2x)(x^2+4-2x)$

$=(x^2+2x+4)(x^2-2x+4)$.

Ex. 2. $16a^4-17a^2b^2+b^4=(16a^4-8a^2b^2+b^4)-9a^2b^2$

$=(4a^2-b^2)^2-(3ab)^2$

$=(4a^2-b^2+3ab)(4a^2-b^2-3ab)$

$=(4a^2+3ab-b^2)(4a^2-3ab-b^2)$.

Exercise LV.

Resolve into factors :

1. $x^4-13x^2y^2+4y^4$.	**2.** a^4+a^2+1.	**3.** $9a^4+14a^2+25$
4. x^4-12x^2+16.	**5.** $a^4-18a^2b^2+b^4$.	**6.** $x^4-7x^2y^2+y^4$
7. $9x^4+38x^2y^2+49y^4$.	**8.** x^4-5x^2+4.	**9.** $16x^4+4x^2+1$.
10. $16a^4+36a^2x^2+81x^4$.	**11.** a^4+4b^4.	**12.** $49a^4-15a^2b^2+121b^4$.
13. $9x^4+21x^2y^2+25y^4$.	**14.** $25a^4-9a^2b^2+16b^4$.	**15.** x^4+4.

135. Sometimes an expression may be resolved into more than two factors.

Ex. 1. $81x^4-1=(9x^2)^2-1^2=(9x^2+1)(9x^2-1)$

$=(9x^2+1)(3x+1)(3x-1)$.

Ex. 2. $x^6-y^6=(x^3+y^3)(x^3-y^3)$

$=(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)$.

Ex. 3. $x^8-a^8=(x^4+a^4)(x^4-a^4)$

$=(x^4+a^4)(x^2+a^2)(x^2-a^2)$

$=(x^4+a^4)(x^2+a^2)(x+a)(x-a)$.

Exercise LVI.

Resolve into elementary factors :—

1. x^6-64.
2. x^8-256.
3. $1-729x^6$.
4. $1-16a^4$.
5. $x^{12}-a^{12}$. (C.E. 1859 & A.E. 1896.)
6. $x^{16}-a^{16}$. (B.M. 1872).
7. $a^4x^4-b^4y^4$.
8. x^8+x^4+1. (M.M. 1868).
9. $x^8+x^4y^4+y^8$.
10. a^6-b^6.
11. x^6-1.
12. $(3a-b)^4-(a-3b)^4$
13. $4a^2b^2-(a^2+b^2-c^2)^2$. (B.E. 1881).
14. $a^4-16(b-c)^4$.

Formulae for Resolution into Factors.

136. The results that we have proved in this *Section* should be committed to memory :—

1. $ax+bx=(a+b)x$.
2. $ac+ad+bc+bd=(a+b)(c+d)$.
3. $ac-ad-bc+bd=(a-b)(c-d)$.
4. $a^2+2ab+b^2=(a+b)^2$.
5. $a^2-2ab+b^2=(a-b)^2$.
6. $a^2-b^2=(a+b)(a-b)$.
7. $a^2+b^2+c^2+2bc+2ca+2ab=(a+b+c)^2$.
8. $x^2+(a+b)x+ab=(x+a)(x+b)$.
9. $x^2-(a+b)x+ab=(x-a)(x-b)$.
10. $x^2+(a-b)x-ab=(x+a)(x-b)$.
11. $x^2-(a-b)x-ab=(x-a)(x+b)$.
12. $acx^2+(bc+ad)x+bd=(ax+b)(cx+d)$.
13. $acx^2-(bc+ad)x+bd=(ax-b)(cx-d)$.
14. $acx^2+(bc-ad)x-bd=(ax+b)(cx-d)$.
15. $acx^2-(bc-ad)x-bd=(ax-b)(cx+d)$.
16. $a^3+b^3=(a+b)(a^2-ab+b^2)$.
17. $a^3-b^3=(a-b)(a^2+ab+b^2)$.
18. $\left.\begin{array}{l} a^3+3a^2b+3ab^2+b^3 \\ \text{or } a^3+3ab(a+b)+b^3 \end{array}\right\}=(a+b)^3$.
19. $\left.\begin{array}{l} a^3-3a^2b+3ab^2-b^3 \\ \text{or } a^3-3ab(a-b)-b^3 \end{array}\right\}=(a-b)^3$.
20. $a^4+a^2b^2+b^4=(a^2+ab+b^2)(a^2-ab+b^2)$.

Exercise LVII.

Miscellaneous Factors (Easy).

Resolve into elementary factors :

1. $-4+4x^2$. **2.** $3a^5b^3c^2-18a^7b^4c^3+24a^4b^4c^2$.
3. $a^3bc+2a^2bc^2+abc^3$. **4** $x^4-2bx^3+b^2x^2$. **5.** $a^4b^{12}-c^8$.
6. $x^6-2a^2x^4+a^4x^2$. **7.** $ab-ac-b^2+bc$. **8.** $a^2+4a-21$.
9. $9a^2b^4-30ab^2c+25c^2$. **10.** $3x^2-21xy+30y^2$. **11.** $2a^2-50$.
12. $5a^3b^2+5a^2bc-60ac^2$. **13.** $4a^2b^2-8abc+3c^2$. **14.** $12-3x^2$.
15. $8x^3+(y-z)^3$. **16** $(a+b)^3-125c^3$. **17.** $(a+b)^3+c^3$.
18. $(x+y)^3+(x-y)^3$. **19.** $(x+y)^3-(x-y)^3$ **20.** $3-3(a-b)^2$.
21. $abx^2+ax+bx+1$. **22.** $17x^2+51x+34$. **23.** $117x^2-13$.
24. $250a^3+2$. **25** $(a+b)^3+1$.
26. $a^4+b^4-3a^2b^2$. **27** $28x^4+64x^3-60x^2$. **28.** $9-(a+b)^2$.
29. $6x^2-11x+3$. **30.** $a^2+6a-135$. **31** $49a^2-121b^2$.
32. $a^2b^2-a^2-b^2+1$. (P. E. 1895). **33** $(a+3b+2c)^2-9(2a+b-c)^2$.
34. $(a+2b+3c)^2-4(a+b-c)^2$. **35.** $(x-2y)^3+y^3$.
36. $11x^2+75x-14$. **37.** $21x^2-13xy-20y^2$.
38. $x^8+16x^4y^4+256y^8$. **39.** $x^2y-15xy^2+36y^3$.
40. $81x^4-625y^4$. **41.** $2(x+y)^2-7(x+y)(a+b)+3(a+b)^2$.
42. $30x^2+23x-143$. **43.** $42x^2-155x+102$.
44. $63x^2+132x-35$. **45.** x^4+16x^2+256.
46. $(5x^2-x-12)^2-(4x^2+x-4)^2$. **47.** $8(2a+b)^3+(a-2b)^3$.
48. $a^3-b^3-a(a^2-b^2)+b(a-b)^2$. **49.** $3(x^2-y^2)-5(x-y)^2$.
50. $a^2-ab+2(b^2-ab)+3(a^2-b^2)-4(a-b)^2$.
51. $5(x^2-y^2)+3(x+y)^2$. **52.** $(x+y)^2+2(x^2+xy)-3(x^2-y^2)$.
53. $2(a^3+a^2b+ab^2)-(a^3-b^3)$. **54.** $(x^2-x)^2-8(x^2-x)+12$.
55. $a^4-b^4+(a^2-b^2)^2-3a^4+3a^2b^2$. **56.** $18x^3-9x^2-2x$.
57. $4-4(2x-1)^2$. **58.** $3x-8x^2+4x^3$. **59.** $7x^2-7$.
60. $a^3b-125b$. **61.** $(ax+by+cz)^2+(bz-cy)^2+(cx-az)^2+(ay-bx)^2$.
62. x^8-16a^8. (C. E. 1887). **63.** $(a+b-3c)^2-a-b+3c$. (A. E. 1894).
64. $(1-c^2)(1+a)^2-(1-a^2)(1+c)^2$. (C. E. 1881).
65. $(a+b)^2+(a+c)^2-(b+d)^2-(c+d)^2$. (B. M. 1892).
66. $a^8+a^4x^4+x^8$. (C. E. 1887).
67. $4(uz-xy)^2-(u^2-x^2-y^2+z^2)^2$. (C. E. 1865 and B. M. 1886).
68. x^4+2x^2+9. (A. E. 1894). **69.** $x^4-(p^2+2)x^2y^2+y^4$. (B. M. 1888).

70. $(x^2+3a^2x)^2-(5ax^2+a^3)^2$.

71. $343x^3+512y^3$. (C. E. 1882).

72. $60-7x-x^2$.

73. $x^2+13x+42$. (C. E. 1882).

74. $(x+1)(x+3)(x+5)(x+7)+15$. (M. M. 1888).

75. x^2+x-42. (C. E. 1882).

76. $x^2-5ax-66a^2$. (C. E. 1881).

77. $12x^4+x^2y^2-y^4$. (M. M. 1883).

78. $3x^2-10x-8$. (B. M. 1882).

79. $6x^2+5x-6$. (B. M. 1882).

80. $10x^2-23x-5$. (B. M. 1884).

81. $39x^2-7x-22$. (A. E. 1894).

82. $8a^3+b^3+(2a+b)^3$.

83. $10x-x^2-24$.

84. x^3+x^2-x-1. (A. E. 1895).

85. $a^2-4b^2-c^2+9d^2-2(3ad-2bc)$. (B. M. 1869).

II. USE OF FACTORS.

137. In this *Section* we should illustrate the *Use of Factors* by the solution of certain typical examples.

Ex. 1. If $x=b+c$, $y=c-a$, $z=a-b$, shew that

$$x^2+y^2+z^2-2xy-2xz+2yz=4b^2. \quad \text{(C. E. 1888).}$$

We have $x-(y+z)=(b+c)-(c-a+a-b)=(b+c)-(c-b)=2b$.

The given expression $=x^2+(y^2+z^2+2yz)-2x(y+z)$

$=x^2+(y+z)^2-2x(y+z)$, Art. 123

$=\{x-(y+z)\}^2=(2b)^2=4b^2$.

Ex. 2. If $x+\frac{1}{x}=p$, express $x^3+\frac{1}{x^3}$ in terms of p. (B. M. 1886).

$$x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right), \text{ Art. 132}$$

$$=\left(x+\frac{1}{x}\right)\left\{\left(x^2+2+\frac{1}{x^2}\right)-3\right\}=\left(x+\frac{1}{x}\right)\left\{\left(x+\frac{1}{x}\right)^2-3\right\},$$

$$=p(p^2-3)=p^3-3p.$$

Ex. 3. Find the continued product of

$$3-x,\ 3+x,\ 9-3x+x^2 \text{ and } 9+3x+x^2.$$

Product $=(3-x)(9+3x+x^2)\times(3+x)(9-3x+x^2)$

$=(27-x^3)\times(27+x^3)=729-x^6$.

Ex. 4. Divide the product of $2x^2+11x-21$ and $3x^2-20x-7$ by x^2-49.

$$\text{Quotient}=\frac{(2x^2+11x-21)(3x^2-20x-7)}{x^2-49}$$

$$=\frac{(2x-3)(x+7)(3x+1)(x-7)}{(x+7)(x-7)}$$

$$=(2x-3)(3x+1)=6x^2-7x-3.$$

Ex. 5. Divide $x^6+y^6-2x^3y^3$ by $(x-y)^2$.

Dividend $=(x^3-y^3)^2=(x-y)^2(x^2+xy+y^2)^2$.

$\therefore$ Quotient $=(x^2+xy+y^2)^2$.

Ex. 6. Divide $(a^2-2bc)^3+27b^3c^3$ by a^2-bc.

Dividend $=(a^2-2bc)^3+(3bc)^3$

$$=(a^2-2bc+3bc)\{(a^2-2bc)^2-3bc(a^2-2bc)+9b^2c^2\}$$
$$=(a^2-bc)(a^4-4a^2bc+4b^2c^2-3a^2bc+6b^2c^2+9b^2c^2)$$
$$=(a^2-bc)(a^4-7a^2bc+19b^2c^2).$$

$\therefore$ Quotient $=a^4-7a^2bc+19b^2c^2$.

Ex. 7. Shew that $(ax+by+cz)^3+(bx+cy+az)^3$ is divisible by $(a+b)x+(b+c)y+(c+a)z$.

Assume $ax+by+cz=X$ and $bx+cy+az=Y$;

then $X+Y=ax+bx+by+cy+cz+az$
$$=(a+b)x+(b+c)y+(c+a)z.$$

Now, the expression $=X^3+Y^3=(X+Y)(X^2-XY+Y^2)$,

which is divisible by $X+Y$ or $(a+b)x+(b+c)y+(c+a)z$.

Ex. 8. The product of two quantities is $(2y-3x)^3-(2x-3y)^3$ and one of them is $5y-5x$; find the other.

Assume $2y-3x=a$ and $2x-3y=b$; then
$$a-b=(2y-3x)-(2x-3y)=5y-5x.$$

Now the expression $=a^3-b^3=(a-b)(a^2+ab+b^2)$;

which when divided $a-b$ or $5y-5x$, gives the other factor

$$=a^2+ab+b^2$$
$$=(2y-3x)^2+(2y-3x)(2x-3y)+(2x-3y)^2$$
$$=(4y^2-12xy+9x^2)+(13xy-6x^2-6y^2)+(4x^2-12xy+9y^2)$$
$$=7x^2-11xy+7y^2.$$

Ex. 9. If $x+y=2a$ and $x-y=2b$, prove that
$$x^4-23x^2y^2+y^4=(7a^2-3b^2)(7b^2-3a^2).$$

We have $x=a+b$ and $y=a-b$; then $xy=a^2-b^2$.

Now $x^4-23x^2y^2+y^4=(x^4+2x^2y^2+y^4)-25x^2y^2=(x^2+y^2)^2-(5xy)^2$
$$=(x^2+y^2+5xy)(x^2+y^2-5xy)$$
$$=\{(x+y)^2+3xy\}\{(x-y)^2-3xy\}$$
$$=\{4a^2+3(a^2-b^2)\}\{4b^2-3(a^2-b^2)\}$$
$$=(7a^2-3b^2)(7b^2-3a^2).$$

Ex. 10. Divide the continued product of $1+x+y$, $1+x-y$, $1-x+y$ and $x+y-1$ by $1+2xy-x^2-y^2$. (C. E. 1865).

Divisor $=1-(x^2+y^2-2xy)=1-(x-y)^2$

$=(1+x-y)(1-x+y)$.

$\therefore$ Quotient $=(1+x+y)(x+y-1)=\{(x+y)+1\}\{(x+y)-1\}$

$=(x+y)^2-1=x^2+2xy+y^2-1$.

Exercise LVIII.

1. If $a=y+z-2x$, $b=z+x-2y$, $c=x+y-2z$, find the value of $b^2+c^2-a^2+2bc$. (B. M. 1892).
2. Find the continued product of x^2-2y^2, $x^2-2xy+2y^2$, x^2+2y^2 and $x^2+2xy+2y^2$. (B. M. 1885)
3. Divide $(2x^2-x-3)(3x^2-x-2)$ by $6x^2-5x-6$.
4. Divide the product of $(b+c)^2-a^2$ and $a^2-b^2-c^2+2bc$ by $b^2-(c-a)^2$. (P. E. 1890).

Divide (by employing factors)

5. $x^6+2x^3y^3+y^6$ by $(x+y)^2$. (C. E. 1859).
6. $x^8+a^4x^4+a^8$ by x^2-ax+a^2. (C. E. 1864.)
7. $(x^3-y^3)(x+y)^2$ by $(x^2+xy+y^2)(x^2-y^2)$. (C. E. 1873).
8. a^8-x^8 by $a+x$. (C. E. 1863).
9. $(x+y)^3-8z^3$ by $x+y-2z$. (A. E. 1894).
10. $(x^2-xy+y^2)^3+(x^2+xy+y^2)^3$ by $2(x^2+y^2)$. (B. M. 1891).
11. $(ax+by+cz)^3+(cx-by+az)^3$ by $(a+c)(x+z)$. (B. M. 1887).
12. $b(x^3+a^3)+ax(x^2-a^2)+a^3(x+a)$ by $(a+b)(x+a)$.
13. $a^8-b^8+a^2b^2(a^4-b^4)$ by $(a^2-ab+b^2)(a^2+ab+b^2)$.
14. $(2a+3b+4c)^3-(a+b+c)^3$ by $a+2b+3c$.
15. $(x^2-5x+6)(x^2+3x-28)$ by $x^2-7x+12$.
16. Subtract $(x^2-7x+13)^2$ from $(x^2-7x-13)^2$.
17. Subtract $(5x^2-5x-11)^2$ from $(6x^2-5x+11)^2$.
18. Shew that $(x-4y)^3-(y-4x)^3+(2y-3x)^3-(2x-3y)^3$ is divisible by $5(x+y)$.
19. Find the divisor when $(4x^2+7xy+5y^2)^2$ is the dividend, $8(x+2y)^3$ the quotient and $y^2(9x+11y)^2$ the remainder. (B. M. 1889).

20. If $x+y=2a$ and $x-y=2b$, find the value of $x^4-6x^2y^2+y^4$ in terms of a and b.

21. If $a+b=x$, $a-b=y$, express $16(a^4+a^2b^2+b^4)$ as the product of two expressions involving x and y. (P. E. 1893).

22. Simplify $(x-y+z)^2+(x+y-z)^2-2(x-y+z)(x+y-z)$.

23. Prove that $(ax+by+cz)^2-(bx+cy+az)^2$ is divisible by $(a-b)x+(b-c)y+(c-a)z$.

24. Shew that $(ax+by)^3+(bx+ay)^3$ is divisible by $(a+b)(x+y)$.

25. Find the quotient when the product of a^3+b^3 and a^3-b^3 is divided by $a^3-2a^2b+2ab^2-b^3$.

26. Multiply $(2x^2+3x+1)^2-(2x^2-3x-1)^2$
by $(x^2+6x-2)^2-(x^2-6x+2)^2$.

27. Divide $7x(x-11)(x^2-x-156)$ by x^3+x^2-132x.

28. Divide $(5x^2-3x-6)^2-(2x^2-7x+9)^2$ by the product of $3x-5$ and $x+3$.

29. Shew that $(7x^2+3x-3)^3+(5x^2-4x-3)^3$ is divisible by $4x-3$ and by $3x+2$.

30. If $x+y=a$ and $x-y=b$, shew that

$$16(x^4-7x^2y^2+y^4)=(5a^2-b^2)(5b^2-a^2).$$

III. EASY IDENTITIES.

138. An **Identity** is a statement that two expressions are equal, whatever numbers the letters stand for.

Thus, $7x-4x=3x$, whatever value x may have.

$a^2-b^2=(a+b)(a-b)$, whatever value a and b may have.

139. The two expressions connected by the sign = are called the **sides** or **members** of the Identity; that to the left is called the **left-hand** side and that to the right, is called the **right-hand** side.

140. In this *Section* we should establish the truth of certain *Easy Identities* with the aid of the foregoing principles.

Ex. 1. Prove that $(x+2a)^2+(x-2a)^2=2(x^2+4a^2)$.

Left side $=(x^2+4ax+4a^2)+(x^2-4ax+4a^2)$, Art. 91

$=2x^2+8a^2=2(x^2+4a^2)$.

Ex. 2. Prove that $(a^2+ab+b^2)^2-(a^2-ab+b^2)^2=4ab(a^2+b^2)$.

Left side $=\{(a^2+b^2)+ab\}^2-\{(a^2+b^2)-ab\}^2$

$=\{(a^2+b^2)^2+2ab(a^2+b^2)+a^2b^2\}$

$-\{(a^2+b^2)^2-2ab(a^2+b^2)+a^2b^2\}$, Art. 91

$=(a^2+b^2)^2+2ab(a^2+b^2)+a^2b^2-(a^2+b^2)^2+2ab(a^2+b^2)$
$-a^2b^2$

$=4ab(a^2+b^2)$.

Otherwise thus: Assume $a^2+b^2=x$.

Then Left side $=(x+ab)^2-(x-ab)^2$

$=\{(x+ab)+(x-ab)\}\{(x+ab)-(x-ab)\}$, Art. 126

$=2x\times 2ab=4abx=4ab(a^2+b^2)$.

Ex. 3. Prove that $x(x+1)(x+2)(x+3)+1=(x^2+3x+1)^2$.

Left side $=x(x+3)\times(x+1)(x+2)+1$

$=(x^2+3x)\times(x^2+3x+2)+1$, Art. 97

$=a(a+2)+1$, (writing a for x^2+3x)

$=a^2+2a+1=(a+1)^2$, Art. 123.

$=(x^2+3x+1)^2$, (restoring the value of a)

Ex. 4. Prove that

$$(b-c)(b+c-a)+(c-a)(c+a-b)+(a-b)(a+b-c)=0.$$

Left side $=\left.\begin{aligned}&(b-c)(b+c)-a(b-c)\\&+(c-a)(c+a)-b(c-a)\\&+(a-b)(a+b)-c(a-b)\end{aligned}\right\}$

$=b^2-c^2-(ab-ac)+c^2-a^2-(bc-ab)+a^2-b^2-(ac-bc)$

$=(b^2-c^2+c^2-a^2+a^2-b^2)-(ab-ac+bc-ab+ac-bc)$

$=0-0=0$.

Ex. 5. Prove that

$$(ax+by)^2+(ay-bx)^2+c^2(x^2+y^2)=(a^2+b^2+c^2)(x^2+y^2).$$

Left side $=a^2x^2+2abxy+b^2y^2+a^2y^2-2abxy+b^2x^2+c^2x^2+c^2y^2$,

$=a^2x^2+b^2x^2+c^2x^2+a^2y^2+b^2y^2+c^2y^2$,

(re-arranging the terms)

$=(a^2+b^2+c^2)x^2+(a^2+b^2+c^2)y^2$,

$=(a^2+b^2+c^2)(x^2+y^2)$.

Ex. 6. Prove that

$$(a-b)^3+(a+b)^3+3(a-b)^2(a+b)+3(a+b)^2(a-b)=8a^3.$$

Assume $a-b=x$ and $a+b=y$. Then $x+y=2a$.

Left side $=x^3+y^3+3x^2y+3y^2x$

$=x^3+y^3+3xy(x+y)=(x+y)^3$, Art. 133.

$=(2a)^3=8a^3$.

Ex. 7. Prove that $(a+2b)a^3-(b+2a)b^3=(a-b)(a+b)^3$.

Left side $=a^4+2a^3b-b^4-2ab^3=(a^4-b^4)+2ab(a^2-b^2)$

$=(a^2+b^2)(a^2-b^2)+2ab(a^2-b^2)$, Art. 124.

$=(a^2-b^2)\{(a^2+b^2)+2ab\}=(a^2-b^2)(a+b)^2$, Art. 123.

$=(a+b)(a-b)(a+b)^2=(a-b)(a+b)^3$.

Ex. 8. Prove that

$$(2x+a)^2+(x+b)^2+4ab=(x-b)^2+4(x+a)(x+b)+a^2.$$

Left side $=(4x^2+4ax+a^2)+(x^2+2bx+b^2)+4ab$, Art. 91.

$=(x^2-2bx+b^2)+4(x^2+ax+bx+ab)+a^2$,

(adding and subtracting $2bx$ and re-arranging the terms)

$=(x-b)^2+4(x+a)(x+b)+a^2$. Arts. 123 and 129.

Ex. 9. If $2s=a+b+c$, prove that

$$(s-a)^2+(s-b)(s-c)+as=a^2+bc$$

Left side $=(s^2-2as+a^2)+s^2-(b+c)s+bc+as$, Arts. 91 and 97.

$=2s^2-(a+b+c)s+a^2+bc$

$=2s^2-2s.s+a^2+bc$ (for $a+b+c=2s$).

$=2s^2-2s^2+a^2+bc=a^2+bc$.

Ex. 10. If $2s=a+b+c+d$, prove that

$$4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2=16(s-a)(s-b)(s-c)(s-d).$$

Left side $=\{2(ab+cd)\}^2-(a^2+b^2-c^2-d^2)^2$

$=\{2(ab+cd)+(a^2+b^2-c^2-d^2)\}$

$\times\{2(ab+cd)-(a^2+b^2-c^2-d^2)\}$, Art. 126.

$=\{(a^2+b^2+2ab)-(c^2+d^2-2cd)\}$

$\times\{(c^2+d^2+2cd)-(a^2+b^2-2ab)\}$

$=\{(a+b)^2-(c-d)^2\}\{(c+d)^2-(a-b)^2\}$, Art. 123

$=\{(a+b)+(c-d)\}\{(a+b)-(c-d)\}$

$\times\{(c+d)+(a-b)\}\{(c+d)-(a-b)\}$, Art. 126.

$=(a+b+c-d)(a+b-c+d)(a-b+c+d)(-a+b+c+d)$,

$=(a+b+c+d-2d)(a+b+c+d-2c)(a+b+c+d-2b)$

$\times(a+b+c+d-2a)$

$=(2s-2d)(2s-2c)(2s-2b)(2s-2a)$, (for $a+b+c+d=2s$)

$=2(s-d)\times 2(s-c)\times 2(s-b)\times 2(s-a)$

$=16(s-a)(s-b)(s-c)(s-d)$.

Exercise LIX.

Prove the truth of the following Identities.

1. $(x+2a)^2-(x-2a)^2=8ax.$

2. $(a^2+ab-b^2)^2-(a^2-ab-b^2)^2=4ab(a^2-b^2).$

3. $(b+c)(b-c)+(c+a)(c-a)+(a+b)(a-b)=0.$

4. $(ac+bd)^2+(ad-bc)^2=(a^2+b^2)(c^2+d^2).$

5. $(a+b+c)^2+a^2+b^2+c^2=(b+c)^2+(c+a)^2+(a+b)^2.$

6. $(a+b+c+d)^2+2(a^2+b^2+c^2+d^2)$
$=(a+b)^2+(a+c)^2+(a+d)^2+(b+c)^2+(b+d)^2+(c+d)^2.$

7. $(a-b)^3+b^3-a^3=3ab(b-a).$

8. $(a+b)^2+2(a^2-b^2)+(a-b)^2=4a^2.$

9. $(b-c)(x-a)+(c-a)(x-b)+(a-b)(x-c)=0.$

10. $(a+b+c)^2+(a+b-c)^2+(a-b+c)^2+(b+c-a)^2=4(a^2+b^2+c^2).$

11. $(ax+by+cz)^2+(ay-bx)^2+(bz-cy)^2+(cx-az)^2$
$=(a^2+b^2+c^2)(x^2+y^2+z^2).$

12. $(a+b)^2(b+c-a)(c+a-b)+(a-b)^2(a+b+c)(a+b-c)=4abc^2.$

13. $(ax+by)^2+(cx+dy)^2+(ay-bx)^2+(cy-dx)^2$
$=(a^2+b^2+c^2+d^2)(x^2+y^2).$

14. $\{(ax+by)^2+(ay-bx)^2\}\{(ax+by)^2-(ay+bx)^2\}=(a^4-b^4)(x^4-y^4).$
(C. E. 1859).

15. $(a+b)^3-(a-b)^3-3(a+b)^2(a-b)+3(a+b)(a-b)^2=8b^3.$

16. $(x+1)(x+2)(x+3)(x+4)+1=(x^2+5x+5)^2.$

17. $(1-a^2)(1-b^2)-(c+ab)^2=1-a^2-b^2-c^2-2abc.$

18. $(1-a^2)(a+bc)-(b+ca)(c+ab)=a(1-a^2-b^2-c^2-2abc).$

19. $(a+b)(a+c)+(b+c)(b+a)+(c+a)(c+b)-(a+b+c)^2$
$=bc+ca+ab.$

20. $(x-y)^2+(z-x)(z-y)=(y-z)^2+(x-y)(x-z)$
$=(z-x)^2+(y-z)(y-x).$

21. $(2a+b)^3+9a(2a+b)(a-b)+(a-b)^3=27a^3.$

22. $(a+b-c)(b+c)+(b+c-a)(c+a)+(c+a-b)(a+b)$
$=2(bc+ca+ab).$

23. $(b+c)^2+(c+a)^2+(a+b)^2+2(b+c)(c+a)+2(b+c)(a+b)$
$+2(c+a)(a+b)=4(a+b+c)^2.$

24. $(a+b)^2+2(a+b)c+c^2=(a+c)^2+2(a+c)b+b^2.$

25. $a(b-c)(1+bc)+b(c-a)(1+ca)+c(a-b)(1+ab)=0.$

26. $(b+c)(c+a)(a+b)=(a+b+c)(bc+ca+ab)-abc.$

27. $a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc=(a+b+c)(bc+ca+ab).$

28. $a(b+c)^2+b(c+a)^2+c(a+b)^2-4abc=(b+c)(c+a)(a+b).$

29. $(a-b)^2(c+d)^2+4ab(c-d)^2=(a+b)^2(c-d)^2+4cd(a-b)^2.$

30. $(1+a)^2(1+b^2)-(1+a^2)(1+b)^2=2(a-b)(1-ab).$

31. $a^3(a-2b)-b^3(b-2a)=(a+b)(a-b)^3.$

32. $(x-y)(x+1)(y+1)-x(y+1)^2+y(x+1)^2$
$=(x-y)(x+y+2xy).$ (M. M. 1887).

33. $(b+c)^2+(c+a)^2+(a+b)^2-(b+c)(c+a)-(c+a)(a+b)-(a+b)(b+c)$
$=a^2+b^2+c^2-bc-ca-ab.$

34. $(b-c)^2+(c-a)^2+(a-b)^2=2(a-b)(a-c)+2(b-c)(b-a)$
$+2(c-a)(c-b).$

35. If $x=a+d$, $y=b+d$, $z=c+d$, prove that
$x^2+y^2+z^2-yz-zx-xy=a^2+b^2+c^2-bc-ca-ab.$

36. If $s=a+b+c$, prove that
$(as+bc)(bs+ac)(cs+ab)=(b+c)^2(c+a)^2(a+b)^2.$
(C. E. 1902 and A. E. 1890)

If $2s=a+b+c$, prove that

37. $4b^2c^2-(b^2+c^2-a^2)^2=16s(s-a)(s-b)(s-c).$

38. $s^2+(s-a)^2+(s-b)^2+(s-c)^2=a^2+b^2+c^2.$

39. $s(s-a)(s-b)+s(s-b)(s-c)+s(s-a)(s-c)=(s-a)(s-b)(s-c)$
$+abc.$ (B. M. 1870).

40. $2(s-a)(s-b)(s-c)+a(s-b)(s-c)+b(s-c)(s-a)+c(s-a)(s-b)$
$=abc.$ (C. E. 1898).

41. If $s=a+b+c$, prove that
$s(s-2b)(s-2c)+s(s-2c)(s-2a)+s(s-2a)(s-2b)$
$=(s-2a)(s-2b)(s-2c)+8abc.$

42. Shew that $(x+y)^3+3(x+y)^2z+3(x+y)z^2+z^3$
$=(x+z)^3+3(x+z)^2y+3(x+z)y^2+y^3.$

CHAPTER V.

EQUATIONS AND SQUARED PAPER.

I. SIMPLE EQUATIONS.

141. When two algebraical expressions are connected by the sign of equality (=), the whole expression is called according to circumstances, an **Identity** or an **Equation**.

142. An **Identity** is merely the statement of the equivalence of two different forms of the same quantity, and is true for *any* values of any of the letters involved in it.

Thus, $(x+y)^2=x^2+2xy+y^2$ or $x^2-y^2=(x+y)(x-y)$, is *always* true, whatever be the values of x and y. Hence each of these expressions is an *Identity*.

143. An **Equation**, however, is the statement of the equality of two *different* algebraical expressions, in which case the equality does not exist for *any*, but only for some particular values of one or more of the symbols contained in it.

Thus, the equation $x-3=5$ will be found true only when we give x the value 8, and $x^2=5x-6$ only when we give x the value 2 or 3.

144. Hence, an equation which is *only* true when the symbols have certain particular values is called an **equation of condition** or a **conditional equation**.

145. The two parts of an equation on either side of the sign of equality are called its **sides** or **members**.

146. The letter whose value is not known and is required to be found is called the **unknown quantity**. The process of finding its value is called **solving the equation**.

147. An equation is said to be **satisfied** by any value of the unknown quantity which makes the values of the *two* sides of an equation the *same*.

Thus, the equation $x+3=7$ is true when $x=4$. The value 4 is said to *satisfy* the equation.

This includes the case when all the terms of an equation lie on one side and zero on the other, as in $x^2-5x+6=0$, which is *satisfied* by 2 or 3, either of which, being put for x, makes the first side $=0$.

148. Those values of the unknown quantity, by which the equation is satisfied are called its **roots**.

Thus, 7 is the *root* of the equation $x-3=4$; 2 and 3 are the *roots* of $x^2-5x+6=0$, and so on.

149. An equation of one unknown quantity is said to be of as many **dimensions** as is denoted by the index of the highest power of the unknown quantity involved in it. Hence, an equation which involves only the first power of the unknown quantity is of one dimension and therefore it is called a **simple equation,** or an equation of the **first degree.**

Thus, $x+3=7$ is of *one dimension*, and therefore it is called an equation of the first degree or a **simple equation**; $x^2=5x-6$ is of *two dimensions*, and therefore it is called an equation of the second degree or a **quadratic equation**; $x^3-8=6x^2$ is of *three dimensions*, and therefore it is called an equation of the third degree or a **cubic equation**; $x^4-6x^2=15$ is of *four dimensions*, and therefore it is called an equation of the fourth degree or a **biquadratic equation**; and so on.

150. The process of solving Simple Equations with one unknown quantity consists mainly in the use of the following **axioms.**

1. *If equals be added to equals, the sums are equal.*

   Thus, if $x=a$, then $x+4=a+4$.

2. *If equals be taken from equals, the remainders are equal.*

   Thus, if $x=a$, then $x-3=a-3$.

3. *If equals be multiplied by equals, the products are equal.*

   Thus, if $x=a$, then $6x=6a$.

4. *If equals be divided by equals, the quotients are equal.*

   Thus, if $7x=14$, then $x=2$.

Ex. 1. Solve the equation $5x=15$.

Dividing both sides by 5, $x=3$.

Ex. 2. Solve the equation $\frac{x}{3}=-4$.

Multiplying both sides by 3, $x=-12$.

Ex. 3. Solve the equation $15x-3x+x=37-11$.

By collecting the terms, we have $13x=26$.

Dividing both sides by 13, $x=2$.

Exercise LX.

Solve the following equations :—

1. $3x=12$.	**2.** $5x=20$.	**3.** $4x=-16$.	**4.** $18x=54$.
5. $11x=-44$.	**6.** $-x=7$.	**7.** $8x=0$.	**8.** $-3x=-18$.

9. $\frac{x}{5}=-4.$ **10.** $\frac{x}{3}=5.$ **11.** $-\frac{x}{3}=6.$ **12.** $\frac{x}{3}=1.$

13. $-5x=0.$ **14.** $3x=8$ **15.** $9x=17.$ **16.** $-3x=-24.$

17. $\frac{2x}{3}=4.$ **18.** $\frac{3x}{4}=\frac{12}{8}.$ **19.** $\frac{5x}{4}=20.$ **20.** $\frac{7x}{5}=-14.$

21. $\frac{4x}{5}=0.$ **22.** $-\frac{x}{4}=\frac{1}{12}.$ **23.** $\frac{5x}{7}=\frac{25}{14}.$ **24.** $\frac{5x}{3}=10.$

25. $-6x+8x=7-5.$ **26.** $-11x+7x=-6+18.$ **27.** $6x-2x=20.$

28. $7x-2x+3x=18-2.$ **29.** $-3x-4x-7x=-40+4.$

30. $-2x-x-3x=-7+4-12.$ **31.** $6x-3x+5x=-35+11+2.$

32. $-7x=\cdot 21.$ **33.** $\cdot 4x=\cdot 16.$ **34.** $6x=\cdot 06.$ **35.** $\cdot 7x=3\cdot 5.$

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151. Principle of Transposition. *A quantity may be transferred from one side of an equation to the other by changing its sign, without destroying the equality expressed by it.*

Thus, if $x-a=y+b$, adding a to each side of the equation (which, of course, will not destroy the equality) we have $x=y+b+a$, and, subtracting b from each side, we have $x-b=y+a$; where we see that the $-a$ has been transferred to the other side with its sign changed to $+$, and so also the $+b$, with its sign changed to $-$.

152. Consider the equation $12x-8=3x+28$.

Subtracting $3x$ from both sides, $12x-3x-8=28$. (*Ax.* 2)

Adding 8 to both sides, $12x-3x=28+8$. (*Ax.* 1)

Thus, we see that $+3x$ has been removed from the right-hand side, and appears as $-3x$ on the left, *i.e.*, with its sign changed; and -8 has been removed from the left-hand side, and appears as $+8$ on the right, *i.e.*, with its sign changed.

Hence, the above Rule.

153. Change of Signs. *If the signs of all the terms of both sides of an equation be changed, the equality expressed by it will not be destroyed.*

Thus, if $a-b=c-d$; multiplying each side by -1,

We have $-1(a-b)=-1(c-d)$, (*Ax.* 3)

i.e., $-a+b=-c+d$.

154. Consider the equation $-4x-15=2x-3$.

Transposing, $-2x+3=4x+15$

or $4x+15=-2x+3$.

which is the original equation with the sign of every term changed.

Hence the above Rule.

II. SIMPLE EQUATIONS NOT INVOLVING FRACTIONS.

155. To solve a simple equation of one unknown quantity.

Rule. *Transpose all the terms involving the unknown quantity to one side of the equation, and the known quantities to the other, changing the sign of every term thus removed. Collect the terms on each side; divide both sides by the coefficient of the unknown quantity, and thus the root required will be found.*

Ex. 1. Solve the equation $4x+2=3x+4$.
Transposing, $4x-3x=4-2$.
Collecting, $x=2$, the root of the equation.

Ex. 2. Solve the equation $4x+5=10x-16$.
Transposing, $4x-10x=-16-5$,
Collecting, $-6x=-21$,
Changing signs, $6x=21$,
Dividing by 6, $x=\frac{21}{6}=3\frac{3}{6}=3\frac{1}{2}$.

To verify the fact that $3\frac{1}{2}$ is a root of the equation $4x+5=10x-16$.

When $x=3\frac{1}{2}$, $4x+5=4\times 3\frac{1}{2}+5=14+5=19$.
... ... $10x-16=10\times 3\frac{1}{2}-16=35-16=19$.
$\therefore$ ... ... $4x+5=10x-16$,
i.e., the equation is then **satisfied**. Q. E. D.

Ex. 3. Solve the equation $5(x+1)-2=3(x-5)$.
Removing brackets, $5x+5-2=3x-15$,
Transposing, $5x-3x=-15-5+2$,
Collecting, $2x=-18$,
Dividing by 2, $x=-9$.

Verification. When $x=-9$, the left side

$$=5(-9+1)-2=5\times -8-2=-40-2=-42.$$

When $x=-9$, the right side $=3(-9-5)=3\times -14=-42$.
$=$the left side. Q. E. D.

Ex. 4. Solve the equation $(x-3)(x-4)-22=(x-5)(x-6)$.
Multiplying out, $x^2-7x+12-22=x^2-11x+30$.
Transposing, $x^2-x^2-7x+11x=30-12+22$,
Collecting, $4x=40$,
Dividing by 4, $x=10$.

Verification. When $x=10$,
the left side $=(10-3)(10-4)-22=7\times 6-22=42-22=20$.
When $x=10$, the right side $=(10-5)(10-6)$
$=5\times 4=20=$ the left side. Q. E. D.

Ex. 5. Shew that $x=4$ *satisfies* the equation
$$(3x+1)(2x-7)=6(x-3)^2+7.$$
When $x=4$, the left side $=(12+1)(8-7)=13\times 1=13$.
.... the right side $=6(4-3)^2+7=6\times 1+7=6+7$
$=13=$ the left side.

Hence $x=4$ **satisfies** the equation.

Exercise LXI.

Solve the following equations :—

1. $4x-2=3x+3$. **2.** $3x+7=9x-5$. **3.** $4x+9=8x-3$.
4. $3+2x=7-5x$. **5.** $x=7+15x$. **6.** $24x-49=19x-14$.
7. $4x-22=34-3x$. **8.** $26-8x=80-14x$. **9.** $3x=7-2x+8$.
10. $3(x-2)+4=4(3-x)$. **11.** $5-3(4-x)+4(3-2x)=0$.
12. $13x-21(x-3)=10-21(3-x)$. **13.** $4(3x-2)-2(4x-3)=3(4-x)$.
14. $12(x-3)-3(2x-1)=22-5x$. **15.** $5(5-2x)-7(2x-5)=12$.
16. $3x+14-5x+15=4x+5$. **17.** $6x+18=4x-8+3x-2$.
18. $6(x-4)=0$. **19.** $5(3x+7)=0$. **20.** $\frac{7}{9}(6x-15)=0$.
21. $\frac{4}{11}(9x-25x)=0$. **22.** $4x-6x+35=5x-3x+7$.
23. $3(x-3)-2(x-2)+x-1=x+3+2(x+2)+3(x+1)$.
24. $2x-1-2(3x-2)+3(4x-3)-4(5x-4)=0$.
25. $5(3x+2)-2(7x-9)=7(5x+4)+11(5-3x)-61$.
26. $11(x-2)-2(4-3x)-4(1-2x)=17(x-1)+7$.
27. $8x+2-4(5-x)=2(10-x)-7+3(5x-7)$.
28. $(x-8)(x+12)=(x-6)(x+1)$.
29. $(4x-3)(3x-4)-(2x-1)(6x+1)=3(3-5x)-2$.
30. $(x-1)(x-2)+(x-2)(x-3)+2=2(x-3)(x-4)-2$.
31. $5(x-2)^2-(3x-7)(4x-19)=42-7(x-3)^2$.
32. $3x(5x-2)-(2x+7)(x-3)=13(x+1)(x-1)-15$.
33. $(x-3)^2=x^2+4x+29$. **34.** $(x-4)^2=(x-1)^2-3$.
35. $3(2+x)(1-x)-(1-3x)(1+x)-2x=15-8x$.

36. $2x^2-7=x(2x-3)$. **37.** $3x^2-5-x(3x+1)=0$.
38. $2(x+1)(x+3)+8=(2x+1)(x+5)$. **39.** $x(x-9)=x^2-49$.
40. $(x-1)^2-(x-5)^2=(x+5)^2-(x+1)^2-16x$.

156. Literal Equations. Known quantities are sometimes denoted by the letters of the alphabet, a, b, c, &c., as well as by numbers; unknown quantities are always denoted by x, y, z.

Ex. 1. Solve $bx+2x-a=3x+2c$.

Transposing, $bx+2x-3x=a+2c$.

Collecting, $bx-x=a+2c$.

Bracketing, $x(b-1)=a+2c$.

Dividing by $b-1$, $x=\dfrac{a+2c}{b-1}$.

Ex. 2. Solve $(a+b)(b-x)+x(a-b)=0$.

Multiplying out, $ab+b^2-ax-bx+ax-bx=0$.

Transposing and reducing, $-2bx=-ab-b^2$.

Changing signs, $2bx=b(a+b)$.

Dividing by $2b$, $x=\frac{1}{2}(a+b)$.

Exercise LXII.

Solve the following equations:—

1. $mx+a=nx+b$. **2.** $a^2+ax=3a^2-4ax$. **3.** $3ax-4ab=3bx-4b^2$.
4. $5(a+x)-2x=3(a-5x)$. **5.** $(2+x)(a-3)=-4-2ax$.
6. $(a+x)(a-x)=2a^2+2ax-x^2$. **7.** $(m+n)(m-x)=m(n-x)$.
8. $(6x+a)(2x-a)=(3x-b)(4x+b)$.
9. $(x-a)(x-b)=x^2-a^2+b^2+ab$.
10. $x(x-a)+x(x-b)=2(x-a)(x-b)$.

III. SIMPLE EQUATIONS INVOLVING FRACTIONS.

157. When the equations are in fractional form, the fractions should be cleared first by multiplying every term by any common multiple of all the denominators. If the L. C. M. be employed, the equations will be expressed in most simple terms.

Ex. 1. Solve the equation $\frac{x}{2}-\frac{2x}{3}+\frac{3x}{4}=11+\frac{x}{8}$.

Here, we first clear the equation of fractions, by multiplying every term by 24, the L. C. M. of the denominators, and thus we get

$$12x-8\times 2x+6\times 3x=264+3x,$$

or $12x-16x+18x=264+3x$.

Transposing, $12x-16x+18x-3x=264$.

Collecting terms, $11x=264$.

Dividing by 11, $x=24$.

Ex. 2. Solve the equation $\frac{14}{3}+\frac{4}{x}=1-\frac{x-1}{6x}$.

Multiplying both sides by $6x$,

$$14\times 2x+4\times 6=6x-(x-1)$$

or $28x+24=6x-x+1$.

Transposing, $28x-6x+x=-24+1$.

Collecting terms, $23x=-23$,

$\therefore x=-1$.

Verification. When $x=-1$,

the left side $=\frac{14}{3}+4\div(-1)=\frac{14}{3}-4=\frac{14-12}{3}=\frac{2}{3}$,

the right side $=1-(-1-1)\div(6\times -1)=1+2\div(-6)$

$=1-\frac{1}{3}=\frac{2}{3}=$ the left side. Q. E. D.

Ex. 3. Solve the equation $\frac{x+1}{2}+\frac{x+2}{3}=16-\frac{x+3}{4}$

Multiplying both sides by 12, the L. C. M. of 2, 3, 4

$$6(x+1)+4(x+2)=192-3(x+3),$$

or $6x+6+4x+8=192-3x-9$,

Transposing, $6x+4x+3x=192-9-6-8$.

Collecting terms, $13x=169$,

$\therefore x=\frac{169}{13}=13$.

Verification. When $x=13$,

the left side $=\frac{13+1}{2}+\frac{13+2}{3}=\frac{14}{2}+\frac{15}{3}=7+5=12$.

the right side $=16-\frac{13+3}{4}=16-\frac{16}{4}=16-4=12$,

$=$ the left side. Q. E. D.

Ex. 4. Solve the equation

$$\frac{3x^2+x}{2}-\frac{2x^2+x}{3}+\frac{x^2+x}{4}-2\frac{3}{20}=x^2+\frac{x^2+x}{6}-\frac{x^2+5x}{12}+\frac{2}{15}.$$

Multiplying both sides by 60, the L. C. M. of the denominators and expressing the mixed number $2\frac{3}{20}$, as an improper fraction $\frac{43}{20}$, we get

$$30(3x^2+x)-20(2x^2+x)+15(x^2+x)-129$$
$$=60x^2+10(x^2+x)-5(x^2+5x)+8,$$

$$\text{or } 90x^2+30x-40x^2-20x+15x^2+15x-129$$
$$=60x^2+10x^2+10x-5x^2-25x+8.$$

Transposing, we find that the terms involving x^2 destroy one another (otherwise the equation would be a quadratic), and we have the result

$$30x-20x+15x-10x+25x=129+8.$$

Collecting terms, $40x=137$,

$$\therefore\ x=3\tfrac{17}{40}.$$

Exercise LXIII.

Solve the following equations :—

1. $3x+\frac{5x}{4}=34.$ 2. $\frac{3x}{4}=\frac{2x}{3}+\frac{1}{3}.$ 3. $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=13.$

4. $\frac{x}{2}+\frac{x}{3}=x-7.$ 5 $\frac{x}{2}-\frac{x}{3}=\frac{x}{4}-1.$ 6. $\frac{x}{3}+\frac{x}{4}=\frac{x}{8}+5\frac{1}{2}.$

7. $\frac{x}{2}-\frac{x}{3}+\frac{x}{4}=2-\frac{x}{6}+\frac{5x}{12}.$ 8. $\frac{2x}{5}+\frac{x-2}{3}=2x-7.$

9. $\frac{2x}{7}+\frac{x-1}{6}=x-4.$ 10. $\frac{9-2x}{2}=\frac{3}{2}-\frac{7x-18}{10}.$

11. $x+\frac{14-x}{3}=\frac{21-x}{2}.$ 12. $2x-\frac{1}{3}=\frac{2(3-2x)}{5}+\frac{x}{2}.$

13. $\frac{2x+7}{7}-\frac{9x-8}{11}=\frac{x-11}{2}.$ 14. $\frac{x-2}{3}-\frac{x-4}{5}=\frac{x-6}{7}.$

15. $\frac{3x-4}{2}-\frac{5x-6}{3}=27-\frac{6x-14}{7}.$ 16. $\frac{x}{2}-\frac{5x+4}{3}=7-\frac{8x-2}{3}.$

17. $\frac{3x-5}{2}-\frac{x+4}{3}=\frac{2x-3}{7}+\frac{x-1}{4}$.

18. $\frac{7x+5}{6}-\frac{5x-6}{4}=\frac{8-5x}{12}$.

19. $\frac{x}{4}-\frac{x-2}{5}+\frac{5x}{12}=5+\frac{14-x}{2}$.

20. $7+\frac{9}{2x}=9+\frac{1}{2x}$.

21. $\frac{1}{x}+\frac{1}{2x}-\frac{3}{4x}-\frac{5}{12}=\frac{7}{24}$.

22. $\frac{x-1}{2}-\frac{x-2}{3}+\frac{x-3}{4}=\frac{2}{3}$.

23. $\frac{3x-1}{2}-\frac{2(x-1)}{3}=\frac{x-3}{4}-\frac{x-5}{6}+5\frac{1}{3}$.

24. $\frac{x}{6}-1\frac{2}{3}=8\frac{3}{5}+2\left(\frac{3x}{5}-1\right)-\frac{x+8}{3}$.

25. $\frac{60-x}{14}-\frac{3x-5}{7}=6-\frac{24-3x}{4}$.

26. $\frac{5x}{7}-\frac{4x-3}{5}-\frac{2x+7}{21}+1=0$.

27. $\frac{1}{3}(x+1)+\frac{1}{4}(5-2x)-\frac{1}{2}(2+5x)+\frac{1}{3}(5-x)=0$.

28. $\frac{1}{2}(x+7)-\frac{1}{3}(x-7)=\frac{1}{4}(x+9)-\frac{1}{5}(x-9)$.

29. $\frac{2x-1}{5}+\frac{6x-4}{7}=\frac{7x+12}{11}$.

30. $\frac{x+1}{3}+\frac{5-2x}{4}-\frac{2+5x}{2}+\frac{5-x}{3}=0$.

31. $\frac{1}{3}(2x+5)+\frac{1}{6}(2x-5)=\frac{1}{4}(3x+1)+\frac{1}{5}(3x-1)$.

32. $\frac{3x-1}{2}-\frac{2x-5}{3}+\frac{x-3}{4}-\frac{x}{6}=x+1$.

33. $\frac{x}{12}-\frac{8-x}{8}-\frac{5+x}{4}+\frac{11}{4}=0$.

34. $\frac{1}{2}x-\frac{2}{3}(x-7)+\frac{3}{5}(x-3)=14\frac{1}{2}$.

35. $\frac{1}{2}(x-1)-\frac{1}{3}(2-x)+\frac{1}{4}(x+1)=x$.

36. $4x-\frac{x-1}{2}=x+\frac{2x-2}{5}+24$. (C. E. 1880).

37. $6\frac{1}{3}-\frac{x-7}{3}=\frac{4x-2}{5}$. (C. E. 1861).

38. $x-\frac{x-2}{2}=5\frac{3}{4}-\frac{x+10}{5}+\frac{x-2}{4}$. (M. M. 1883).

39. $\frac{7x-1}{4}-\frac{1}{3}\left(2x-\frac{1-x}{2}\right)=6\frac{1}{3}$. (C. E. 1872).

40. $\frac{5-3x}{4}+\frac{5x}{3}=\frac{3}{2}-\frac{3-5x}{3}$. (C. E. 1866).

41. $\frac{x+3}{8}-\frac{x-3}{10}=\frac{x+5}{6}-\frac{x-7}{3}$. (M. M. 1880).

42. $\frac{2(2x-1)}{9} - \frac{3x-2}{13} = 1$. (A. E. 1889).

43. $x - \frac{3-x}{5} = 3\cdot\frac{x-1}{2} + \frac{x+1}{5} - \frac{3}{10}$. (C. E. 1891).

44. $\frac{x-1}{\text{[illegible]}} - \frac{x-9}{2} + \frac{3x-2(x-2)}{7} = 4\frac{1}{2}$. (C. E. 1864).

45 $x - \left(3x - \frac{2x+5}{10}\right) = \frac{1}{6}(2x+57) + \frac{1}{3}$. (A. E. 1890).

46. $\frac{4x+3}{9} + \frac{13x}{108} = \frac{8x+19}{18}$. (C. E. 1878).

47. $\frac{x+2\frac{1}{2}}{15} + \frac{x+3\frac{1}{4}}{25} = \frac{x+4\frac{1}{6}}{55}$. (C. E. 1888)

48. $\frac{x+5}{6} + \frac{1}{9}\left(\frac{x}{2} + \frac{2}{5}\right) - \frac{2}{3}(3+2x) = \frac{4x-14}{3} + \frac{x+10}{10}$. (C. E. 1894).

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158. Sometimes, it happens, that the L. C. M. of all the denominators is too large to be conveniently employed. In such cases, we may see whether two or three of the denominators have a simple common multiple, and get rid of their fractions first, observing to collect terms, and simplify as much as possible, after each step.

Ex. Solve the equation

$$\frac{2x+3}{11} - \frac{x-12}{3} + \frac{3x+1}{4} = 5\tfrac{1}{3} + \frac{4x+3}{12}.$$

Here, the L. C. M. of all the denominators would be 132; but as 12 will include three of them, multiplying by it, (having first changed $5\frac{1}{3}$ to $\frac{16}{3}$), we get

$$\tfrac{12}{11}(2x+3) - 4(x-12) + 3(3x+1) = 64 + 4x + 3,$$

$$\text{or } \tfrac{12}{11}(2x+3) - 4x + 48 + 9x + 3 = 64 + 4x + 3.$$

Hence, transposing and collecting terms, we have

$$\tfrac{12}{11}(2x+3) - 4x + 9x - 4x = 64 + 3 - 48 - 3,$$

or $\frac{12}{11}(2x+3) + x = 16$. Now, multiplying by 11,

$$12(2x+3) + 11x = 176, \text{ or } 24x + 36 + 11x = 176.$$

Transposing, $24x + 11x = 176 - 36$,

or $35x = 140$ and $\therefore x = \frac{140}{35} = 4$.

Exercise LXIV.

Solve the following equations :—

1. $\dfrac{x-2}{8}-\dfrac{x-4}{7}=\dfrac{2x-3}{12}-2\tfrac{3}{4}$. (C. E. 1869).

2. $\dfrac{2x-13}{9}-\dfrac{x-1}{11}=\dfrac{x}{8}+\dfrac{x}{7}-9$. (C. E. 1876).

3. $\dfrac{x-8}{5}+\dfrac{x+4}{4}+\dfrac{x-1}{7}=7-\dfrac{23-x}{5}$. (C. E. (1892).

4. $\dfrac{2x-3}{12}-\dfrac{3x-2}{5}=\dfrac{4x-3}{8}-3\tfrac{5}{24}$.

5. $\dfrac{5(x-9)}{7}+\dfrac{7(x-5)}{9}=\dfrac{9(x-7)}{5}+1\tfrac{2}{3}$.

6. $\dfrac{2x-1}{15}-\dfrac{3x-2}{16}=\dfrac{x-12}{18}-\dfrac{x+12}{24}$.

7. $\dfrac{7x+20}{8}-\dfrac{3(3x+4)}{16}=\dfrac{3x+1}{10}-\dfrac{29-8x}{20}$.

8. $\dfrac{3-x}{2}-\dfrac{1}{3}\left(\dfrac{3-2x}{4}\right)=\dfrac{2x+3}{7}+\left(\dfrac{1}{4}-\dfrac{1+3x}{2}\right)$. (C. E. 1894).

9. $\dfrac{3x-2}{5}+\dfrac{4x-1}{7}-\dfrac{10x}{9}=5(x-9)+3-\dfrac{x}{3}$. (M. M. 1891).

10. $\dfrac{9x-10}{11}-\dfrac{2x-7}{15}=\dfrac{2x}{3}-\dfrac{5+x}{33}$. **11.** $\dfrac{28x-7}{15}-\dfrac{10x+5}{24}=5\dfrac{1}{4}-\dfrac{x}{32}$.

12. $\dfrac{2x+7}{27}-\dfrac{2x-7}{15}=1\dfrac{5}{6}-\dfrac{3x+4}{20}$.

13. $\dfrac{4x-21}{7}+7\dfrac{5}{6}+\dfrac{7x-28}{3}=x+3\dfrac{3}{4}-\dfrac{9-7x}{8}+\dfrac{1}{12}$.

14. $\dfrac{x^2-2\frac{1}{2}}{4}-\dfrac{x-3\frac{1}{2}}{5}=\dfrac{2x^2-3}{8}-\dfrac{x-5\frac{1}{2}}{3}$. (C. E. 1883).

15. $\dfrac{x-3}{7}-\dfrac{\frac{1}{2}x-3}{3}=\dfrac{\frac{1}{6}x+2}{2}-\dfrac{x-6}{3}+\dfrac{x}{8}$. (C. E. 1866).

16. $\dfrac{7x+5}{23}+\dfrac{9x-1}{10}-\dfrac{x-9}{5}+\dfrac{2x-3}{15}=\dfrac{70}{3}$. (P. E. 1837).

17. $\dfrac{3(a-2x)}{4}-\dfrac{2(2a-x)}{3}+\dfrac{x-a}{8}=\dfrac{15(a+x)}{32}$.

18. $\dfrac{2a-2(b-x)}{3}-\dfrac{3x-3(b-a)}{4}-\dfrac{4b-4(a+x)}{5}=\dfrac{5x+5(a-b)}{6}$.

19. $\dfrac{x-a}{5}-\dfrac{m-(a-x)}{24}=\dfrac{3(m+x)}{16}-\dfrac{15a+16m}{80}$.

20. $\frac{1}{3}\{x-(2a-3c)\}-\frac{5}{72}\{7a-5(x-2c)\}=\frac{1}{48}\{8(a+10c)-(2c-x)\}$.

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159. To ensure accuracy in solving equations whose coefficients are decimals, it is advisable to express all the decimals as vulgar fractions, and proceed as before; but it is often found more simple to work entirely in decimals.

Ex. 1. Solve $\cdot 7x-3\cdot 35=6\cdot 4-3\cdot 2x$.

Transposing, $\cdot 7x+3\cdot 2x=6\cdot 4+3\cdot 35$.

Collecting terms, $(\cdot 7+3\cdot 2)x=9\cdot 75$, or $3\cdot 9x=9\cdot 75$.

$$\therefore\ x=9\cdot 75\div 3\cdot 9=2\cdot 5,\ i.e.,\ x=2\tfrac{1}{2}.$$

Ex. 2. Solve $\cdot 15x+\dfrac{\cdot 135x-\cdot 225}{\cdot 6}=\dfrac{\cdot 36}{\cdot 2}-\dfrac{\cdot 09x-\cdot 18}{\cdot 9}$.

Multiplying all the terms by $1\cdot 8$, we have

$$\cdot 27x+\cdot 405x-\cdot 675=3\cdot 24-\cdot 18x+\cdot 36.$$

Transposing, $\cdot 27x+\cdot 405x+\cdot 18x=\cdot 675+3\cdot 24+\cdot 36$.

Collecting terms, $\cdot 855x=4\cdot 275$,

$$\therefore\ x=4\cdot 275\div\cdot 855=5.$$

Exercise LXV.

Solve the following equations:—

1. $\cdot 5x+\cdot 6x-\cdot 8=\cdot 75x+\cdot 25$.

2. $\cdot 09x-\cdot 01x=\cdot 14-\cdot 06x$.

3. $\cdot 2x+\cdot 005x=11\cdot 7+\cdot 01x$.

4. $\cdot 4x+\cdot\dot{3}=\cdot 7+\cdot 8\dot{3}x$.

5. $\dfrac{x-1}{\cdot 25}-\dfrac{x-2}{\cdot 125}=4\cdot 2$.

6. $\dfrac{2x-3}{2\cdot 5}-\dfrac{3x-4}{12\cdot 5}=\cdot 24$.

7. $\cdot 011x+\dfrac{\cdot 001x-\cdot 125}{\cdot 6}=\dfrac{5-x}{\cdot 03}-\cdot 145$. (C. E. 1886).

8. $\dfrac{x-2}{\cdot 05}-\dfrac{x-4}{\cdot 0625}=56$. (P. E. 1889).

9. $\frac{x}{\cdot 5}-\frac{1}{\cdot 05}+\frac{x}{\cdot 005}-\frac{1}{\cdot 0005}=0.$ (C. E. 1883).

10. $\cdot 5x+\frac{\cdot 02x+\cdot 07}{\cdot 03}-\frac{x+2}{9}=9\cdot 5.$ (C. E. 1866).

11. $\frac{5\cdot 2x}{13}-\frac{1-\cdot 21}{5}\left(\frac{3}{5}-\cdot 1\right)=\cdot 1x-\frac{5x-2}{4}+\cdot 028.$ (P. E. 1891).

12. $\cdot 65x+\frac{\cdot 585x-\cdot 975}{\cdot 6}=\frac{1\cdot 56}{\cdot 2}-\frac{\cdot 39x-\cdot 78}{\cdot 9}.$ (C. E. 1882).

160. Approximate Solutions. In finding approximate values, if the first figure neglected is **5** or more than **5**, increase by **one** the last figure retained (See Arith., Art. 385).

In solving the equation, $7x=33$,

dividing both sides by 7, $x=4\cdot 714285$.........

$\therefore x=5$, to the nearest integer,

$=4\cdot 7$ correct to one decimal place,

$=4\cdot 71$ ....two.............places,

$=4\cdot 714$ ....three...................

$=4\cdot 7143$..............four....................

Exercise LXVI.

Find approximate values of x in the following equations :—

1. $7(3x+9)-6(8x+4)=5(6x-3)$, correct to the nearest integer.
2. $5(x-7)+63=18x$, correct to two decimal places.
3. $9(x-16)=16(x+4)$, correct to one decimal place.
4. $(x-2)^2=(x-5)^2+7$, correct to three decimal places.
5. $(x-4)(x+4)=(x-9)(x+9)+13x$, correct to two decimal places.
6. $(x+3)(x-5)-x^2=0$, correct to the nearest integer.
7. $\frac{x}{4}=\frac{x}{7}+5$, correct to the nearest integer.
8. $\frac{x-1}{6}+\frac{2x-1}{7}=\frac{2}{4}\frac{3}{2}$, correct to two decimal places.
9. $\frac{x-1}{2}+\frac{x-2}{3}-\frac{x-9}{4}=0$, correct to two decimal places.

10. $\frac{4}{5}(x-1)+\frac{2x}{7}-\frac{x-7}{14}=\frac{x-1}{5}+13$, correct to one decimal place.

11. $4\frac{3}{4}-\frac{3}{4}(14x-31)=5-\frac{3}{1\frac{1}{2}}x$, correct to two decimal places.

12. $\frac{5x+1}{[\text{illegible}]}+\frac{x+3}{[\text{illegible}]}=x$, correct to the nearest integer.

Exercise LXVII.

Miscellaneous Equations (Easy).

Solve the following equations :–

1 $4x+3=8x-9$. (C. E. 1861). 2. $2x+11=7x-14$. (C. 1862).

3 $\frac{2x}{3}-\frac{x-1}{15}+\frac{\frac{1}{5}x-7}{6}=3$. 4. $\frac{x-3}{5}-\frac{x-5}{4}=\frac{3}{4}$. (C. E. 1870).

5 $\frac{4-5x}{6}-\frac{1-2x}{3}=\frac{13}{42}$. 6. $\frac{1}{14}(3x+\frac{3}{5})-\frac{1}{7}(4x-6\frac{2}{5})=\frac{1}{5}(5x-6)$.

7. $75-\frac{2}{3}(2x-7)-\frac{1}{10}(x-4)=5x-\frac{1}{4}(3x-2)$.

8. $x-5-(5-x)(x+1)=(x-5)(1+x)+4(5-x)$. (P. E. 1888).

9. $(x+\frac{3}{2})(x-\frac{1}{2})-(x+5)(x-3)+\frac{3}{4}=0$. (C. E. 1867).

10. $(6x+9)^2+(8x-7)^2=(10x+3)^2-71$. (C. E. 1882).

11. $120x-4[5x-2\{6x+7(x-8)\}]=16-4[3x-2\{x-6(x-1)\}]$. (C. E. 1893).

12. $(x-1)(x-2)(x-6)=(x-3)^3$. (M. M. 1881).

13. $r(q+x)-pr=t(q+x)-pt$. (M. M. 1882).

14. $x^2+a(2a-x)-\frac{1}{4}k^2=(x-\frac{1}{2}k)^2+a^2$. (P. E. 1889).

15. $\frac{1}{3}(x-a)-\frac{1}{5}(2x-3b)-\frac{1}{2}(a-x)=0$.

16. $(a+b-x)(a-b+x)+(a+x)(b+x)-a^2=0$. (E. M. 1892)

17. $\frac{2x}{5}+\frac{x-2}{3}=2x-7$. (C. E. 1863).

18. $\frac{x-\frac{1}{2}}{5}-\frac{7x-3}{6}+\frac{3}{7}=0$. (C. E. 1875).

19. $\frac{a-x}{a}+\frac{2a-x}{2a}=\frac{3a-x}{3a}$. (C. E. 1870).

20. $\frac{7x+1}{5}-\frac{17-2x}{3}=\frac{5x+1}{4}$.

21. $\frac{17-3x}{5}-\frac{4x+2}{3}=5-6x+\frac{7x+14}{3}$. (B. M. 1883).

22. $x-\left(3x-\frac{2x-5}{10}\right)=\frac{1}{6}(2x-57)-\frac{5}{3}$. (C. E. 1889).

23. $\frac{1}{3}(x+1)+\frac{1}{2}(x-1)-\frac{1}{4}(3x-7)=2$. (C. E. 1890).

24. $\frac{1}{3}(\frac{1}{2}x-2)-2(x-30)=\frac{1}{7}(x-6)-7$. (M. M. 1880).

25. $\frac{2x-3}{6}+\frac{3x-8}{11}=\frac{4x+15}{33}+\frac{1}{2}$. (C. E. 1877).

26. $\frac{15-\frac{2}{5}x}{5}-\frac{2x+5}{2\frac{1}{2}}=\frac{17-\frac{14}{5}x}{3}$. (C. E. 1874).

27. $\frac{2-x}{3}+\frac{3-x}{4}+\frac{4-x}{5}+\frac{5-x}{6}+\frac{3}{4}=0$. (C. E. 1900).

28. $\frac{1}{2}(x-\frac{1}{3}a)-\frac{1}{3}(x-\frac{1}{4}a)+\frac{1}{4}(x-\frac{1}{5}a)=0$. (C. E. 1859-65).

29. Find the value of x which makes the two expressions
$(3x+1)(2x-7)$ and $6(x-3)^2+7$ equal.

30. What value of x will make the expression
$5x-(4x-7)(3x-5)$ equal to $6-3(4x-9)(x-1)$?

31. What value of x will make
$\frac{2x-3}{5}-\frac{4x-6}{3}+\frac{6x+16}{10}$ equal to zero?

32. What do you deduce about the equation
$(2x-3)(3x-4)=(6x-5)(x-2)$?

33 What value of a will make the product of $3-8a$ and $3a+4$ equal to the product of $6a+11$ and $3-4a$? (B. M. 1891).

34. Find the approximate value of x in the following equation
$x-\frac{2x-\cdot 3}{\cdot 7}=\frac{5-x}{\cdot 35}$, correct to two decimal places.

Solve the following equations :—

35. $\frac{x-\frac{1}{3}(x-1)}{3}+\frac{31}{36}=\frac{3-\frac{1}{4}(x-2)}{5}$.

36. $\frac{3x-\frac{2}{3}(1+x)}{4}+\frac{1-\frac{1}{3}x}{5\frac{1}{2}}=\frac{2\frac{2}{3}+\frac{1}{27}(x-1)}{2\frac{1}{3}}$.

37. $\frac{11x-13}{25}+\frac{19x+3}{7}-\frac{5x-25\frac{1}{4}}{4}=28\frac{1}{7}-\frac{17x+4}{21}$.

38. $\frac{x-1\frac{25}{28}}{2}-\frac{2-6x}{13}=x-\frac{5x-\frac{1}{4}(10-3x)}{39}$.

39. $\frac{x+1\frac{1}{2}}{3}-\frac{10-x}{3\frac{2}{3}}=\frac{4-\frac{2}{3}x}{11}-\frac{1}{11}$.

40. $\frac{7x+6\frac{1}{2}}{10}+\frac{11x-\frac{1}{3}(x-1\frac{1}{3})}{12}=\frac{3x+1}{5}+\frac{43x-\frac{1}{2}(3-8x)}{22}$.

41 $4x-\frac{1}{2}(x-2)-[2x-(\frac{1}{4}x-\frac{1}{18}\{16-\frac{1}{2}(x+4)\})]=\frac{2}{3}(x+\frac{5}{2})$.

42. $\frac{1}{3}\{4a(1+x)-\frac{9}{4}(a-x)\}=\frac{1}{4}\{3a(1-x)-\frac{16}{3}(a+x)\}.$

43. $\frac{3x-2}{5}+\frac{4x-1}{7}-\frac{10x}{9}=5(x-9)+3-\frac{x}{3}.$ (M. M. 1891).

44. $x-\frac{x-2}{2}=5\frac{3}{4}-\frac{x+10}{5}+\frac{x-2}{4}.$ (M. M. 1883).

45. $\frac{6x+18}{13}-4\frac{5}{6}-\frac{11-3x}{36}=5x-48-\frac{13-x}{12}-\frac{21-2x}{18}.$ (M. M. 1865).

IV. SYMBOLICAL EXPRESSION.

161. The principles of Algebra are largely employed in solving problems of various kinds, but the chief difficulty lies in representing *symbolically*, *i. e.*, in algebraical form the statements containing relations of quantities expressed in ordinary language. This process of representation is called **Symbolical Expression** and we give here a few instances illustrating its use.

(1) The excess of 9 over 5 is 4, for $9-5$ is 4.
So the excess of x over 5 is $x-5$,
and the excess of 9 over x is $9-x$.

(2) The defect of 7 from 10 is 3, for $10-7$ is 3.
So the defect of x from 10 is $10-x$.

(3) The number which is 5 greater than 7 is 12, for $7+5$ is 12.
So the number which is 5 greater than x is $x+5$.

(4) The number which is 5 less than 7 is 2, for $7-5$ is 2.
So the number which is a less than x is $x-a$.

(5) If 5 is one part of 15, the other part is 10, for $15-5$ is 10.
So if x is one part of 15, the other part is $15-x$.

(6) If 5 is one factor of 15, the other factor is 3, for $\frac{15}{5}$ is 3.
So if x is one factor of 15, the other factor is $\frac{15}{x}$.

(7) 2×5 is a number which is double of 5.
So $2\times x$ or $2x$ is double of x.
Similarly, $3x$ is treble of x and so on.

(8) 9 oranges at 2 pice each, cost (9×2) pice.
So x oranges at 2 pice each, cost $(x\times 2)$ pice or $2x$ pice.

(9) 5 rupees and 9 annas $=(5\times16+9)$ annas.

So x rupees and y annas $=(16x+y)$ annas.

(10) 3 rupees $+$ 5 annas $+$ 8 pies $=(3\times192+5\times12+8)$ pies.

So x rupees $+y$ annas $+z$ pies $=(192x+12y+z)$ pies

(11) A man who walks 4 miles an hour walks (6×4) miles in 6 hours.

So he walks $(x\times4)$ or $4x$ miles in x hours.

Also he walks 29 miles in $\frac{29}{4}$ hours.

So he walks y miles in $\frac{y}{4}$ hours

(12) If Rs. 20 be equally divided amongst 6 men, each man gets Rs $\frac{20}{6}$.

So if x Rs.... ....6... ....Rs. $\frac{x}{6}$.

... x Rs........ ... . y... , ...Rs. $\frac{x}{y}$.

(13) An even number is a number which is divisible by 2.

$\therefore$ if x be any whole number,

$2x$ is an even number.

So if x is any whole number,

$2x+1$ and $2x-1$ are odd numbers.

Ex. 1. Write down three consecutive numbers, the middle one of which is x.

Consecutive numbers differ from each other by 1.

If the middle number is x, the next greater number is 1 greate and is thus $x+1$.

Also the next smaller number is less than x by 1, and is thus $x-1$.

Hence the three numbers are $x-1$, x and $x+1$.

Ex. 2. The difference of two numbers is x and the less of them is 8; what is the other?

The greater number $-$ the smaller number $=x$.

$\therefore$ the greater number $-8=x$.

$\therefore$ the greater number $=x+8$.

Ex. 3. A man is now x years of age; (1) how old will he be in 8 years? (2) how old was he 10 years ago? (3) how old was he y years ago? (4) how old will he be z years hence?

(1) 8 years hence his age will be 8 years more than now, and now his age is x years.

$\therefore$ his age will then $= x$ years $+ 8$ years $= (x+8)$ years.

(2) 10 years ago his age was 10 years less than now.

$\therefore$ his age then $= x$ years $- 10$ years $= (x-10)$ years.

(3) y years ago his age was y years less than now.

$\therefore$ his age then $= x$ years $- y$ years $= (x-y)$ years.

(4) In z years' time his age will be z years more than now.

Hence his age will then $= x$ years $+ z$ years $= (x+z)$ years

Ex. 4. A man had originally *Rs.*20 in his pocket; he spent *Rs.* x, lost *Rs.* y and had *Rs.* z given him. How much has he left?

After spending *Rs.* x, he had *Rs.* $(20-x)$ left.

Then after losing *Rs.* y, he had left *Rs.* $(20-x) -$ *Rs.* y, *i. e.* *Rs.* $(20-x-y)$.

Then after receiving *Rs.* z, he had *Rs.* $(20-x-y) +$ *Rs.* z, *i. e.* *Rs.* $(20-x-y+z)$.

Ex. 5. **A** and **B** commence to gamble; to begin with they had respectively *Rs.* x and *Rs.* y; **A** wins *Rs.*10 from **B**; what has each at the end?

At the beginning **A** had *Rs* x.

After winning *Rs.*10 from **B**, he has *Rs.* $x +$ *Rs.*$10 = (x+10)$*Rs.*

Also after losing *Rs.*10, **B** has *Rs.* $y -$ *Rs.*$10 = (y-10)$*Rs.*

Exercise LXVIII.

1. What number exceeds x by 9?

2. What number exceeds 9 by x?

3. What number is less than x by 16?

4. What number is less than 16 by x?

5. One part of x is 15; what is the other part?

6. One part of 20 is x; what is the other part?

7. By what must 6 be multiplied to make a?

8. What number multiplied by x will give 35?

9. What number divided by x will give 20?

10. By how much does x exceed 7?

11. The sum of two numbers is x and one of them is 25; wha is the other?

12. The difference of two numbers is 12 and x is the greater; what is the other?

13. The sum of two numbers is x and one of them is y; what is the other?

14. The difference of two numbers is x and the less of them is 9; what is the other?

15. The sum of three numbers is 75. One of them is x, another y; what is the third?

16. How many times is x contained in 80?

17. How many times is x contained in $3y$?

18. If x oranges cost 9 pies, what is the price of one?

19. By how much does 15 exceed y?

20. If a book costs 9 annas, how many can be bought for y annas? How many for z rupees?

21. The sum of 15 equal numbers is $75x$; what is the value of each number?

22. If there are 7 numbers each equal to a, what is their product?

23. If there are x numbers each equal to m, what is their product?

24. If x books of equal value cost y Rupees, what does each cost?

25. The sum of two numbers is $2a+5b$ and one of them is $a+3b$; what is the other?

26. I am x years old now; how old shall I be in 5 years? How old was I 10 years ago?

27. Find a number half as much again as x?

28. If I walk x miles in 9 hours, how many do I walk in one hour? How long do I take to walk one mile?

29. If I can walk x miles in y days, what is my rate per day?

30. What is the price in pence of x eggs at six-pence a score?

31. What is the price of x mangoes at 13 annas a dozen?

32. If eggs sell at x annas a dozen, how much does each egg cost? How many will you get for y Rupees.

33. If 5 lbs. of sugar cost 12 annas, what will x lbs. cost?

34. How many days must a man work in order to earn 20 Rupees at the rate of 5 annas a day?

35. If I spend x shillings out of a sum of £7, how many shillings have I left?

36. If 35 contains x five times, what is the value of x?

37. What is the cost of x articles at y shillings each?

38. A man has x crowns and y florins, how many shillings has he? How many pounds?

39. The sum of two numbers is $x+y$; one of them is $x-y$; what is the other?

40. By how much does $3x+y$ exceed $x-y$?

41. What number added to $a-5b$ will make $a+2b$?

42. I walk x miles at the rate of y miles an hour; how many hours do I take to do it?

43. How far can I walk in p hours at the rate of q miles an hour? How long shall I take to walk qx miles?

44. What is the daily wages in shillings of a man who earns 15 Rupees in x weeks, working 6 days a week? (One rupee = 1*s.* 6*d.*).

45. Write down three consecutive numbers of which x is the least.

46. Write down three consecutive numbers of which x is the greatest.

47. Write down four consecutive numbers of which x is the least.

48. Write down five consecutive numbers of which x is the middle one.

49. The greatest of four consecutive numbers is $x+3$; what are the others?

50. What is the next odd number after $2x-1$?

51. What is the even number next before $2x$?

52. Write down three consecutive even numbers, the middle one of which is $2x$.

53. A purse contains £a, b shillings, and c pence; what is the total amount of pence in it?

54. In $2x$ years a man will be a years old, what is his present age? How old was he y years ago?

55. In 8 years a boy will be x years old; what is the present age of his father if he is twice as old as his son?

56. How many miles can a man walk in 25 minutes if he walks x miles in y minutes?

57. How long will it take a man to walk x miles if he walks y miles in p hours?

58. A man travels x miles by boat and y miles by train, how

long will the journey take if the train goes 40 miles and the boat 8 miles an hour?

59. How far is it from **A** to **B** if a man, bicycling at the rate of 6 miles an hour, does the journey in y hours?

60. A square has sides x feet long; what is its area?

61. What is the area in square feet of a room x feet long and y feet wide?

62. A room is x feet long, y feet broad and z feet high; what is the total area of the floor and four walls?

63. If x men do a piece of work in 5 hours, how many men will be required to do the same work in y hours?

64. What is the remainder if x divided by y gives a quotient z?

65. What is the number which when divided by x gives a quotient y and remainder z?

66. What is the quotient if when x is divided by y there is a remainder z?

67. A man has x Rupees in his pocket, he pays away 14 annas and receives 9 pies; express in annas the sum he has left?

68. A horse eats x maunds a week. How many days will it take him to eat 56 maunds? How many days will it take y horses to eat the same amount?

69. A train travels at the rate of x miles an hour; how many yards does it go per minute?

70. How old is a man now who x years ago was m times as old as his son, who is y years old at the present time?

Express the following statements in the form of equations:—

71. When x is divided by y, the quotient is 12 and the remainder 5.

72. A man is x years old, and his son y years younger. The sum of their ages is m years.

73. **A** has £x and **B** has y shillings; after **A** has won 3 shillings from **B**, each has the same amount.

74. The excess of x over 50 is y.

75. The fifth part of $x-7$ is equal to the ninth part of $2x+3$.

76. The product of three consecutive numbers, of which x is the middle one, is a^2.

77. There are x pence in £a, b half-crowns and c shillings.

78 The area of a room a ft. long and b ft. wide is x square yds.

79. The product of x and y is five times the excess of a over b.

80. A is x years old, B is 10 years older. The sum of their ages is p.

81. y exceeds one-quarter of x by 20.

82. A boy possesses x marbles and he buys y more on each of 7 consecutive days. He had finally a marbles.

83. An army had x men originally, it lost one quarter of its men in an action, y men died of their wounds after the battle and 600 men deserted. There were a soldiers left.

84. The simple interest of a Rupees for p years at 3 per cent. is x.

85. The cost of x mangoes at y annas a-piece is b Rupees.

V. EASY PROBLEMS.

162. We shall now apply the methods explained in the above and preceding *Sections* to the solution of many entertaining problems, and thus exhibit to the student specimens of the practical use of Algebra.

In treating these problems, however, after having observed the methods used in the following examples, the student must be left very much to his own ingenuity, as no *general* rule can be stated for their solution. The only advice that can be given is to read over carefully and consider well the meaning of the question proposed; then it will always appear that some quantity, at present unknown, is required to be found from the *data* furnished by it: put x to represent this quantity, and now set down in algebraical language the statement made in the question, using x whenever this unknown quantity is wanted in it. We shall thus (in the problems we are now considering) arrive at a simple equation, by means of which the value of x may be found.

Ex. 1. Find two numbers whose sum is 31 and whose difference is 5.

Le x be the smaller number, then $x+5$ is the greater

Their sum is $x+(x+5)$, which is to be equal to 31.

Hence, $x+x+5=31$;

$\therefore\ 2x+5=31$, or $2x=31-5=26$;

$\therefore\ x=13$ and $x+5=18$.

Thus the numbers are 18 and 13.

Verification. $18+13=31$ and $18-13=5$. Q. E. D.

Ex. 2. The sum of two numbers is 20; and if three times the smaller be added to five times the greater the sum is 84. Find the numbers.

Let x be the greater number, then $20-x$ is the smaller.

Five times the greater is $5x$ and three times the smaller is $3(20-x)$.

Their sum is $5x+3(20-x)$, which is to be equal to 84.

Hence, $5x+3(20-x)=84$;

$\therefore\ 5x+60-3x=84$, or $5x-3x=84-60$;

$\therefore\ 2x=24$ and $\therefore\ x=12$, and $20-x=8$.

Thus the numbers are 12 and 8.

Ex. 3. What number is that to which if 8 be added, one-fourth of the sum is equal to 29?

Let x represent the number required.

Adding 8 to it, we have $x+8$, one-fourth of this is $\frac{1}{4}(x+8)$, and this is equal to 29.

Hence, $\frac{1}{4}(x+8)=29$;

Multiplying by 4, $x+8=116$,

$\therefore\ x=116-8=108$.

Thus the required number is 108.

Ex. 4. What number is that, the double of which exceeds its half by 6?

Let $x=$ the number.

Then the double of x is $2x$ and the half of x is $\frac{1}{2}x$.

Hence, $2x-\frac{1}{2}x=6$;

Multiplying by 2, $4x-x=12$, or $3x=12$.

$\therefore\ x=4$.

Thus the number required is 4.

Ex. 5. A cask, which held 270 gallons, was filled with a mixture of brandy, wine and water. There were 30 gallons of wine in it more than of brandy, and 30 of water more than there were of wine and brandy together. How many were there of each?

Let x $=$ number of gals. of brandy;

then $x+30=$ .... wine,

and $2x+30=$ .... wine and brandy together.

$\therefore\ 2x+30+30$ or $2x+60=$ no. of gals. of water;

but the whole number of gallons was 270

Hence, $x+(x+30)+(2x+60)=270$.

$\therefore\ 4x=270-90=180.\ \therefore\ x=45$.

and $x+30=75$ and $2x+60=150$.

Thus, the no. of gals. of brandy$=45$, wine$=75$ and water$=150$.

Ex. 6. How old is a man whose age 10 years ago was three-eighths of what it will be in 15 years?

Let x be the required age in years.

Then 10 years ago, his age was $(x-10)$ years and 15 years hence his age will be $(x+15)$ years.

Hence, $x-10=\frac{3}{8}(x+15)$.

Multiplying by 8, $8x-80=3x+45$;

$\therefore\ 8x-3x=80+45$, or $5x=125$;

$\therefore\ x=25$.

Thus the age of the man is 25 years.

Ex. 7. A sum of £50 is to be divided among **A**, **B** and **C**, so that **A** may have 13 guineas more than **B**, and **C** £5 more than **A**; determine their shares.

In questions of this kind it is of essential importance to have all quantities expressed in the same denomination; in the present instance it will be convenient to express the money in shillings.

Let $x=$ **B**'s share in *shillings*:

Since 13 guineas$=273$ shillings and £5$=100$ shillings,

$\therefore\ x+273=$ **A**'s share, and $(x+273)+100$ or $x+373=$ **C**'s.

Hence, $(x+273)+x+(x+373)=1000$; (for £50$=1000s.$).

$\therefore\ 3x+646=1000$, or $3x=1000-646=354$.

$\therefore\ x=118$ and $x+273=391$, and $x+373=491$.

Thus, **A**'s share$=391s.=$£19. 11$s.$

B's ...... $=118s.=$£ 5. 18$s.$

and **C**'s ... .. $=491s.=$£24. 11$s.$

Ex. 8. Two trains, one of which travels half as fast again as the other, start at the same time from two places 300 miles apart, and meet in 5 hours. Find their rates of travelling.

Suppose the slow train travels x miles an hour,

then the fast .... $(x+\frac{1}{2}x)$ or $\frac{3}{2}x$ ....

In 5 hours, the slow train goes $5x$ miles

$\therefore$ ....fast train goes $5 \times \frac{3}{2}x$ miles.

But the total distance travelled by both in 5 hours is 300 miles.

Hence $5x + 5 \times \frac{3}{2}x = 300$;

Multiplying by 2, $10x + 15x = 600$, or $25x = 600$.

$\therefore$ $x = 24$ and $\frac{3}{2}x = \frac{3}{2} \times 24 = 36$.

Thus the trains travel 24 and 36 miles per hour.

Ex. 9. **A**, **B**, **C** divide among themselves 620 apples, **A** taking 4 to **B**'s 3, and 6 to **C**'s 5; how many did each take?

Let x = **A**'s share;

Then $\frac{3}{4}x$ = **B**'s share and $\frac{5}{6}x$ = **C**'s share.

Hence, $x + \frac{3}{4}x + \frac{5}{6}x = 620$;

Multiplying by 12, $12x + 9x + 10x = 7440$;

$\therefore$ $31x = 7440$, and $\therefore$ $x = 240$,

and $\frac{3}{4}x = 180$ and $\frac{5}{6}x = 200$.

Thus the shares of **A**, **B** and **C** are 240, 180 and 200 apples respectively.

Alternative Solution.

We might have avoided fractions by assuming $12x$ for **A**'s share, when we should have had $9x$ = **B**'s, and $10x$ = **C**'s.

Hence, $12x + 9x + 10x = 620$; whence $x = 20$.

Thus the shares are 240, 180 and 200, as before.

163. It will sometimes be found easier not to put x equal to the quantity directly required, but to some other quantity involved in the question; by this means the equation is often simplified.

Ex. 10. A person bought a number of oranges for 3*s*. 9*d*., and finds that 12 of them cost as much over 5*d*. as 16 of them cost under 2*s*. 6*d*.; how many oranges were bought?

Let x be the price of an orange in pence;

Then 12 oranges cost $12x$ pence and 16 oranges cost $16x$ pence.

Hence, $12x - 5 = 30 - 16x$; (for 2*s*. 6*d*. = 30*d*.)

$\therefore$ $12x + 16x = 30 + 5$, or $28x = 35$; $\therefore$ $x = \frac{5}{4} = 1\frac{1}{4}$.

Thus the price of an orange is $1\frac{1}{4}d.$, and the number of oranges = $45 \div 1\frac{1}{4} = 36$; (for 3*s*. 9*d*. = 45*d*.)

Exercise LXIX.

1. What two numbers are those, whose sum is 48 and difference 12?

2. What number is that, to which if 7 be added, twice the sum will be equal to 32?

3. At an election where 979 votes were given, the successful candidate had a majority of 47; what were the numbers for each?

4. The sum of the ages of two brothers is 49, and one of them is 13 years older than the other: find their ages.

5. What number is that which exceeds its sixth part by 10?

6. The difference of two numbers is 54, and their sum is 88. What are the numbers?

7. The sum of two numbers is 100, and the greater exceeds three times the less by 4. Find the numbers.

8. Three times a certain number exceeds 50 by as much as its double falls short of 40. What is the number?

9. Divide Rs. 140 among **A**, **B**, and **C**, so that **A** may have twice as much as **B**, and **B** three times as much as **C**.

10. Find a number such that its half, third, and fourth parts shall be together greater than its fifth part by 106.

11. Divide 150 into two parts, so that one of them shall be two-thirds of the other.

12. There is a number such that, if 8 be added to its double, the sum will be five times its half. Find it.

13. A bookseller sold 10 books at a certain price, and afterwards 15 more at the same price, and at the latter time received *Rs.* 17 *8a.* more than at the former: what was the price per book?

14. Divide 87 into three parts, such that the first may exceed the second by 7, and the third by 17.

15. Find a number such that if increased by 10, it will become five times as great as the third part of the original number.

16. Find a number such that, if 10 be taken from its double, and 20 from the double of the remainder, there may be 40 left.

17. Find a number whose half is as much less than 100 as its double is greater than 99.

18. Find a number such that, when diminished by 3, one fourth of the remainder may be greater by 2 than one-fifth of the original number.

19. The sum of two numbers is 35 and their difference exceeds one-fifth of the smaller number by 2. Find the numbers

20. Find two consecutive numbers, such that one-half and one-fifth of the first taken together are equal to one-third and one-fourth of the second taken together.

21. **A** is twice as old as **B**; twenty-two years ago he was three times as old. Required **A**'s present age.

22. **A** and **B** began to play with equal sums; **A** won Rs. 30, and then 7 times **A**'s money was equal to 13 times **B**'s: what had each at first?

23. **A** spent Re. 1. 4*a.* in oranges, and says, that 3 of them cost as much under 8*a.*, as 9 of them cost over 8*a.*: how many did he buy?

24. A market woman being asked how many eggs she had, replied, If I had as many more, half as many more, and one egg and a half, I should have 104 eggs; how many had she?

25. **A** and **B** play together for a stake of Rs. 5; if **A** win, he will have thrice as much as **B**; but if he lose, he will have only twice as much. What has each at first?

26. **A** is twice as old as **B**, and seven years ago their united ages amounted to as many years as now represent the age of **A**. Find the ages of **A** and **B** respectively.

27. The sum of the ages of two persons is now 46 years, and the difference of their ages 10 years ago was 12 years. Find the present age of each.

28. A father is 30, and his son 2 years old. In how many years will the father be eight times as old as his son?

29. How much money have I in my purse when a fourth and a fifth of it together make a guinea?

30. Two boys have 240 marbles between them; one arranges his in heaps of six, and the other in heaps of nine; the whole number of heaps thus got is 36. How many marbles has each boy?

31. Divide Rs. 640 among three persons, so that the first may have three times as much as the second, and the third, one-third as much as the first and second together.

32. Two sums of money are together equal to £54. 12*s.*, and there are as many pounds in the one as shillings in the other. What are the sums? (C. E. 1885 and A. E. 1895).

33. Divide Rs. 1000 among **A**, **B**, and **C**, so that **B** shall ha Rs. 100 more than **A**, and **A** four times as much as **C**.

34. **A** had Rs. 20 more than **B**, and after each has spent Rs. **A** has five times as much as **B**. What had **A** and **B** at first?

35. A house and garden cost Rs. 10,000, and ten times the price of the house was equal to fifty times the price of the garden. Find the price of each.

36. The sum of £7. 3*s*. 6*d*. is made up of a number of half-sovereigns, three times as many florins, twice as many shillings, and five times as many six-pences. Find how many coins there are in all.

37. A person buys four horses, for the second of which he gives Rs. 120 more than for the first, for the third Rs. 60 more than for the second, and for the fourth Rs. 20 more than for the third The paid for all was Rs. 2300. How much did each cost?

38. If I subtract from the double of my present age, the treble of my age six years ago, the result is my present age. What is my present age? (A. E. 1893).

39. **A** is twice as old as **B** and 4 years older than **C**. The sum of the ages of **A**, **B** and **C** is 96 years. Find **B**'s age. (C. E. 1866).

40. I bought 25 yards of cloth for Rs. 223. 8*a*.; for a part I paid Rs.8. 8*a*. a yard, and for the rest Rs.9. 8*a*. a yard; how many yards of each were there? (C. E. 1859).

41. A labourer is engaged for 30 days, on condition that he receives 2*s*. 6*d*., per each day he works, and loses 1*s*. for each day he is idle: he receives £2. 7*s*. in all. How many days he works and how many days is he idle? (C. E. 1869, B. M. 1893).

42. A sum of Rs.63. 4*a*. was paid in Rupees and two anna pieces. The total number of coins being 100, how many of each kind were used? (M. M. 1890).

43. A post is a fourth of its length in the mud, a third of its length in the water and 10 feet above the water? What is its length? (C. E. 1863).

44. A person bought a picture at a certain price and paid the same price for the frame. if the frame had cost £1 less and the picture 15*s*. more, the price of the frame would have been only half that of the picture. Find the cost of the picture. (C. E. 1860.)

45. From two towns 561 miles apart, two men start, one from each, at the same time, with a design to meet; one goes 24 and the other 27 miles a day: in how many days will they meet? (C. E. 1879).

46. **A**, who travels $3\frac{1}{2}$ miles an hour, starts $2\frac{1}{2}$ hours before **B** who goes the same road at $4\frac{1}{2}$ miles an hour; when will he overtake **A**? (A. E. 1889).

47. A father's age is 40 and his son's 8; in how many years will the father's age be triple of the son's?

48. What was the total amount of a person's debts, who when he had paid a half, and then a third, and then a twelfth of them, had still Rs. 15. 12*a*. to pay?

49. **A** and **B** have together Rs. 8, **A** and **C** have Rs. 10, **B** and **C** have Rs. 12. What have they each?

50. **A** and **B** compared their monthly incomes and found that **A**'s income was to that of **B** as 7 to 9, and that the third of **A**'s income was Rs. 30 greater than the difference of their incomes. Find what each received (C. E. 1871).

51. A person bought 166 mangoes for 10 Rupees; some he bought at the rate of 18 per Rupee, and the rest at 15 per Rupee. How many did he buy of each sort? (M. M. 1889).

52. A person bought 20 yards of cloth for 10 guineas, for part of which he gave 11*s*. 6*d*. a yard, and for the rest 7*s*. 6*d*. a yard. How many yards of each did he buy?

53 Two coaches start at the same time from Calcutta and Rajmahal, a distance of 200 miles, travelling one at $9\frac{1}{2}$ miles an hour, the other at $9\frac{1}{4}$; where will they meet, and in what time from starting?

54. A workman is engaged for 28 days at *Re.*1. 4*a.* per day, but instead of receiving anything, is to pay 8*a.* per day on all days upon which he is idle: he receives altogether *Rs.*26. 4*a.*; for how ny idle days did he pay?

55. A cistern is filled in 20 min by 3 pipes, one of which conveys 10 gallons more and another 5 gallons less than the third per minute. The cistern holds 820 gallons. How much flows through each pipe in a minute?

56. A garrison of 1500 men was victualled for 36 days; after 16 days it was reinforced, and then the provisions were exhausted in 12 days. Find the number of men in the reinforcement.

(B. M. 1870).

57. **A** starts upon a walk at the rate of 4 miles an hour, and after 15 minutes **B** start at the rate of $4\frac{3}{4}$ miles an hour; when and where will he overtake **A**?

58. How much tea at *Rs.*2. 4*a.* per lb. must be mixed with 50 lbs. at *Rs.*3. per lb., that the mixture may be sold at *Rs.*2. 12*a.* per lb.?

59. A bill of £1. 10*s*. 6*d*. is paid with 13 coins, partly in half-crowns and partly in florins. How many coins were there of each?

60. Divide a yard into two parts such that half of one part with 22 inches may be double the other part.

61. **A** is now 12 years older than **B**; twelve years ago he was twice as old as **B** then was. How old is **A** now and how many years ago is it since he was three times as old as **B** then was?

62. **A**, **B** and **C** have *Rs.*66 divided among them in such a way, that for every *Rs.*3 which **A** receives, **B** receives *Rs* 2, and the share of **C** is *Rs.*6 more than the difference of the shares of **A** and **B**. Find the share of each.

63. The sum of three consecutive numbers is 159; find them.

64. A, travelling half as fast again as B, and starting 9 miles behind him, catches him up in 6 hours; find their rates of travelling.

65. In a cricket match A made 35 runs, C half as many as B, and D one-third as many B, and B's score was just as much below A's as C's above B's. Find the scores of B, C and D.

66. A man walks one-half of a journey at the rate of 4 miles an hour bicycles one third at 12 miles an hour and rides the remainder on horseback at 9 miles an hour, completing the journey in 6 hours and 10 minutes. Find the length of the journey.

67. A starts at noon to travel from P to Q at the rate of 6 miles an hour, and B starts at 1 P.M. to travel from Q to P at the rate of 5 miles an hour. If they meet at 4-30 P.M., find the distance from P to Q.

68. I bought a certain number of apples at 4 a penny, and three-fifths of that number at 3 a penny; by selling them at 16 for five pence I gained 4*d*; how many apples did I buy?

69. How many sheep must a person buy at £7 each that, after paying one shilling a score for folding them at night, he may gain £79. 16*s*. by selling them at £8 each?

70. A person meeting a company of beggars gave 4 pice to each, and had 4*a* left; he found that he should have required 3*a*. more to enable him to give the beggars 6 pice each: how many beggars were there?

71. The numerator of a fraction is 4 less than the denominator; if 10 be subtracted from the numerator, or if 30 be added to the denominator, the resulting fractions are equal. Find the original fraction.

72. Two men invest *Rs.*1000 in a boat. Their claim of the profit is proportionate to the capital invested by each. They make a profit of *Rs.*50, of which one gets *Rs.*5 more than the other. What did each contribute?

73. Five-sixths of the fish in a pond weigh 1 seer each, ten weigh 8 seers each, and the remainder 16 seers each, the total weight being 40 maunds. Find the number.

74. Find two numbers, whose sum is 72, such that their product increased by the square of the smaller number is 864.

75. A and B can perform a task in 30 days, working together. After 11 days, however, B is called away, and A finished it by himself 28 days after. How long would it take A to do the whole of the work by himself?

VI. USE OF SQUARED PAPER.

164. The following Solutions will serve as specimens of the methods to be employed in using **squared paper**. The paper is ruled with faint horizontal and vertical lines which divide the sheet into a number of equal small squares. The most convenient paper for beginners is that ruled to show inches and tenths of an inch.

Ex. 1. A man travels 9 miles west, then 11 miles south, and finally 4 miles east; how far from the starting point, to the nearest mile, is he at the finish?

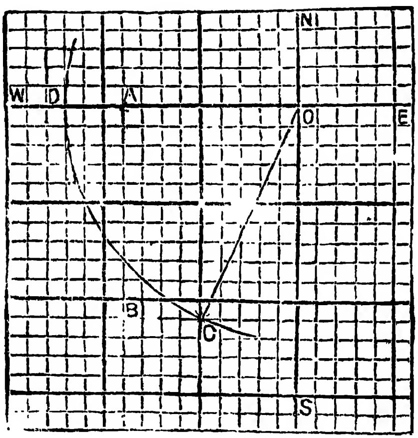


Fig. 1.

On squared paper, take each side of a square to represent one mile, and **O** the starting point. Then

9*m*. west brings him from **O** to **A**,
11*m*. south.................**A** to **B**,
and 4*m*. east............ ....**B** to **C**.

Join **OC**. It is required to find the length of **OC**. (Fig. 1.)

With centre **O** and radius **OC** describe an arc cutting **OW**, the horizontal line through **O**, at **D**. The reqd. distance $=$ **OC** $=$ **OD** $=$ 12 miles, to the nearest mile, as shewn in Fig. 1.

Ex 2. Two vertical posts, 6 ft. and 9 ft. high, are 4 ft. apart ; find the length of the straight line joining their upper ends.

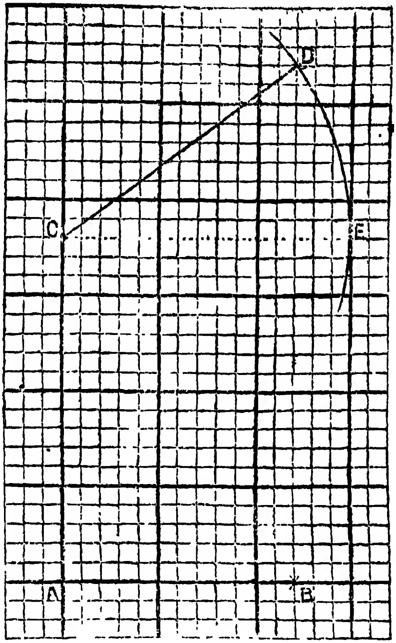


Fig. 2.

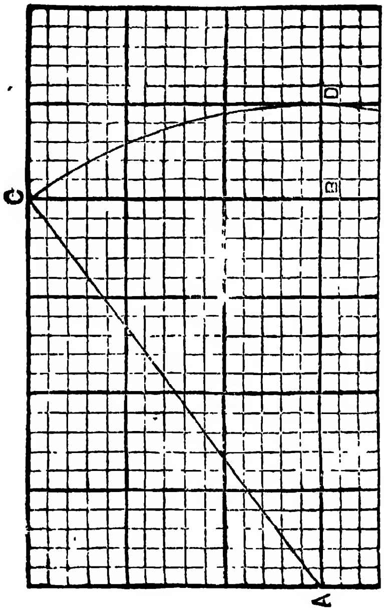
On squared paper, take 3 sides of a square to represent one foot. Then mark the points **A** and **B** 12 sides of a square apart, also the point **C** 18 sides of a square vertically above **A**, and the point **D** 27 sides of a square vertically above **B**. Join **CD**. It is required to find the length of **CD**. (Fig. 2).

With centre **C** and radius **CD** describe an arc of a circle cutting the horizontal line through **C** at **E**.

Then **CD** = **CE** = 15 sides of a square, as shewn in Fig. 2.

Hence **CD** = 5 ft.

Ex 3. A ladder with its foot at a horizontal distance of 20 ft. from a vertical wall, just reaches a point on the wall 15 ft. from the ground; find the length of the ladder.



Fig

Take a side of a square to represent one foot. (Fig. 3.)

Let **A** be the foot of the ladder. Take a point **B** 20 sides of a square in a horizontal line from **A**, so that **B** is the foot of the wall Mark the point **C** 15 sides of a square vertically above **B**.

Join **AC**. It is required to find the length of **AC**.

With centre **A** and radius **AC** describe an arc cutting the horizontal line through **A** at **D**.

Then **AC**=**AD**=25 sides of a square, as shewn in Fig. 3.

Hence **AC**=25 ft.

Ex 4. On a base of 3 inches, describe a triangle whose other sides are 4 inches and $4\frac{1}{2}$ inches long: find the altitude of the triangle to the nearest tenth of an inch.

Let 5 sides of a square represent one inch.

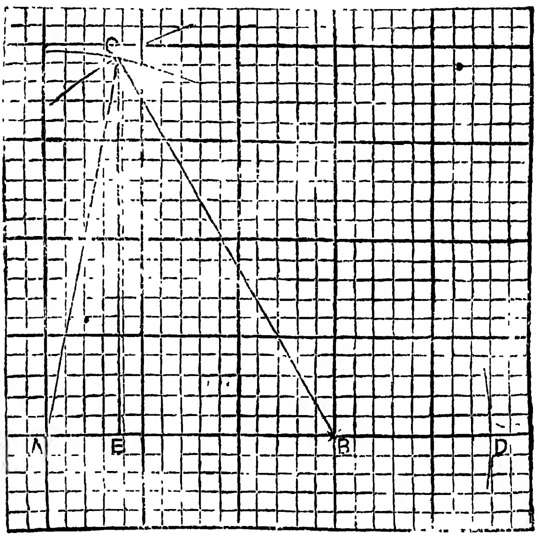


Fig. 4.

Take two points **A** and **B** 15 sides of a square apart. With centre **A** and radius 20 sides of a square describe an arc; also with centre **B** and radius $22\frac{1}{2}$ sides of a square describe an arc cutting the former arc in **C**. Join **AB**, **AC**, **BC**. Then **ACB** is the required triangle. From **C** draw **CE** perpendicular to **AB**. It is required to find the length of **CE**. (Fig. 4)

With centre **E** and radius **EC** describe an arc cutting the horizontal line through **AB** at **D**.

Then **CE** = **ED** = $19\frac{1}{4}$ sides of a square, as shewn in Fig. 4.

Hence **CE** = $(19\frac{1}{4} \times 2 \div 10)$ in. = 3·9 in.

Ex 5. A travels east at 12 miles an hour, and B, starting at the same time from the same place, travels north-east at 20 miles an hour. Find, to the nearest mile, their distance apart at the end of one hour.

Take one-tenth of an inch to represent one mile.

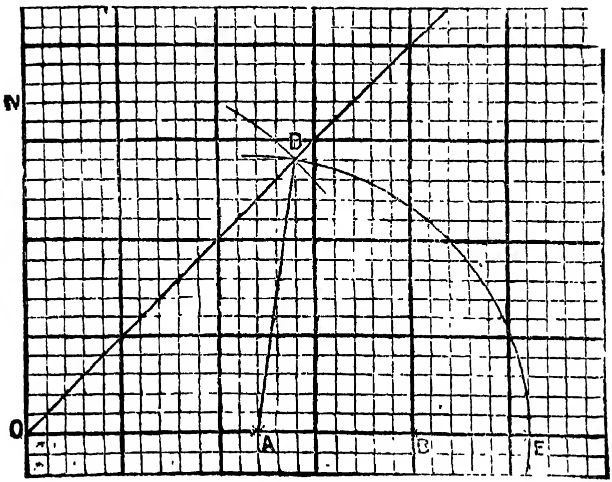


Fig. 5.

Let O be the starting point. Along the horizontal line through O take $OA = 1\cdot2$ inches (representing 12 miles) and $OB = 2$ inches (representing 20 miles). With centre O and radius OB describe an arc cutting the diagonal of the square on OB at D. Then $OD = 2$ inches (representing 20 miles in the north-east direction). Join AD It is required to find the length of AD. (Fig. 5)

With centre A and radius AD describe an arc cutting the horizontal line through O at E.

Then $AD = AE = 1\cdot4$ inches, as shewn in Fig. 5.

Hence $AD = 14$ miles.

Ex. 6. Multiply 2·3 by 3·5 by means of squared paper.

Let **OA** be the unit (5 sides of a square). (Fig. 6).

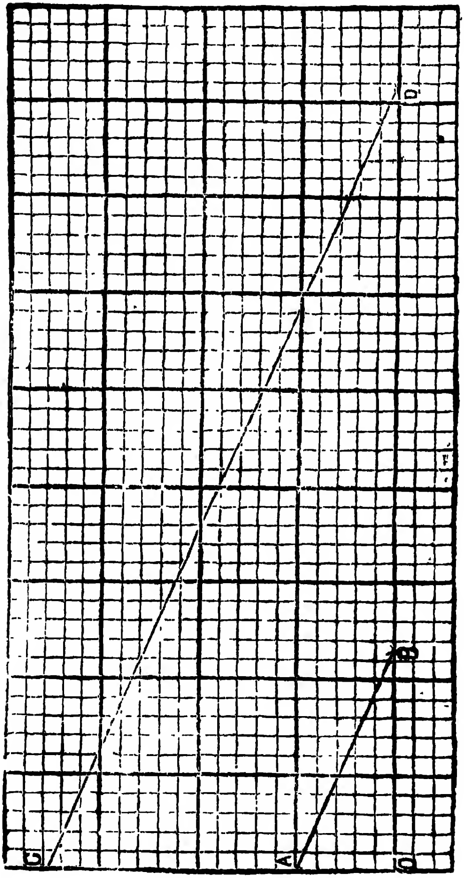


Fig. 6.

Take **OB** = 2·3 and **OC** = 3·5. Join **AB** and through **C** draw **CD** parallel to **AB** meeting the horizontal line through **O** at **D**.

$$\text{Since } \frac{OA}{OB} = \frac{OC}{OD}, \quad \therefore OA \times OD = OB \times OC = 2 \cdot 3 \times 3 \cdot 5.$$

Hence, unit $\times$ **OD** represents the required product.

By measurement, we find **OD** $= 8\frac{1}{20}$ units, as shewn in Fig. 6.

Hence the reqd. product $= 8{\cdot}05$.

Exercise LXX.

1. A man travels 12 miles due east, then 15 miles north, then 20 miles west and finally 22 miles south. Find to the nearest half-mile his distance at the finish from the starting point.

2. Two vertical posts, 24 ft. and 39 ft. high, are 20 ft. apart. Find the length of the straight line joining their upper ends.

3. A ship steaming at the rate of 8 miles an hour due east, and drifting due north with a current is found to be 17 miles from its starting point in 2 hours. Find the rate at which the current flows.

4. Two sides of a right-angled triangle containing the right angle are 2·4 ft. and 3·2 ft. Find, without actual measurement, the length of the third side.

5. A ladder 40 ft. long being placed at the opposite side of a street 24 ft. wide, just reaches the top of a house; how high is the house?

6. A room is 3·6 feet long and 2·7 feet broad. Required the length of a line drawn diagonally through it.

7. A ball rolls 6 ft. east, then 10 ft. north, then 2 ft. west and lastly 6 ft. in a direct line towards its starting point. How far is it then from its starting point?

8. A tower is 96 feet high, and a ladder 100 feet long slopes to the top of it; how far is the foot of the ladder from the bottom of the wall?

9. A man walks 4 miles east, then 6 miles north-east; how far is he then from his starting point?

10. A man walks 3·7 miles south, and then in a direction due west, until he is 5 miles in a straight line from his starting point. Find, without actual measurement, the distance he walked in a westerly direction, to the nearest tenth of a mile.

11. A man rides 2·7 miles east, and then 3·4 miles north; how far is he then from his starting point, to the nearest half-mile?

12. The height of a wall was 31 ft., and the breadth of a ditch surrounding it was 24 ft.: what must be the length of a ladder that will reach from the edge of the ditch to the top of the wall?

13. A room is $5 \cdot 6$ ft. long and $3 \cdot 4$ ft. wide; find the distance between two opposite corners, as accurately as you can.

14. A straight wire joins the top ends of two vertical posts, 17 ft. and 24 ft. high respectively, 35 feet apart. Find the length of the wire to the nearest foot, without actual measurement.

15. A man walks 8 miles west, $6 \cdot 8$ miles north, and then straight towards his starting point until he is two miles from it. How far has he walked?

16. A cow tethered to a post can graze over a circle of 40 feet radius. The shortest distance from the post to a straight hedge is 25 feet. Over what length of hedge can the cow graze?

17. A man walks $2 \cdot 6$ miles west, then $3 \cdot 5$ miles north, and then 2 miles south-east. How far is he then from his starting point?

18. Multiply the following by means of squared paper:—

(i) $3 \cdot 6$ by $2 \cdot 4$. (ii) $4 \cdot 5$ by $3 \cdot 6$. (iii) $3 \cdot 4$ by $4 \cdot 7$.

19. Divide the following by means of squared paper:—

(i) $0 \cdot 75$ by $3 \cdot 9$. (ii) $19 \cdot 08$ by $5 \cdot 3$. (iii) $15 \cdot 98$ by $4 \cdot 7$.

20. **A** travels west at 12 miles an hour, and **B**, starting at the same time from the same place, travels north-west at 20 miles an hour. Find, to the nearest mile, their distance apart at the end of 2 hours.

21. Draw two circles of radii 3 in. and $3\frac{3}{4}$ in., with centres $4\frac{1}{2}$ in apart. Find the length of the line joining their points of intersection.

22. A man, having walked a certain distance in a north-easterly direction, finds that he is 30 miles east of his starting point; how far has he walked?

23. A man walks 5 miles east, then 6 miles north. He then walks due south-west until he is due north of his starting point. How far is he then from home? and how far has he walked?

24. A man walks due east from a town **P** which lies 6 miles north of a town **Q**. How far from **Q** is he when he has walked $7\frac{1}{2}$ miles?

25. **A** and **B** are two places 9 miles apart, **B** lying due east of **A**. One man walks at 3 miles an hour from **A** towards north-east, another man, starting at the same time, walks north-west from **B** at $4\frac{1}{2}$ miles a hour. Find their distance apart to the nearest tenth of a mile in one hour.

CHAPTER VI.

INVOLUTION AND EVOLUTION.

I. INVOLUTION.

165. **Involution** is the name given to the operation by which we find the **powers** of quantities; but all cases of Involution are merely examples of multiplication, where the factors are all the *same*.

166. The following remarks are evident from the *Rule of Signs*.

(i) Since any *even* number of like signs, whether all + or all −, will give + in Multiplication, it follows that any *even* power of a quantity is positive, whether that quantity be taken positively or negatively.

Thus, $(+a)^2$ and $(-a)^2$ are each $= +a^2$, and $(1-x+x^2)^4$ is the same as $\{-(1-x+x^2)\}^4$, or $(-1+x-x^2)^4$.

(ii) No *even* power of any quantity can be *negative*.

(iii) Any *odd* power of a quantity will have *the same sign* as the quantity itself.

167. By the Rules of Multiplication, we have

$$(a^2)^3 = a^2 \times a^2 \times a^2 = a^{2+2+2} = a^6 = a^{2\times 3}.$$

$$(-a^3)^2 = (-a^3) \times (-a^3) = a^{3+3} = a^6 = a^{3\times 2}.$$

$$(-a^4)^3 = (-a^4) \times (-a^4) \times (-a^4) = -a^{4+4+4} = -a^{12} = -a^{4\times 3}.$$

Hence, any *power of a power* of a quantity is obtained by multiplying together the indices of the two powers.

168. To obtain any power of a simple expression we have the following Rule.

Rule. *Multiply the index of every factor in the expression by the number denoting the power, and give the proper sign to the result*

Ex. 1. $(2x^3y^2)^2 = 2^2x^6y^4 = 4x^6y^4.$

Ex. 2. $(-2xy^2z^3)^3 = -2^3x^3y^6z^9 = -8x^3y^6z^9$

Ex. 3. $\left(-\dfrac{3a^2b}{c^3}\right)^4 = \dfrac{3^4a^8b^4}{c^{12}} = \dfrac{81a^8b^4}{c^{12}}.$

Exercise LXXI.

Write down the square of each of the following :—

1. $2ab^2$. 2. $-3a^2b^3$. 3. $4a^2b^3c^4$. 4. $-5x^2y^4$.

5. $\dfrac{3x^2}{4y^3}$. 6. $\dfrac{3a^3}{4c^2}$. 7. $-\dfrac{5a^2}{7b^2c}$. 8. $-\dfrac{3a^2b^2c^3}{5a^3x}$.

Write down the cube of each of the following :—

9. $2a^2b^3c^4$. 10. $-2a^2b^2c^3$. 11. $-3a^2b^2c^4$. 12. $-5a^2b^3$.

13. $-\dfrac{2x^3}{ab^2c^3}$. 14. $-\dfrac{3x^3}{4yz^2}$. 15. $-\dfrac{b^2cd^3}{3a^3}$. 16. $\dfrac{1}{a^4b^3}$.

Write down the value of each of the following :—

17. $\left(-\dfrac{3ab^2}{4c^3}\right)^4$. 18 $\left(-\dfrac{x^2y^3z^4}{2}\right)^6$. 19. $\left(\dfrac{1}{3a^2}\right)^5$. 20. $\left(-\dfrac{x^3}{y^3z^2}\right)^4$.

21. $(-x^2y^4)^5$. 22. $(-3a^2b^3c^4)^4$. 23. $(-2ab^2c^3)^6$. 24. $(-2a^2)^7$.

---

169. We have already had occasion to notice the square and cube of a binomial See Arts. 91 and 100.

Thus, (i) $(a+b)^2=a^2+2ab+b^2$

(ii) $(a-b)^2=a^2-2ab+b^2$.

(iii) $(a+b)^3=a^3+3a^2b+3ab^2+b^3=a^3+3ab(a+b)+b^3$.

(iv) $(a-b)^3=a^3-3a^2b+3ab^2-b^3=a^3-3ab(a-b)-b^3$.

The student may, for exercise, obtain the fourth, fifth, &c., powers of $a+b$. It will be found that

$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$,

and $(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$.

170. Since $(a^3)^2=a^6=(a^2)^3$, it follows that the *square of the cube* of any quantity is the same as the *cube of the square*.

Similarly, $(a^3)^4=a^{12}=(a^4)^3$,

$\{(a-b)^3\}^2=(a-b)^6=\{(a-b)^2\}^3$, and so on.

Hence, we may shorten the operation of finding the 4th power of a quantity by squaring its square; and similarly, to find the 6th, 8th, &c., powers, we may square the 3rd, 4th, &c. So also to find the cube, or 3rd power, we may take the product of the 1st and 2nd, *i.e.*, of the quantity itself and its square; to find the 5th, we may take that of the square and cube; to find the 7th, of the cube and 4th; and so on.

Thus, we shall have -

$$(\boldsymbol{a}+\boldsymbol{b})^4=(a^2+2ab+b^2)^2=\boldsymbol{a}^4+4\boldsymbol{a}^3\boldsymbol{b}+6\boldsymbol{a}^2\boldsymbol{b}^2+4\boldsymbol{a}\boldsymbol{b}^3+\boldsymbol{b}^4.$$

$$(\boldsymbol{a}+\boldsymbol{b})^5=(a^2+2ab+b^2)(a^3+3a^2b+3ab^2+b^3)$$
$$=\boldsymbol{a}^5+5\boldsymbol{a}^4\boldsymbol{b}+10\boldsymbol{a}^3\boldsymbol{b}^2+10\boldsymbol{a}^2\boldsymbol{b}^3+5\boldsymbol{a}\boldsymbol{b}^4+\boldsymbol{b}^5.$$

and $(\boldsymbol{a}+\boldsymbol{b})^6=(a^3+3a^2b+3ab^2+b^3)^2$

$$=\boldsymbol{a}^6+6\boldsymbol{a}^5\boldsymbol{b}+15\boldsymbol{a}^4\boldsymbol{b}^2+20\boldsymbol{a}^3\boldsymbol{b}^3+15\boldsymbol{a}^2\boldsymbol{b}^4+6\boldsymbol{a}\boldsymbol{b}^5+\boldsymbol{b}^6.$$

In like manner, the expansion of $a-b$ may be obtained.

Putting $-b$ for b in the above results, we have

$(\boldsymbol{a}-\boldsymbol{b})^4=\boldsymbol{a}^4-4\boldsymbol{a}^3\boldsymbol{b}+6\boldsymbol{a}^2\boldsymbol{b}^2-4\boldsymbol{a}\boldsymbol{b}^3+\boldsymbol{b}^4$ (*Signs alternately* $+$, $-$).

$$(\boldsymbol{a}-\boldsymbol{b})^5=\boldsymbol{a}^5-5\boldsymbol{a}^4\boldsymbol{b}+10\boldsymbol{a}^3\boldsymbol{b}^2-10\boldsymbol{a}^2\boldsymbol{b}^3+5\boldsymbol{a}\boldsymbol{b}^4-\boldsymbol{b}^5,$$

and $(\boldsymbol{a}-\boldsymbol{b})^6=\boldsymbol{a}^6-6\boldsymbol{a}^5\boldsymbol{b}+15\boldsymbol{a}^4\boldsymbol{b}^2-20\boldsymbol{a}^3\boldsymbol{b}^3+15\boldsymbol{a}^2\boldsymbol{b}^4-6\boldsymbol{a}\boldsymbol{b}^5+\boldsymbol{b}^6.$

171. The above results should be committed to memory and applied in the solution of similar Examples. The expansions of higher powers are generally best obtained by the *Binomial Theorem* without the labour of actual multiplication.

Ex. Expand $(2x-3)^4$.

$$(2x-3)^4=(2x)^4-4.3.(2x)^3+6.3^2.(2x)^2-4.3^3(2x)+3^4$$
$$=16x^4-96x^3+216x^2-216x+81.$$

172. We have already noticed in Art. 93, how to find the *square* of any trinomial expression. We now proceed to find the *cube* of the trinomial $a+b+c$.

$$(a+b+c)^3=\{a+(b+c)\}^3$$
$$=a^3+3a^2(b+c)+3a(b+c)^2+(b+c)^3, \quad \text{Art. 100.}$$
$$=a^3+3a^2(b+c)+3a(b^2+2bc+c^2)+b^3+3b^2c+3bc^2+c^3,$$
$$=a^3+b^3+c^3+3a^2(b+c)+3b^2(c+a)+3c^2(a+b)+6abc.$$

or $(a+b+c)^3=\{(a+b)+c\}^3$

$$=(a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3, \quad \text{Art. 100.}$$
$$=a^3+3ab(a+b)+b^3+3(a+b)^2c+3(a+b)c^2+c^3,$$
$$=a^3+b^3+c^3+3(a+b)\{ab+(a+b)c+c^2\}$$
$$=a^3+b^3+c^3+3(a+b)(a+c)(b+c).$$

Thus, $(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})^3=\boldsymbol{a}^3+\boldsymbol{b}^3+\boldsymbol{c}^3+3\boldsymbol{a}^2(\boldsymbol{b}+\boldsymbol{c})+3\boldsymbol{b}^2(\boldsymbol{c}+\boldsymbol{a})+3\boldsymbol{c}^2(\boldsymbol{a}+\boldsymbol{b})$
$+6\boldsymbol{abc}$.........................(i)
$=\boldsymbol{a}^3+\boldsymbol{b}^3+\boldsymbol{c}^3+3(\boldsymbol{b}+\boldsymbol{c})(\boldsymbol{c}+\boldsymbol{a})(\boldsymbol{a}+\boldsymbol{b})$........(ii)

an important formula, which should be committed to memory.

Ex. Expand $(1-3x+3x^2)^3$.

$$(1-3x+3x^2)^3 = 1^3+(-3x)^3+(3x^2)^3+3.1^2(-3x+3x^2)+3(-3x)^2(3x^2+1)$$
$$+3(3x^2)^2(1-3x)+6.1(-3x)3x^2$$
$$=1-27x^3+27x^6-9x+9x^2+81x^4+27x^2+27x^4$$
$$-81x^5-54x^3$$
$$=1-9x+36x^2-81x^3+108x^4-81x^5+27x^6.$$

Exercise LXXII.

Write down the expansions of :—

1. $(x+2)^7$. 2. $(x-2)^4$. 3. $(x+3)^6$. 4. $(1+2x)^5$.
5. $(2a-1)^6$. 6. $(3x+1)^4$. 7. $(2x-a)^4$. 8. $(3x+2a)^5$.
9. $(4a-3b)^6$. 10. $(ax-y^2)^6$. 11. $(ax+x^2)^4$. 12. $(2ab-b^2)^5$.
13. $(a-b+c)^3$. 14. $(a-b-c)^3$. 15. $(1-x+x^2)^3$. 16. $(1+x+x^2)^3$.
17. $(a+bx+cx^2)^3$. 18. $(1-2x+x^2)^4$. 19. $(1-2x+3x^2)^3$.
20. $(a-2b+c)^3$. 21. $(1-x-x^2)^3$. 22. $(1+3x+2x^2)^3$.

Find the value of

23. $(2+3x+4x^2)^2+(2-3x+4x^2)^2$. 24. $(1-x+x^2)^3(1+x+x^2)^3$.
25. $(2+3x+4x^2)^3-(2-3x+4x^2)^3$. 26. $(1+x)^3(1-x+x^2)^3$.

173. The following result is worthy of notice, as it exhibits the form of the square of any Multinomial.

$(a+b+c+d+\&c.)^2 = a^2+2a(b+c+d+\&c.)+(b+c+d+\&c.)^2$. Art. 91.

Again, in like manner,

$$(b+c+d+\&c.)^2 = b^2+2b(c+d+\&c.)+(c+d+\&c.)^2,$$
$$(c+d+\&c.)^2 = c^2+2c(d+\&c.)+(d+\&c.)^2, \text{ and so on.}$$

$$\therefore\ (a+b+c+d+\&c.)^2 = a^2+2a(b+c+d+\&c.)$$
$$+b^2+2b(c+d+\&c.)$$
$$+c^2+2c(d+\&c.)$$
$$+d^2+\&c.$$

Thus, we see that the square of any multinomial may be formed by setting down *the square of each term* and then *the product of the double of each term by the sum of all the terms that follow it.*

Ex. 1. $(1+2x+3x^2)^2 = 1^2+(2x)^2+(3x^2)^2+2.1(2x+3x^2)+2.2x(3x^2)$
$$=1+4x^2+9x^4+4x+6x^2+12x^3$$
$$=1+4x+10x^2+12x^3+9x^4.$$

Ex. 2. $(1-6x+12x^2-8x^3)^2 = 1^2+2.1(-6x+12x^2-8x^3)$
$+(-6x)^2+2(-6x)(12x^2-8x^3)$
$+(12x^2)^2+2.12x^2(-8x^3)$
$+(-8x^3)^2$
$= 1-12x+24x^2-16x^3+36x^2-144x^3+96x^4+144x^4-192x^5+64x^6$
$= 1-12x+60x^2-160x^3+240x^4-192x^5+64x^6.$

Ex. 3. Find the coefficient of x^4 in $(1-2x+3x^2-4x^3+5x^4)^2$. Evidently, of the square quantities, we must take $(3x^2)^2$, which only contains x^4; of the products, we must take $2.1.5x^4$ and $2.(-2x)(-4x^3)$, which are the only terms involving x^4.

Hence coefficient reqd. $=9+10+16=35$.

Exercise LXXIII.

Find the expansions of :—

1. $(1-2ax-a^2x^2)^2$. **2.** $(2a^2-a-2)^2$. **3.** $(a^2-2ab+3b^2)^2$.
4. $(1-x+x^2-x^3)^2$. **5.** $(x^3-2x^2+3x+4)^2$. **6.** $(1+2x-3x^2+4x^3)^2$.
7. $(1+x)^6$. **8.** $(1+2x)^6$. **9.** $(a-x)^5$.
10. $(a^3-2a^2b+2ab^2-b^3)^2$. **11.** $(1-2x+3x^2-2x^3+x^4)^2$.
12. $(a^4-2a^3x+a^2x^2-2ax^3+x^4)^2$. **13.** $(1+4x+6x^2+4x^3+x^4)^2$.
14. Find the coefficient of x^4 in $(2-3x+4x^2-5x^3)^2$.
15. Find the coefficient of x^6 in $(1-2x+3x^2-4x^3+5x^4)^2$.

II. EVOLUTION.

174. **Evolution** is the name given to the operation by which we find the **roots** of quantities. It is the inverse of Involution.

175. It follows from Art. 166, that

(i) Any *even* root of a *positive* quantity may be either *positive* or *negative*, *i.e.*, will have a double sign $\pm$.

(ii) There can be no *even* root of a negative quantity.

(iii) Any *odd* root of a quantity will have the same sign as the quantity itself.

176. To obtain any root of a simple expression, we have the following Rule.

Rule. *Divide the index of every factor in the expression by the index of the proposed root and give the proper sign to the result.*

Ex. 1. $\sqrt{(4a^2b^4)} = \sqrt{(2^2a^2b^4)} = \pm 2ab^2.$

Ex. 2. $\sqrt[3]{(-8x^3y^6)} = \sqrt[3]{(-2^3x^3y^6)} = -2xy^2.$

Ex. 3. $\sqrt{\left(\frac{16a^2}{9b^4}\right)} = \sqrt{\left(\frac{4^2a^2}{3^2b^4}\right)} = \pm\frac{4a}{3b^2}.$

Ex. 4. $\sqrt[4]{\left(\frac{625a^4y^8}{81x^4}\right)} = \sqrt[4]{\left(\frac{5^4a^4y^8}{3^4x^4}\right)} = \pm\frac{5ay^2}{3x}.$

177. If, however, the index of the quantity cannot be exactly divided by that of the root (as *e.g.* in the 5th root of a^2, where the 2 cannot be divided by 5), then we cannot find the root of it; but can only *indicate* that the root *is to be extracted*, by writing down the quantity, and the sign $\sqrt{}$ before it, with the index of the root in question; as $\sqrt[5]{a^2}$, $\sqrt[3]{a^4}$. Such quantities are called **Surds** or **Irrational quantities.**

Exercise LXXIV.

Find the square roots of the following :—

1. $25x^2y^4z^6$.
2. $121a^6b^4$.
3. $144a^4b^6c^8$.
4. $4a^2b^4c^6$.
5. $49x^4y^6z^2$.
6. $100a^8b^{12}c^{16}$.
7. $9a^4b^4c^8$.
8. $16x^8y^6$.
9. $\frac{9a^2x^4y^6}{25z^2}$.
10. $\frac{49x^2y^4}{64a^2b^2}$.
11. $\frac{25x^6y^{10}}{16a^2b^4}$.
12. $\frac{49a^{10}b^6c^8}{16x^{14}y^{16}}$.

Extract the cube roots of the following :—

13. $8x^6y^9$.
14. $-27a^6b^3$.
15. $8x^{27}$.
16. $-64a^9b^6$.
17. $-\frac{8a^3y^6}{27x^9}$.
18. $\frac{64b^6c^9}{125a^{12}}$.
19. $-\frac{216a^3b^3c^{15}}{343x^6}$.
20. $\frac{64a^{27}b^6}{x^9}$.

Write down the values of each of the following :—

21. $\sqrt[4]{(a^4x^4y^8)}$.
22. $\sqrt[6]{(a^{24}x^{18})}$.
23. $\sqrt[7]{(128a^{21})}$.
24. $\sqrt[5]{(-x^{20}y^{30})}$.
25. $\sqrt[4]{\left(\frac{16x^4y^8}{625a^{12}}\right)}$.
26. $\sqrt[4]{\left(\frac{81a^4b^8c^{12}}{256x^{16}}\right)}$.
27. $\sqrt[5]{\left(\frac{243a^{10}b^5}{x^{15}}\right)}$.
28. $\sqrt[5]{\left(\frac{32a^5b^{10}}{c^{15}}\right)}$.
29. $\sqrt[6]{\left(\frac{64x^{12}y^6}{729z^{18}}\right)}$.
30. $\sqrt[7]{\left(\frac{a^{21}b^{14}}{c^7}\right)}$.

---

178. *To find the square root of a compound expression.*

We *know* that the square of $a+b$ is $a^2+2ab+b^2$; let us see then how from $a^2+2ab+b^2$, we might deduce its square root $a+b$.

Arranging the terms in descending powers of a, we see that the first term is a^2, and its square root is a, which we set down as the first term of the root on the right. Subtract a^2 from the whole expression, and bring down the remainder $2ab+b^2$. If we now divide $2ab$ by $2a$, we get $+b$, the other term in the root; lastly, if we add this b to the $2a$, multiply the $2a+b$, thus formed, by b, and subtract the product $2ab+b^2$, there is no remainder.

$$\begin{array}{r|l}
 & a^2+2ab+b^2\;(\,a+b \\
 & a^2 \\
\hline
2a+b & 2ab+b^2 \\
 & 2ab+b^2
\end{array}$$

Now we may follow this plan in any other case, and if we find no remainder, the root will be exactly obtained.

Ex. 1. Find the square root of $9x^2+6xy+y^2$.

$$\begin{array}{r|l}
 & 9x^2+6xy+y^2\;(\,3x+y \\
 & 9x^2 \\
\hline
6x+y & 6xy+y^2 \\
 & 6xy+y^2
\end{array}$$

Ex. 2. Find the square root of $16a^2-56ab+49b^2$.

$$\begin{array}{r|l}
 & 16a^2-56ab+49b^2\;(\,4a-7b \\
 & 16a^2 \\
\hline
8a-7b & -56ab+49b^2 \\
 & -56ab+49b^2
\end{array}$$

179. If the root consist of more than two terms, a similar process will enable us to find it, as in the following Example; where it will be seen that the divisor at any step is obtained by doubling the expression already found in the root, or (which amounts to the same thing and is more convenient in practice) by *doubling the last term of the preceding divisor*, and then annexing the new term of the root

Ex. 3. Find the square root of $16x^6-24x^5+25x^4-20x^3+10x^2-4x+1$.

The expression is arranged according to the descending powers of x.

$$\begin{array}{r|l}
 & 16x^6-24x^5+25x^4-20x^3+10x^2-4x+1\;(\,4x^3-3x^2+2x-1 \\
 & 16x^6 \\
\hline
8x^3-3x^2 & -24x^5+25x^4 \\
 & -24x^5+9x^4 \\
\hline
8x^3-6x^2+2x & 16x^4-20x^3+10x^2 \\
 & 16x^4-12x^3+4x^2 \\
\hline
8x^3-6x^2+4x-1 & -8x^3+6x^2-4x+1 \\
 & -8x^3+6x^2-4x+1
\end{array}$$

In the above work, having obtained two terms in the root, $4x^3-3x^2$, we have a remainder

$$16x^4-20x^3+10x^2-4x+1.$$

Double the terms of the root already obtained and place the result $8x^3-6x^2$, as the first part of the divisor. Divide $16x^4$, the first term of the remainder, by $8x^3$, the first term of the divisor; we thus get $+2x$, which we annex both to the root and divisor. Now multiply the complete divisor by $2x$ and subtract. There is still a remainder

$$-8x^3+6x^2-4x+1.$$

Proceed as before, and we find -1 as the last term in the root and there is no remainder. Thus the root is found.

180. It should be noticed as in Art. 175 that all *even* roots have *double signs.*

Thus, the square root of $a^2+2ab+b^2$ may be $-(a+b)$, that is $-a-b$, as well as $a+b$: and, in fact, the first term in the root, which we found by taking the square root of a^2, might have been $-a$ as well as $+a$, and by using this we should have obtained also $-b$.

So in Art. 178, **Ex. 1**, the root may also be $-3x-y$; in Art. 179, **Ex. 3**, $-4x^3+3x^2-2x+1$; and in all these cases we should get the two roots by giving a double sign to the first term in the root.

Exercise LXXV.

Find the square roots of :—

1. $4x^2+4xy+y^2$. **2.** $25a^2-30ab+9b^2$. **3.** $25x^4+30x^3y+9x^2y^2$.

4. $49a^2b^2-14a^3b+a^4$. **5.** $16x^2y^2+40xy^2z+25y^2z^2$.

6. $25a^4b^2c^2+10a^2bc^5+c^8$. **7.** $1+4x+10x^2+12x^3+9x^4$.

8. $9x^4+12x^3+22x^2+12x+9$. **9.** $4a^4-12a^3+25a^2-24a+16$.

10. $9a^2+12ab+4b^2+6ac+4bc+c^2$.

11. $x^4-8x^3y+24x^2y^2-32xy^3+16y^4$.

12. $16x^4-16abx^2+16b^2x^2+4a^2b^2-8ab^3+4b^4$.

13. $x^6-4x^5+10x^4-20x^3+25x^2-24x+16$.

14. $x^6+8x^4-2x^3+16x^2-8x+1$. (C. E. 1867).

15. $x^4-2ax^3+5a^2x^2-4a^3x+4a^4$. (M. M. 1885).

16. $4x^4+8ax^3+4a^2x^2+16b^2x^2+16ab^2x+16b^4$. (C. E. 1870).

17. $1-4x+10x^2-20x^3+25x^4-24x^5+16x^6$. (A. E. 1893).

18. $9a^2-6ab+30ac+6ad+b^2-10bc-2bd+25c^2+10cd+d^2$.

19. $x^6-4x^5y+8x^4y^2-10x^3y^3+8x^2y^4-4xy^5+y^6$.

20. $1-6x+15x^2-20x^3+15x^4-6x^5+x^6$.

21. $4x^6-12x^5+13x^4-22x^3+25x^2-8x+16$. (C. E. 1893).

22. $x^6-12x^5+60x^4-160x^3+240x^2-192x+64$. (A. E. 1891).

23. $x^2(x^2+y^2+z^2)+2x(y+z)(yz-x^2)+y^2z^2$. (M. M. 1890).

24. $p^2+2pqx+(2pr+q^2)x^2+2(ps+qr)x^3+(2qs+r^2)x^4+2rsx^5+s^2x^6$

25. $4-12a+5a^2+14a^3-11a^4-4a^5+4a^6$.

26. $x^4+2(y+z)x^3+(3y^2+2yz+3z^2)x^2+2(y^3+y^2z+yz^2+z^3)x+y^4$
$+2y^2z^2+z^4$. (C. E. 1888).

---

181. *To find the square root of fractional expressions*

Rule. Proceed exactly as in Arts. 178 and 179

Ex. 1. Find the square root of $2x^3-\frac{31}{4}x^2+4x^4-2x+4$.

Arrange the expression according to the descending powers of x; thus

$$
\begin{array}{r|l}
 & 4x^4+2x^3-\tfrac{31}{4}x^2-2x+4\;(\,2x^2+\tfrac{1}{2}x-2 \\
 & 4x^4 \\ \hline
4x^2+\tfrac{1}{2}x & 2x^3-\tfrac{31}{4}x^2 \\
 & 2x^3+\tfrac{1}{4}x^2 \\ \hline
4x^2+x-2 & -8x^2-2x+4 \\
 & -8x^2-2x+4
\end{array}
$$

182. As the *fourth power* of a quantity is the square of its square (Art. 170), so the *fourth root* of a quantity is the square root of its square root, and may therefore be found by the preceding Rule. Similarly, the *eighth root* may be found by extracting the square root of the fourth root, and so on.

Thus, if it be required to find the fourth root of $a^4+4a^3x+6a^2x^2+4ax^3+x^4$, the square root will be found to be $a^2+2ax+x^2$, and the square root of this to be $a+x$, which is therefore the fourth root of the given expression.

183. Sometimes we meet with algebraical expressions whose square root cannot be found *exactly*. In such cases an *approximation* to the root can only be obtained to any degree of accuracy.

Ex. Find to four terms the square root of $1+x$.

$$
\begin{array}{r|l}
 & 1+x\left(1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\&c.\right. \\
 & 1 \\
\hline
2+\frac{x}{2} & x \\
 & x+\frac{x^2}{4} \\
\hline
2+x-\frac{x^2}{8} & -\frac{x^2}{4} \\
 & -\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64} \\
\hline
2+x-\frac{x^2}{4}+\frac{x^3}{16} & \frac{x^3}{8}-\frac{x^4}{64} \\
 & \frac{x^3}{8}+\frac{x^4}{16}-\frac{x^5}{64}+\frac{x^6}{256} \\
\hline
 & -\frac{5x^4}{64}+\frac{x^5}{64}-\frac{x^6}{256}
\end{array}
$$

Thus the square root of $1+x$ to four terms is $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3$ there is a remainder $-\frac{5}{64}x^4+\frac{1}{64}x^5-\frac{1}{256}x^6$.

Exercise LXXVI.

Find the square roots of :—

1. $\frac{x^2}{9}+\frac{9}{x^2}+2$. 2. $\frac{4x^2}{y^2}+\frac{4y^2}{x^2}+8$. 3. $\frac{25a^2}{9b^2}-2+\frac{9b^2}{25a^2}$.

4. $\frac{1}{4}x^4-\frac{2}{3}x^3-\frac{11}{36}x^2+x+\frac{9}{16}$. (M.M. 1886).

5. $9x^4-2x^3y+\frac{163}{9}x^2y^2-2xy^3+9y^4$. (C.E. 1874).

6. $1-xy-\frac{15}{4}x^2y^2+2x^3y^3+4x^4y^4$. (M.M. 1889).

7. $\frac{1}{4}x^4-\frac{1}{3}ax^3+\frac{14}{45}a^2x^2-\frac{2}{15}a^3x+\frac{1}{25}a^4$.

8. $x^4-2x^2-\frac{2}{x^2}+\frac{1}{x^4}+3$. (C.E. 1871).

9. $9x^2-24x+19-\frac{4}{x}+\frac{1}{4x^2}$. (C.E. 1890).

10. $x^2+4x+10+\frac{12}{x}+\frac{9}{x^2}$. (M.M. 1880).

11. $\frac{x^2}{a^2}-\frac{2x}{a}+3-\frac{2a}{x}+\frac{a^2}{x^2}$. (B.M. 1890).

12. $\frac{x^4}{4}+4x^2+\frac{ax^2}{3}+\frac{a^2}{9}-2x^3-\frac{4ax}{3}$. (C.E. 1889).

13. $\frac{4x^2}{9y^2}-\frac{x}{z}-\frac{16x^2}{15yz}+\frac{9y^2}{16z^2}+\frac{6xy}{5z^2}+\frac{16x^2}{25z^2}$.

14. $\frac{x^4}{y^4}+\frac{y^4}{x^4}-4\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)+6$.

Extract the fourth roots of the following :—

15. $1-4x+6x^2-4x^3+x^4$. **16.** $a^4-8a^3+24a^2-32a+16$.

17. $16a^4-96a^3b+216a^2b^2-216ab^3+81b^4$.

18. $1-4x+10x^2-16x^3+19x^4-16x^5+10x^6-4x^7+x^8$.

Extract the eighth roots of the following :—

19. $x^8-16x^7+112x^6-448x^5+1120x^4-1792x^3+1792x^2-1024x+256$.

20. $a^8-8a^7b+28a^6b^2-56a^5b^3+70a^4b^4-56a^3b^5+28a^2b^6-8ab^7+b^8$.

Find the square root of each of the following to four terms :—

21. $1-2x$. **22.** a^2+x^2. (C.E. 1877).

23. a^2-b. **24.** $1-x-x^2$. (C.E. 1885).

184. Square Roots by Inspection. When an Algebraical expression can be put into the form of a square of a binomial, its square root can be obtained by inspection alone.

Ex. 1. Find the square root of $a^2+b^2+c^2-2ab+2ac-2bc$.

Arrange in powers of a, and the expression

$$=a^2-2a(b-c)+b^2-2bc+c^2=a^2-2a(b-c)+(b-c)^2,$$
$$=\{a-(b-c)\}^2.$$

Hence the square root $=a-(b-c)=a-b+c$.

Ex. 2. Find the square root of $\left(x^2+\frac{1}{x^2}\right)^2-4\left(x+\frac{1}{x}\right)^2+12$, (C.E. 1866).

$$\text{The given expression }=\left(x^2+\frac{1}{x^2}\right)^2-4\left(x^2+2+\frac{1}{x^2}\right)+12$$
$$=\left(x^2+\frac{1}{x^2}\right)^2-4\left(x^2+\frac{1}{x^2}\right)+4$$
$$=\left\{\left(x^2+\frac{1}{x^2}\right)-2\right\}^2.$$

Hence the square root $=\left(x^2+\frac{1}{x^2}\right)-2=x^2-2+\frac{1}{x^2}$.

Exercise LXXVII.

Find (*by inspection*) the square root of the following :—

1. $\left(x+\frac{1}{x}\right)^2-4\left(x-\frac{1}{x}\right)$. (P.E. 1887; C.E. 1907).

2. $4-4c+2b+c^2-bc+\frac{b^2}{4}$. (C.E. 1876).

3. $\frac{a^2}{b^2}+\frac{b^2}{a^2}+3+\frac{2a}{b}+\frac{2b}{a}$. (A.E. 1890).

4. $\frac{x^2}{y^2}+\frac{y^2}{x^2}-\frac{x}{y}+\frac{y}{x}-1\frac{3}{4}$. (B.M. 1893).

5. $x^4+x^2yz+\frac{1}{4}y^2z^2-2x^2z^2-yz^3+z^4$. (B.M. 1892).

6. $x^4-4x^3+6x^2-4x+1$. (M.M. 1880).

7. $(ab+ac+bc)^2-4abc(a+c)$. (C.E. 1888).

8. $a^2+b^2+c^2+d^2-2a(b-c+d)-2b(c-d)-2cd$. (C.E. 1868).

9. $a^4+b^4+c^4+d^4-2(a^2+c^2)(b^2+d^2)+2a^2c^2+2b^2d^2$. (C.E. 1867).

10. $(a^2+b^2+c^2)(x^2+y^2+z^2)-(bz-cy)^2-(cx-az)^2-(ay-bx)^2$. (C. E. 1878).

11. $(a-b)^2\{(a-b)^2-2(a^2+b^2)\}+2(a^4+b^4)$. (B. M. 1887).

12. $4\{(a^2-b^2)cd+ab(c^2-d^2)\}^2+\{(a^2-b^2)(c^2-d^2)-4abcd\}^2$.

13. $3(3a^2-2ab+b^2)(a^2+3b^2)+b^2(a+4b)^2$. (M. M. 1898).

14. $a^4+2(2b-c)a^3+4a^2b^2+3a^2c^2+4abc^2-4a^2bc+c^4-2ac^3$.

15. $\frac{x^4}{y^4}+\frac{y^4}{x^4}-2\left(\frac{x^3}{y^3}+\frac{y^3}{x^3}\right)+3\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)-4\left(\frac{x}{y}+\frac{y}{x}\right)+5$. (M. M. 1882).

185. *To find the cube root of a compound expression.*

Let us consider the expression $a^3+3a^2b+3ab^2+b^3$, which we know to be the cube of $a+b$, and see how we may get the root from it.

Arranging the terms in descending powers of a, we see that the first term is a^3, of which the cube root is a, and this we set down as the first term of the root. Subtract a^3 from the given expression and the remainder is

$$3a^2b+3ab^2+b^3 \text{ or } (3a^2+3ab+b^2)\times b.$$

Now dividing the first term of this remainder by $3a^2$ *i. e.* by three times the square of the term already found, we get b, the other term in the root. Having found b, we can complete the divisor thus :

To three times a, add b, the new term in the root, and thus we get $3a+b$. We then multiply this quantity by b, and add the product to $3a^2$, *i.e.* to three times the square of the root already found.

The work may be arranged as follows in three columns :—

$$\begin{array}{ll|l}
 & & a^3+3a^2b+3ab^2+b^3\ \big(\ a+b \\
 & & a^3 \\
\hline
3a+b & 3a^2 & 3a^2b+3ab^2+b^3 \\
 & +(3a+b)b & \\
\cline{2-2}
 & 3a^2+3ab+b^2 & 3a^2b+3ab^2+b^3 \\
\end{array}$$

Pursuing the same course in any other case, if there be no remainder, we conclude that we have obtained the exact cube root. This process may be extended so as to find the cube root of any multinomial expression consisting of more than four terms.

Ex. 1. Find the cube root of $8x^3+12x^2y+6xy^2+y^3$.

$$\begin{array}{ll|l}
 & & 8x^3+12x^2y+6xy^2+y^3\ \big(\ 2x+y \\
 & & 8x^3 \\
\hline
6x+y & 12x^2 & 12x^2y+6xy^2+y^3 \\
 & +6xy+y^2 & \\
\cline{2-2}
 & 12x^2+6xy+y^2 & 12x^2y+6xy^2+y^3 \\
\end{array}$$

Ex. 2. Find the cube root of $x^6-3x^5-3x^4+11x^3+6x^2-12x-8$.

$$\begin{array}{ll|l}
 & & x^6-3x^5-3x^4+11x^3+6x^2-12x-8\ \big(\ x^2-x-2 \\
 & & x^6 \\
\hline
3x^2-x & 3x^4 & -3x^5-3x^4+11x^3 \\
 & -3x^3+x^2 & \\
\cline{2-2}
 & 3x^4-3x^3+x^2 & -3x^5+3x^4-x^3 \\
-2x & x^2 & \\
\hline
3x^2-3x-2 & 3x^4-3x^3+3x^2 & -6x^4+12x^3+6x^2-12x-8 \\
 & -6x^2+6x+4 & \\
\cline{2-2}
 & 3x^4-3x^3-3x^2+6x+4 & -6x^4+12x^3+6x^2-12x-8 \\
\end{array}$$

186. Cube Roots by Inspection. When an expression, which is arranged in descending powers of some letter, consists of four terms and is a perfect cube, its cube root will be the algebraical sum of the cube roots of the first and last terms.

Ex. 1. Find the cube root of $27x^3-135ax^2+225a^2x-125a^3$.

Observing that $\sqrt[3]{(27x^3)}+\sqrt[3]{(-125a^3)}=3x-5a$, we have the given Exp. $=(3x)^3-3.3x.5a(3x-5a)-(5a)^3=(3x-5a)^3$.

$\therefore$ the cube root $=3x-5a$.

187. The above process can easily be extended to the case in which the cube root consists of more than two terms.

Ex. 2. Find the cube root of $x^6-3ax^5+5a^3x^3-3a^5x-a^6$.

Here, $\sqrt[3]{x^6}=x^2$ and $\sqrt[3]{(-a^6)}=-a^2$. Also $-3ax^5$ divided by three times the square of x^2, *i. e.* $3x^4$, gives $-ax$.

Hence, the cube root, *if there be one* $=x^2-ax-a^2$.

On verification, we find that the cube of x^2-ax-a^2 is equal to the given expression.

Exercise LXXVIII.

Find by inspection or otherwise the cube root of :—

1. $x^3+6x^2y+12xy^2+8y^3$.
2. $a^3-9a^2+27a-27$.
3. $x^3+12x^2+48x+64$.
4. $8a^3-36a^2b+54ab^2-27b^3$.
5. $a^3+24a^2b+192ab^2+512b^3$.
6. $8x^3-84x^2y+294xy^2-343y^3$.
7. $m^3-12m^2nx+48mn^2x^2-64n^3x^3$.
8. $a^3x^3-15a^2bx^3+75ab^2x^3-125b^3x^3$.
9. $a^6+6a^5+15a^4+20a^3+15a^2+6a+1$.
10. $x^6-12x^5+54x^4-112x^3+108x^2-48x+8$.
11. $x^6+6x^5+21x^4+44x^3+63x^2+54x+27$. (C.E. 1866)
12. $a^6-3a^5b+6a^4b^2-7a^3b^3+6a^2b^4-3ab^5+b^6$.
13. $8x^6+48x^5y+60x^4y^2-80x^3y^3-90x^2y^4+108xy^5-27y^6$.
14. $x^9-3x^8+6x^7-10x^6+12x^5-12x^4+10x^3-6x^2+3x-1$.
15. $x^3+3x^2(a+b)+3x(a^2+2ab+b^2)+(a^3+3a^2b+3ab^2+b^3)$.
16. $\dfrac{a^3}{27b^3}+4\dfrac{b}{a}-\dfrac{8b^3}{a^3}-\dfrac{2a}{3b}$.
17. $a^6+\dfrac{1}{a^6}+6\left(a^4+\dfrac{1}{a^4}\right)+15\left(a^2+\dfrac{1}{a^2}\right)+20$.
18. $a^3-b^3+c^3-3(a^2b-a^2c-ab^2-ac^2-b^2c+bc^2)-6abc$.
19. $1-6x+21x^2-56x^3+111x^4-174x^5+219x^6-204x^7+144x^8-64x^9$.
20. $(1+3x^2)^3-x^2(3+x^2)^2$.
21. $x^2(x^2+3y^2)^2-y^2(3x^2+y^2)^2$.
22. $b^3(a^3+b^3-c^3)+3b^4(a-c)^2+3b^3(b^2-ac)(a-c)$. (M. F. A. 1887).

Find the sixth root of the following expressions —

23. $1+12x+60x^2+160x^3+240x^4+192x^5+64x^6$.
24. $x^6-12ax^5+60a^2x^4-160a^3x^3+240a^4x^2-192a^5x+64a^6$.

CHAPTER VII.

MEASURES AND MULTIPLES.

I. HIGHEST COMMON FACTOR.

188. When one quantity divides another without remainder, it is said to **measure** it, and is called a **measure** of it.

Thus, 3, a, b, $3a$, ab, a^2, &c., are all *measures* of $3a^2b$.

189. A **Common Measure** of two or more quantities is one which divides each of them without remainder.

Thus, a, b, $3a$, $3b$, ab, $3ab$, are all *common measures* of $3a^2b$ and $15abc$.

190. The **Greatest Common Measure** of two or more expressions is the expression of highest or greatest dimensions or degree which divides each of them without remainder.

Thus, $3ab$ is the *greatest common measure* of $3a^2b$ and $15abc$.

191. The term **Greatest Common Measure** (briefly G. C. M.) is not very appropriate in Algebra, for the terms *greater* and *less* are seldom applicable to those algebraical expressions in which definite numerical values have not been assigned to the various letters which occur. It would be better to speak of the **Highest Common Factor** (briefly H. C. F.), or of the **Highest Common Divisor** (briefly H. C. D.).

192. To find, by *inspection*, the H. C. F. of two or more simple expressions.

Rule. *Find by Arithmetic, the* H. C. F. *of the numerical coefficients; after this number write the lowest powers of the letters common to all the expressions.*

Ex. 1. Find the H. C. F. of $9ab^2c^3$ and $12a^2b^3c^4$.

Here, the H. C. F. of 9 and 12 is 3. The letters common to both expressions are a, b, and c. The lowest power of a is a; the lowest power of b is b^2, and the lowest power of c is c^3.

Hence the required H. C. F. $= \underline{3ab^2c^3}$.

Ex. 2. Find the H. C. F. of $9ab^2c^3x^2y$, $12a^3bc^2y^2$ and $15a^2b^3c^4$.

Here, the H. C. F. of 9, 12 and 15 is 3. The letters common to all the expressions are a, b and c, and the indices of their lowest powers are respectively 1, 1 and 2.

Hence, the required H. C. F. $= \underline{3abc^2}$.

Exercise LXXIX.

Find the H. C. F. of :—

1. a^3b^2 and a^3b. 2. ax^3y^4 and x^2y^3z. 3. $3x^3$ and $12x^2y$.
4. $4a^2b^3$ and $-6ab^3$. 5. $-12x^2y^3z^4$ and $8y^5z^3$. 6. $10a^4b^3$ and $15a^3b^4$.
7. $12a^3bc^2$ and $18a^2b^2c^3$. 8. $8ab^2x^3y^4$ and $-12a^2x^4z^5$.
9. $8ax$, $6a^2y$ and $10ab^2x^3$. 10. $a^4b^5c^4$, $a^3b^4c^5$ and $a^2b^3c^6$.
11. $8a^3bc^2$, $12a^2b^3c$ and $4a^4b^4c^4$. 12. $15a^3b$, $60a^5b^2c^3$ and $25a^3c^4$.
13. $10x^2y^3z^2$ $15x^4y^2z^2$ and $20x^3z^3$. 14. $12x^2y^2z^3$, $18x^3y^6z^4$ and $24y^4z^3$.
15. $77x^3y^6z^2$, $33x^2y^3z^5$, $143xy^2z^6$, and $11x^3y^2z^4$.

---

193. With the aid of the preceding Article, we may sometimes find by *inspection* the H. C. F. of two or more compound expressions, if it happens to be easy to separate them into their component factors.

Ex. 1. Find the H. C. F. of $6a^2b^2-8a^4b$ and $2a^2c^2+5a^2bc$,

$$6a^2b^2-8a^4b=2a^2b(3b-4a^2),$$
$$\text{and } 2a^2c^2+5a^2bc=a^2c(2c+5b).$$

Hence, a^2, which is the H. C. F. of $2a^2b$ and a^2c is the reqd. H.C.F.

Ex. 2. Find the H. C. F. of $6a^2x^2(a^2-x^2)$ and $4a^3x(a+x)^2$.

$$6a^2x^2(a^2-x^2)=2a^2x.3x(a+x)(a-x),$$
$$\text{and } 4a^3x(a+x)^2=2a^2x.2a(a+x)(a+x).$$
$$\therefore \text{ H. C. F.}=2a^2x(a+x).$$

Ex. 3. Find the H.C.F. of $a^2(a^2x^2-3ax^3+2x^4)$ and $x^2(a^4-4a^2x^2)$.

$$a^2(a^2x^2-3ax^3+2x^4)=a^2x^2(a^2-3ax+2x^2)=a^2x^2(a-x)(a-2x).$$
$$\text{and } x^2(a^4-4a^2x^2)=a^2x^2(a^2-4x^2)=a^2x^2(a+2x)(a-2x).$$
$$\therefore \text{ H. C. F.}=a^2x^2(a-2x).$$

Ex. 4. Find the H. C. F. of $x(a+1)^2$, $x^2(a^2-1)$ and $2x(a^2-a-2)$.

$$x(a+1)^2=x(a+1)(a+1),$$
$$x^2(a^2-1)=x.x(a+1)(a-1),$$
$$\text{and } 2x(a^2-a-2)=2.x(a+1)(a-2).$$
$$\therefore \text{ H. C. F.}=x(a+1).$$

Exercise LXXX.

Find the H. C. F. of :—

1. $3x^3y+6xy^3$ and $3xy-9x^2y^2$. 2. $3ax^2-2a^2x$ and a^2x^2-3abx
3. $3a^3+2a^2b-5ab^2$ and $2a^2b+2ab^2$.

4. a^2+ab and a^2-ab.
5. $6a^2-3a^2b$ and $2ab-ab^2$.
6. a^2-b^2 and a^3-b^3.
7. a^3+x^3 and $(a+x)^3$.
8. $4x^3(a+x)^2$ and $10(a^2x-x^3)^2$.
9. $x^2(a^2-x^2)^2$ and $(a^2x+ax^2)^2$.
10. $(a^2b-ab^2)^2$ and $ab(a^2-b^2)^2$.
11. $6(x^2-1)$ and $8(x^2-3x+2)$.
12. $(x^2+x)^2$ and $x^3(x^2-x-2)$.
13. $4(x^3+a^3)$ and $6(x^2-2ax-3a^2)$.
14. $6x^3y-12x^2y^2+3xy^3$ and $4ax^2+4axy+4a^2x$.
15. x^2+x-12 and x^2-2x-3.
16. $x^2+5xy+6y^2$ and $x^2-xy-12y^2$.
17. $a^2-2ax-3x^2$ and $a^2+3ax+2x^2$.
18. $12(x^2+y^2)^2$ and $8(x^4-y^4)$.
19. $a^2(x^2+12x+11)$ and $a^2x^2-11a^2x-12a^2$.
20. $9(a^2x^2-4)$ and $12(a^2x^2+4ax+4)$.
21. a^2+5a+4, a^2+2a-8 and $a^2+7a+12$.
22. x^2-y^2, $(x+y)^2$ and $x^2+3xy+2y^2$
23. x^2-y^2, x^3-y^3 and $x^2-7xy+6y^2$
24. a^2-2a-3, a^2-a-6 and $a^2-7a+12$
25. a^3-1, a^2-1 and a^2+a-2
26. $a^2+b^2-c^2+2ab$, $a^2-b^2-c^2+2bc$ and $ac+bc-c^2$.

194. But if the H. C. F. of two compound expressions be a *compound* expression, it cannot generally be thus easily found by inspection, but may always be obtained by a method we are now about to explain, the proof of which will be given hereafter.

195. An algebraical expression is said to be of so many **dimensions** as is indicated by the highest index of its letter of reference.

Thus $x^2-7x+10$ is of *two* dimensions, x^3+1 of *three*.

If it also involve other letters, it is said to be of so many dimensions in each of them, according to the highest indices of each. If the dimensions of each term are the *same*, the expression is said to be **homogeneous**, and so many dimensions as is indicated by the *sum* of the indices in each term.

Thus, $x^4y+3x^3y^2+x^2y^3$ is of *four* dimensions in x, and *three* in y. It is also *homogeneous* and of *five* dimensions.

196. To find the H. C. F. of two compound expressions which cannot readily be resolved into their prime factors.

Rule. *Arrange the expressions according to powers of some common letter, and divide the one of higher dimensions by the other; or if the highest index happen to be the same in each, take either of*

them for dividend. Take now, as in Arithmetic, the remainder after this division for divisor, and the preceding divisor for dividend, and so on until there is no remainder. The last divisor will be the H. C. F. *required.*

Ex. 1. Find the H. C. F. of $x^2-7x+10$ and $4x^3-25x^2+20x+25$.

$$
\begin{array}{l}
x^2-7x+10\,)\,4x^3-25x^2+20x+25\,(\,4x+3 \\
\qquad\qquad\quad\ \underline{4x^3-28x^2+40x} \\
\qquad\qquad\qquad\qquad 3x^2-20x+25 \\
\qquad\qquad\qquad\qquad \underline{3x^2-21x+30} \\
\qquad\qquad\qquad\qquad\qquad x-5\,)\,x^2-7x+10\,(\,x-2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{x^2-5x} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad -2x+10 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{-2x+10}
\end{array}
$$

$\therefore$ H. C. F. $= x-5$.

The above work may be shown in short thus —

$$
\begin{array}{r|r|r|l}
x & x^2-7x+10 & 4x^3-25x^2+20x+25 & 4x \\
 & x^2-5x & 4x^3-28x^2+40x & \\
-2 & -2x+10 & 3x^2-20x+25 & 3 \\
 & -2x+10 & 3x^2-21x+30 & \\
 & & x-5 &
\end{array}
$$

$\therefore$ H. C. F. $= x-5$

197. If the given expressions have *both* or *either* of them, in any case, *simple* factors, as in Art. 192, these must be struck out, and the Rule applied to the resulting expressions. Then the H. C. F. of these, being found as above, will be the same as that of the given ones; except it should happen that we have to strike factors out of *both* of them, and that *these factors themselves* have a common factor. In this case the H. C. F. found, as above, of the resulting expressions, must be multiplied by this common factor, in order to produce that of the given expressions.

So also, whenever we convert a remainder, according to the Rule, into a divisor, we may strike out of it any simple factor it may contain. Here, however, there is no restriction, as in the former case; because no part of such a simple factor *can* be common also to the new dividend, which, being the same as the former divisor, will be already clear of simple factors. It is only with the *first* pair, or *given expressions*, that we shall have to attend to this.

And, if, moreover, the first term of any such remainder is negative, we may, for the sake of neatness, before taking it as a new divisor, change the signs of all its terms, which is equivalent to dividing it by -1. This can only affect the *signs* of the H. C. F.

Ex. 2. Find the H. C. F. of

$$2x^5-8x^4+12x^3-8x^2+2x \text{ and } 3x^5-6x^3+3x.$$

We have $2x^5-8x^4+12x^3-8x^2+2x=2x(x^4-4x^3+6x^2-4x+1)$,

and $3x^5-6x^3+3x=3x(x^4-2x^2+1)$.

Also $2x$ and $3x$ have a common factor x. Removing the simple factors $2x$ and $3x$, and *reserving* their common factor x, we continue as in Art. 192.

$$\begin{array}{r|r|rl}
x^2 & x^4 \qquad -2x^2 \qquad +1 & x^4-4x^3+6x^2-4x+1 & \\
 & x^4-2x^3+\ x^2 \qquad\quad & x^4 \qquad -2x^2 \qquad +1 & \\
\hline
2x & 2x^3-3x^2 \qquad +1 & -4x \mid -4x^3+8x^2-4x & \\
 & 2x^3-4x^2+2x \qquad & x^2-2x+1 & \\
\hline
1 & x^2-2x+1 & & \\
 & x^2-2x+1 & &
\end{array}$$

$\therefore$ H. C. F. reqd. $=x(x^2-2x+1)$.

198. If now, having first attended to the directions of the above Art., we find at any step of our process, that the first term of the dividend is not *exactly* divisible by the first of the divisor, then, in order to avoid fractions in the quotient, we may multiply the whole dividend by such a simple factor, as will make its first term so divisible.

Ex. 3. Find the H. C. F. of $6x^2y+4xy^2-2y^3$

and $8x^3+4x^2y-4xy^2$.

We have $6x^2y+4xy^2-2y^3=2y(3x^2+2xy-y^2)$,

and $8x^3+4x^2y-4xy^2=4x(2x^2+xy-y^2)$.

Now, noting that $2y$ and $4x$ contain the common factor 2 and keeping it for the reqd. H. C. F., we proceed with the remaining expressions as follows.

$$\begin{array}{r|r|r|l}
2x & 2x^2+xy-y^2 & 3x^2+2xy-y^2 & \\
 & 2x^2+2xy \quad & 2 \qquad\qquad\quad & \\
\hline
-y & -xy-y^2 & 6x^2+4xy-2y^2 & 3 \\
 & -xy-y^2 & 6x^2+3xy-3y^2 & \\
\hline
 & & y \mid \ xy+y^2 & \\
 & & x+y &
\end{array}$$

$\therefore$ H. C. F. required $=2(x+y)$.

Ex. 4. Find the H. C. F. of

$$6x^3+33x^2+36x-27 \text{ and } 6x^4+31x^3+33x^2-19x-3.$$

We have $6x^3+33x^2+36x-27=3(2x^3+11x^2+12x-9)$.

Rejecting 3, we proceed as follows :—

$$\begin{array}{r|r|r|r}
x & 2x^3+11x^2+12x-9 & 6x^4+31x^3+33x^2-19x-3 & 3x \\
 & 2x^3+5x^2-3x & 6x^4+33x^3+36x^2-27x & \\
\hline
3 & 6x^2+15x-9 & -2x^3-3x^2+8x-3 & -1 \\
 & 6x^2+15x-9 & -2x^3-11x^2-12x+9 & \\
\hline
 & & 4 \;\; 8x^2+20x-12 & \\
 & & 2x^2+5x-3 &
\end{array}$$

$\therefore$ H. C. F. reqd $= 2x^2+5x-3$.

Exercise LXXXI.

Find the H. C. F. of :

1. $3x^2+x-2$ and $3x^2+4x-4$.
2. $6x^2+7x-3$ and $12x^2+16x-3$.
3. $9x^2-25$ and $9x^2+3x-20$.
4. $6x^2+13x+6$ and $8x^2+6x-9$.
5. $15x^2-x-6$ and $9x^2-3x-2$.
6. $6x^2-x-2$ and $21x^3-26x^2+8x$
7. $8x^2+14x-15$ and $8x^3+30x^2+13x-30$.
8. $4x^2+3x-10$ and $4x^3+7x^2-3x-15$.
9. $2x^3+6x^2+6x+2$ and $6x^3+6x^2-6x-6$.
10. $2y^3-10y^2+12y$ and $3y^4-15y^3+24y^2-24$.
11. $x^3-6ax^2+12a^2x-8a^3$ and $x^4-4a^2x^2$.
12. $2x^3+10x^2+14x+6$ and $x^3+x^2+7x+39$.
13. $6x^3-6x^2+2x-2$ and $12x^2-15x+3$.
14. $3x^3-22x-15$ and $5x^4+x^3-54x^2+18x$.
15. $3x^3-3x^2y+xy^2-y^3$ and $4x^3-x^2y-3xy^2$.
16. x^3-8x+3 and x^6+3x^5+x+3.
17. x^3-5x-2 and $2x^4+3x^3-8$.
18. $3x^3+3x^2-15x+9$ and $3x^4+3x^3-21x^2-9x$.
19. $x^3+x^2y+xy^2+y^3$ and $x^4+x^3y+xy^3-y^4$.
20. $2a^4+a^3b-4a^2b^2-3ab^3$ and $4a^4+a^3b-2a^2b^2+ab^3$
21. $5x^3+2x^2-15x-6$ and $-7x^3+4x^2+21x-12$.
22. $x^3+6x^2+11x+6$ and $x^3+9x^2+27x+27$. (C. E. 1866 & B. M. 1893.)
23. $x^3-7x^2-80x+576$ and $3x^2-14x-80$. (C. E. 1882).
24. $x^3-5ax^2+7a^2x-3a^3$ and $3x^2-10ax+7a^2$. (C. E. 1880).
25. $x^3-8x^2-12x+144$ and $3x^2-16x-112$. (A. E. 1889).
26. x^3+3x^2-9x+5 and $x^3-19x+30$. (C. E. 1859).
27. x^3+4x^2-5 and x^3-3x+2. (C. E. 1868).
28. $2x^2-5x-39$ and $x^4-21x-18$.

29. $x^3-x^2-8x+12$ and $3x^2-2x-8$. (A. E. 1892).

30. $15x^3-4x^2-53x+30$ and $15x^3-x^2-31x-15$. (C. E. A. 1882)

31. $2x^4+x^3-20x^2-7x+24$ and $2x^4+3x^3-13x^2-7x+15$.

32. $3a^6+15a^5b-3a^4b^2-15a^3b^3$ and $10a^5-30a^4b-10a^3b^2+30a^2b^3$.

33. $x^4-2x^3y+2xy^3-y^4$ and $x^4-2x^3y+2x^2y^2-2xy^3+y^4$.

34. $x^4+6x^3+11x^2+4x-4$ and $x^4+2x^3-5x^2-12x-4$.

35. $20x^4+x^2-1$ and $25x^4+5x^3-x-1$. (P. E. 1896).

36. $x^3-2ax^2-5a^2x-12a^3$ and $x^3-7ax^2+13a^2x-4a^3$. (A. E. 1891).

37. $2x^3+9x^2+4x-15$ and $4x^3+8x^2+3x+20$. (C. E. 1867).

38. $4x^3-8ax^2-20a^2x+24a^3$ and $6x^3+24ax^2+6a^2x-36a^3$. (C. E. 1888).

39. $3x^3-17x^2+19x+11$ and $6x^3-25x^2+17x-22$. (C. E. 1872).

40. $2x^4-x^3-9x^2+13x-5$ and $7x^3-19x^2+17x-5$. (M. M. 1889).

41. x^6+x^4+1 and $x^6-2x^4+x^2-1$. (C. E. 1890).

42. $6x^3+7x^2-9x+2$ and $8x^4+6x^3-15x^2+9x-2$. (M. M. 1890).

43. $x^4-6x^3+7x^2+6x-8$ and $2x^3-11x^2+11x+4$. (M. M. 1887).

44. $2x^3+3x^2+2x-2$ and $4x^4-2x^3+2x-1$. (C. E. 1876).

45. $x^4-9a^2x^2+10a^3x$ and $ax^3-a^2x^2-4a^4$. (C. E. 1873).

46. $3x^3-5x^2+5x-2$ and $2x^4-2x^3+3x^2-x+1$. (C. E. 1893).

47. $x(6x^2-8y^2)-y(3x^2-4y^2)$ and $2xy(2y-x)+4x^3-2y^3$. (C. E. 1883).

48. $x^4-2x^3-4x^2-55x$ and $x^5+8x^4+25x^3+52x^2-11x$. (C. E. 1894).

49. $x^4-3x+20$ and $5x^4-3x^3+64$. (B. M. 1882).

50. $x^2+\frac{7}{6}x+\frac{1}{3}$ and $x^2+\frac{1}{3}x+\frac{1}{12}$. (C. E. 1879).

51. $6x^4+x^3-6x^2-5x-2$ and $2x^4+3x^3+2x^2-7x-6$. (C. E. 1875).

52. $x^4+x^3-11x^2-9x+18$ and $x^4-10x^3+35x^2-50x+24$. (C. E. 1877).

53. $20a^4-3a^3b+b^4$ and $64a^4-3ab^3+5b^4$. (C. E. 1891).

54. $x^4-8x^3+28x^2-53x+42$ and $x^4+6x^3-42x^2+129x-154$. (M. M. 1891).

55. $2x^4-2x^3+x^2+3x-6$ and $4x^4-2x^3+3x-9$. (A. E. 1893).

56. $x^4-2x^3+x^2-8x+8$ and $4x^3-12x^2+9x-1$.

57. $6x^4-x^3y-3x^2y^2+3xy^3-y^4$ and $9x^4-3x^3y-2x^2y^2+3xy^3-y^4$.

58. $12x^5-12x^3y^2+12x^2y^3-3xy^4$ and $12x^5+8x^4y-18x^3y^2-6x^2y^3+4xy^4$.

59. $3x^4-10x^3y+22x^2y^2-22xy^3+15y^4$ and $2x^4-7x^3y+16x^2y^2-17xy^3+12y^4$. (M. M. 1882).

60. $7x^4-10ax^3+3a^2x^2-4a^3x+4a^4$ and $8x^4-13ax^3+5a^2x^2-3a^3x+3a^4$. (B.M. 1884).

61. $2x^6 - 11x^2 - 9$ and $4x^6 + 11x^4 + 81$. (C.E. 1865-66 & P.E. 1890 & A.E. 1898).

62. $4x^6 - 209x^3 + 15$ and $15x^6 - 209x^3 + 4$. (M.F.A. 1891).

63. $x^5 + 3x^4 + 46x^3 + 89x^2 + 127x + 164$ and $x^6 + 3x^5 + 46x^4 + 89x^3 + 132x^2 + 169x + 205$. (C.E. 1895).

64. What value other than zero must be given to a in order that $x^3 - x - a$ and $x^2 + x - a$ may have a common factor, and what is their H.C.F. when a has this value?

65. Find the value of y which will make the expressions $2(y^2+y)x^2 + (11y-2)x + 4$ and $2(y^3+y^2)x^3 + (11y^2-2y)x^2 + (y^2+5y)x + 5y - 1$ have a common measure other than unity.

199. To find the H.C.F. of more than two expressions which cannot easily be resolved into their prime factors.

Rule. *Find the* H.C.F. *of any two of them. Then the* H.C.F. *of this answer and the third, and so on. The last* H.C.F. *will be the required* H.C.F.

Ex. Find the H.C.F. of $x^3 - x^2 + 3x + 5$, $x^3 - 5x^2 + 11x - 15$ and $2x^3 - 7x^2 + 16x - 15$.

By the usual process, it will be found that the H.C.F. of $x^3 - x^2 + 3x + 5$ and $x^3 - 5x^2 + 11x - 15 = x^2 - 2x + 5$.

Proceeding as before, it will be found that the H.C.F. of $x^2 - 2x + 5$ and $2x^3 - 7x^2 + 16x - 15 = x^2 - 2x + 5$.

$\therefore$ H.C.F. of the three expressions $= x^2 - 2x + 5$.

Exercise LXXXII.

Find the H.C.F. of :—

1. $x^3 - 4x^2 + 9x - 10$, $x^3 + 2x^2 - 3x + 20$ and $x^3 + 5x^2 - 9x + 35$.

2. $x^4 + 2x^2 + 1$, $x^6 + x^4 - x^2 - 1$ and $x^4 - 1$. (C.E. 1869).

3. $20x^4 + x^2 - 1$, $25x^4 + 5x^3 - x - 1$ and $25x^4 - 10x^2 + 1$.

4. $x^3 - 2x^2 - 19x + 20$, $x^3 + 2x^2 - 23x - 60$ and $x^4 + 7x^3 - 4x^2 - 52x + 48$.

5. $3x^3 - 14x^2 + 16x$, $x^3 - 7x^2 + 16x - 12$ and $5x^3 - 10x^2 + 7x - 14$.

6. $8x^2 + 14x + 3$, $20x^3 + 9x^2 - 3x - 1$ and $12x^3 - 5x^2 + 2x + 1$.

7. $3x^3 - 8x^2 + 7x - 2$, $x^3 - 4x^2 + 5x - 2$ and $2x^3 - 4x^2 + 3x - 1$.

8. $x^6 + x^4 - 4x^3 + 2x^2 + 6x - 9$, $x^4 - x^2 + 6x - 9$ and $x^4 + 2x^3 - 5x^2 - 6x + 9$. (B.M. 1886).

200. In order to prove the Rule given above, it will be necessary to shew first the truth of the following statements.

(1) *If* P *be a measure or factor of* A, *it will be a measure or factor of* mA.

For let a denote the quotient when A is divided by P; then $A = aP$; therefore $mA = maP$, so that P is a factor or measure of mA.

(2) *If* P *be a common measure or factor of* A *and* B, *it will also be a factor of the sum or difference of any multiples of* A *and* B, *as* $mA \pm nB$.

For let P be contained p times in A, and q times in B; then $A = pP$ and $B = qP$, and $mA \pm nB = mpP \pm nqP = (mp \pm nq)P$; hence P is contained $mp \pm nq$ times in $mA \pm nB$, and therefore P is a factor of $mA \pm nB$.

201. *To prove the Rule for finding the* H.C.F. *of two compound Algebraical expressions.*

First, let the two given expressions, denoted by A and B, have neither of them any simple factor.

Let A be that which is not of lower dimensions than the other: and suppose A divided by B, with quotient p and remainder C; B by C, with quotient q and remainder D, and C by D, with quotient r and no remainder.

$$\begin{array}{l} B\,)\,A\,(\,p \\ \quad\;\; pB \\ \quad\;\; \overline{C}\,)\,B\,(\,q \\ \qquad\quad qC \\ \qquad\quad \overline{D}\,)\,C\,(\,r \\ \qquad\qquad\; rD \end{array}$$

Thus, we have the following results.—

$$A = pB + C,\ B = qC + D,\ C = rD.$$

Then, by Art. 200, all the common factors of A and B are also factors of $A - pB$ or C, and are therefore common factors of B and C; and, conversely, all the common factors of B and C are also factors of $pB + C$ or A and are therefore common factors of A and B. Hence it is plain that B and C have precisely the *same* common factors as A and B.

In like manner, it may be shewn that C and D have the same common factors as B and C, and therefore the *same* as A and B.

And so we might proceed if there were more remainders, the quantities A, B, C, D, &c., getting lower and lower, yet still being such that A and B, B and C, C and D, &c., have the same common factors.

But, here, since D divides C without remainder, then D is itself the greatest expression that divides both C and D, (for no quantity

higher than **D** will divide **D**), that is, **D** is the *greatest* of the common factors of **C** and **D**, and therefore is the *Highest Common Factor* of **A** and **B**.

202 *Next*, let A and B have simple factors, and let $A=a\mathbf{A}$, $B=b\mathbf{B}$, where a denotes the product of *all* the simple factors in A, and b of those in B, and **A**, **B** are the resulting expressions, when these simple factors are struck out : then **A**, **B**, having neither of them any simple factor, will have no factor in common with a or b. Now A or $a\mathbf{A}$ is made up only of the factors in a and **A**, and B or $b\mathbf{B}$ only of those in b and **B**. Hence, *if* a be *prime* to b, (that is, *if* a *have no factor in common with* b), the only factors which A can have in common with B must be those which **A** may have in common with **B**, that is, the H. C. F. of A and B will be the same as that of **A** and **B**. But, if a and b have any common factor, then this will also be common to A and B, besides what may be common to **A** and **B**, that is, the H. C. F. of A and B will be obtained by multiplying the H. C. F. of **A** and **B** by the common factor of a and b.

Hence, this case also is reduced to finding the H. C. F. of two expressions **A** and **B**, *which have no simple factors.* And, of course, the above reasoning holds if either a or b be unity, that is, if *one* only of the given expressions have a simple factor to be struck out.

203. Having shewn that we may strike any simple factors out of the original expressions, we shall now shew that we may strike them also out of any of the remainders.

Let then **A**, **B**, represent expressions having no simple factors, (either the original expressions **A**, **B**, if they have no simple factor, or else **A**, **B**, reduced, as above) ; and let us apply the Rule to **A**, **B** dividing **A** by **B**, and obtain the first remainder **C** : then we know that the H. C. F. of **A** and **B** is the same as that of **B** and **C**. Suppose now that $\mathbf{C}=cC$, where c is a simple factor, and C a compound expression, having no simple factor. Then **C** is made up of the factors in c and C; and **B** (having no *simple* factor) can have no factor in common with c, and therefore can have none in common with **C** but such as it may have in common with C; that is, the H. C. F. of **B** and **C** is the same as that of **B** and C. And, of course, the same reasoning holds with the other remainders.

204. *Lastly*, if, at any step (supposing simple factors struck out) the first term of the dividend should not be exactly divisible by the first of the divisor, as, for instance, in the case of **A** and **B**, we may multiply the dividend **A** by any simple factor a', which will make it so divisible : for, since the divisor **B** has no *simple* factor, it can have no factor in common with a', nor therefore any in common with the dividend $a'\mathbf{A}$, but what it may have in common with **A**, that is, the H.C.F. of **A** and **B** will be the same as that of $a'\mathbf{A}$ and **B**.

205. *Every common factor of two expressions is a factor of their* H.C.F.

Let **A** and **B** be the two expressions, and **D** their H.C.F. Let d be any common factor of **A** and **B**, so that

$$A = md \text{ and } B = nd.$$

Then $C = A - pB = md - pnd = (m - pn)d$;
also $D = B - qC = nd - q(m - pn)d = (n - mq + pqn)d$; (Art. 201).
and therefore d is a factor of **D**.

Also, since **D** cannot contain a factor which is not in common with **A** and **B**, (for then it will not divide them exactly), therefore **D** contains all the common factors of **A** and **B** and nothing more.

206. *To prove the Rule for finding the* H.C.F. *of more than two expressions.*

Let **A**, **B** and **C** be the three expressions, and suppose the H.C.F. of **A** and **B** to be **D**: then the H.C.F. of **D** and **C** will be the H.C.F. of **A**, **B** and **C**.

For, since **D** contains all the factors common to **A** and **B**, and nothing more (Art. 205), therefore all the common factors of **D** and **C** must contain all the factors common to **A**, **B** and **C**, and nothing more. Hence the H.C.F. of **D** and **C** is the same as the H.C.F. of **A**, **B** and **C**.

In the same manner, whatever be the number of expressions, their H.C.F. will be determined by a continuation of this process.

Or we may obtain the H.C.F. of **A**, **B**, **C**, and **D**, by finding **X** the H.C.F. of **A** and **B**, and **Y** the H.C.F. of **C** and **D**; then the H.C.F. of **X** and **Y** will be that required: and so in other cases.

207. Nomenclature. We have already noticed in Art. 191 that the term G. C. M. is not appropriate in Algebra. The reason for this will appear from the following consideration.

The H. C. F. of $3x^2+7x-6$ and $3x^2+13x-10$ is $3x-2$; but if we put $x=3$, these quantities become 42 and 56, whose G. C. M. is 14, whereas the numerical value of $3x-2$ would be 7. The fact is that $3x^2+7x-6=(3x-2)(x+3)$, and $3x^2+13x-10=(3x-2)(x+5)$; and here, besides the common factor $3x-2$, the two factors $x+3$ and $x+5$, which algebraically have no common factor, will have the common factor 2, whenever we give x the value of any odd number.

208. The principle explained in Art. 200 can often be employed to shorten the work of finding the H. C. F.

Ex. 1. Find the H.C.F. of x^3+6x^2-8x-7 and $x^3+8x^2+10x+21$.

The common factor must be a factor of

$3(x^3+6x^2-8x-7)+(x^3+8x^2+10x+21)$,

i.e. of $4x^3+26x^2-14x$, or $2x(2x^2+13x-7)$,

i.e. of $2x(2x-1)(x+7)$.

Now $2x$ is not a common factor, nor is $2x-1$, as is evident.

$\therefore$ $x+7$ must be the H. C. F. if there is one.

Ex. 2. Find the H.C.F. of $2x^3-5x+6$ and $4x^3+x^2-12x+4$

The common factor must be a factor of

$4x^3+x^2-12x+4-2(2x^3-5x+6)$,

i.e. of x^2-2x-8, or $(x+2)(x-4)$.

Now $x-4$ is not a common factor, for 4 will not divide exactly both 6 and 4.

$\therefore$ $x+2$ must be the H. C. F. if there is one.

II. LOWEST COMMON MULTIPLE.

209. When one quantity *contains* another, as a divisor without remainder, it is said to be a **Multiple** of it ; and a **Common Multiple** of two or more quantities is one that contains each of them without remainder.

Thus, $6x^3y$ is a *Common Multiple* of $2x^2$, $3xy$, $6x^3$, &c., and any quantity is a *multiple* of any of its measures.

210. The **Lowest Common Multiple** (briefly L. C. M.) of two or more expressions is the lowest expression that can be formed, so as thus to contain each of them.

Thus, $6x^3y$ is the L. C. M. of $2x^2$, $3xy$ and $6x^3$.

211. The term *Least Common Multiple*, as *Greatest Common Measure* is objected to in Algebra, as being inappropriate, for reasons similar to those said in Art. 191. It would be better to speak of *Lowest Common Multiple*, and we shall use this expression.

212. To find, by *inspection*, the L. C. M. of two or more simple expressions.

Rule. *Find by Arithmetic, the* L. C. M. *of the numerical coefficients ; after this number write the highest powers of all the letters that occur in the given expressions.*

Ex. 1. Find the L. C. M. of $4a^2b^2c^2$ and $6ab^3c^4$.

Here, the L. C. M. of the numerical coefficients 4 and 6 is 12. The letters which occur are a, b, and c. The highest power of a is a^2, of b is b^3 and of c is c^4.

$\therefore$ the L. C. M. required $=12a^2b^3c^4$.

Ex. 2. Find the L. C. M. of $4a^3b^2c$, $6ab^3c^2d^2$ and $18a^2bc^3$.

Here, the L. C. M. of the numerical coefficients is 36. The letters which occur are a, b, c and d; and the indices of their highest powers are respectively 3, 3, 3 and 2.

Hence the required L. C. M. $= 36a^3b^3c^3d^2$.

Exercise LXXXIII.

Find the L. C. M. of :—

1. $4a^2bc$ and $6ab^2c$.
2. $9x^3y$ and $12xy^3$.
3. $8a^2x^2y^3$ and $12b^2x^3y^2$.
4. $20xyz$, $5ax^2y$ and $10a^2y^2z^2$.
5. a^4b^2, b^3c and a^2c^3.
6. $4a^2b$, $2ab^2$ and $6bc^2$.
7. $2b^2x$, $6abx^2y$ and $3a^2cx$.
8. $8a^4$, $10a^3b$ and $12a^2b^2$.
9. a^6, $5a^4b$, $10a^3b^2$, $10a^2b^3$, $5ab^4$ and b^6.
10. $9x^2$, $6ax$, $8a^2$, $36x^3$, $3ax^2$, $50a^2x$ and $24a^3$.
11. $12a^3b^2c^4x^2$, $28a^6bc^8xy^2$ and $77ab^6c^2x^7y$.
12. $77a^2b^3c^4x^6$, $91a^6b^7c^2x^9$ and $143a^3b^2c^8x^4$.

213. With the aid of the preceding Article, we may sometimes find by *inspection* the L. C. M. of two or more compound expressions, if it happens to be easy to separate them into their component factors.

Ex. 1. Find the L. C. M. of a^2b+ab^2 and $a^4b^2-a^2b^4$.

We have $a^2b+ab^2=ab(a+b)$,

and $a^4b^2-a^2b^4=a^2b^2(a^2-b^2)=a^2b^2(a+b)(a-b)$.

Here, the highest power of a is a^2, of b is b^2, of $a+b$ is $a+b$ and of $a-b$ is $a-b$.

Hence the L. C. M. $=a^2b^2(a+b)(a-b)=a^2b^2(a^2-b^2)$.

Ex. 2. Find the L. C. M. of $2a^2(a+x)$, $4ax(a-x)$ and $6x^2(a+x)$.

Here, the L. C. M. of the simple factors $2a^2$, $4ax$ and $6x^2$ is $12a^2x^2$; that of the compound factors is $(a+x)(a-x)=a^2-x^2$.

Hence, reqd. L. C. M. $=12a^2x^2(a^2-x^2)$.

Ex. 3. Find the L. C. M. of $3(x^2+4x-5)$, $6(x^2+2x-15)$ and x^2-4x+3.

We have $3(x^2+4x-5)=3(x-1)(x+5)$,

$6(x^2+2x-15)=2.3(x-3)(x+5)$,

and $x^2-4x+3=(x-1)(x-3)$.

Select the highest power of each factor,

$\therefore$ L. C. M. $=2.3(x-1)(x-3)(x+5)=6(x^3+x^2-17x+15)$.

Exercise LXXXIV.

Find the L. C. M. of :—

1. axy and $a(xy-y^2)$.
2. $ab+ad$ and $ab-ad$.
3. $2(a+b)$ and $3(a^2-b^2)$.
4. $4(a^2-a)$ and $6(a^2+a)$.
5. a^3+b^3 and $a+b$.
6. x^2-x-20 and x^2+x-12.
7. x^2-3x-4 and x^2-x-12.
8. a^2-1, a^2+1 and a^4-1.
9. $6(x^2+xy)$, $8(xy-y^2)$ and $10(x^2-y^2)$.
10. $4(a^3-a^2b)$, $12(ab^2+b^3)$ and $8(a^3-ab^2)$.
11. $6(x^2y+xy^2)$, $9(x^3-xy^2)$ and $4(y^3+xy^2)$.
12. x^2+x-30, x^2-x-30 and $x^2-11x+30$.
13. a^2b-ab^2, $a^2+ab-2b^2$ and $a^2-ab-2b^2$.
14. $x^2-12x+35$, $x^2-10x+21$ and $x^2-8x+15$.
15. a^2+5a+4, a^2-a+1 and a^3+1.
16. xy, $x-y$ and y^3-x^2y. (C. E. 1866 and B.M. 1893.)
17. a^2-x^2, $a^2+3ax+2x^2$ and a^2-4x^2.
18. x^2-9, x^3-27, $x-3$ and x^2+3x+9.
19. $(3a^2-3ab)^2$, $18(a^3b^2-ab^4)$ and $24(a^3b^3-b^6)$.
20. $18(a+b)^2(a^3-b^3)$, $24(a-b)^2(a^3+b^3)$ and $36(a^2-b^2)^2$.
21. $5x(x^2+2x+1)$, $10x^2(x^2-1)$, and $15(x+1)(x^2-2x+1)$.
22. $2(x-2)^2$, $2x^2-8$, x^3+2x and $2x^2-4x$. (C. E. 1871).
23. x^3+a^3, x^3-a^3, x^2+ax+a^2 and x^2+a^2. (A. E. 1889).
24. x^2-1, x^2+1, $(x-1)^2$, $(x+1)^2$, x^3-1 and x^3+1. (C. E. 1885).
25. $6x^2-x-1$, $3x^2+7x+2$ and $2x^2+3x-2$. (C. E. 1869).
26. $a^2+6ab+5b^2$, $a^3-a^2b-ab^2+b^3$ and $a^2+5ab-6b^2$. (B. M. 1885).
27. $x^3+x^2y+xy^2+y^3$ and $x^3-x^2y+xy^2-y^3$. (C. E. 1870).
28. $1+4x+4x^2-16x^4$ and $1+2x-8x^3-16x^4$. (C. E 1874 & P. E. 1888).
29. $9x^4-28x^2+3$, $27x^4-12x^2+1$, $27x^4+6x^2-1$ and x^4-6x^2+9. (C. E. 1886).
30. x^3-3x^2+3x-1, x^3-x^2-x+1 and x^4-2x^3+2x-1. (M. M. 1890).
31. x^2-4a^2, $x^3+2ax^2+4a^2x+8a^3$ and $x^3-2ax^2+4a^2x-8a^3$.
32. $x^2-(a+b)x+ab$, $x^2+3ax-3ab-b^2$ and $x^2-(2a+b)x-ab-3a^2$.

214. To find the L. C. M. of two compound expressions which cannot easily be resolved into factors

Let A and B represent the two expressions, and D their H. C. F. found according to the Rule of Art. 196. Since each of the expressions A and B is exactly divisible by D, let them be divided by D and let a and b denote the quotients respectively. Then

$$A = aD \text{ and } B = bD.$$

Now it is clear that a and b have no common factor. Therefore the lowest expression which contains a and b is ab, and therefore the lowest expression which contains aD and bD is abD, which is consequently the L. C. M. required of A and B.

Thus, the L. C. M. $= abD = \dfrac{aD \times bD}{D} = \dfrac{A \times B}{D}$.

Hence, we obtain the following Rule :—

Rule. *Divide the product of the two expressions by their* H.C.F.; *or, which is more simple in practice, divide either of them by their* H.C.F. *and multiply the quotient by the other.*

215. Since L. C. M. of two expressions $= \dfrac{\text{their product}}{\text{H.C.F.}}$,

$\therefore$ **L.C.M. × H.C.F. = Product of the two expressions**

Ex 1. Find the L. C. M. of $9x^3 - x - 2$ and $3x^3 - 10x^2 - 7x - 4$.

The H.C F. of the two expressions will be found to be $3x^2 + 2x + 1$.

By division, we shall find that

$(9x^3 - x - 2) \div (3x^2 + 2x + 1) = 3x - 2$,

and $(3x^3 - 10x^2 - 7x - 4) \div (3x^2 + 2x + 1) = x - 4$.

$\therefore$ L. C. M. required $= (3x - 2)\ (x - 4)\ (3x^2 + 2x + 1)$.

216. To find the L. C. M. of three or more expressions which cannot easily be resolved into factors.

Let A, B and C be the proposed expressions, and let M be the L. C. M. of A and B; then the L. C. M. of M and C will be the L. C. M. of A, B and C.

For, since M is the L. C. M. of A and B, therefore M contains all the factors that occur in A and B, raised each to the highest power it has in any of them, and nothing more. Hence the L. C. M. of M and C must contain all the factors in A, B and C, raised each to the highest power it has in any of them and nothing more, and is therefore the L. C. M. of A, B and C.

Similarly, for four or more expressions.

Hence, we obtain the following Rule :—

Rule. *Find the* L.C.M. *of any two of the expressions, and then the* L.C.M. *of this answer and the third and so on. The last result so obtained will be the required* L.C.M.

Ex. 2. Find the L. C. M. of $2x^2+xy-6y^2$, $3x^2+8xy+4y^2$ and $6x^2-5xy-6y^2$.

First, find the L. C. M. of $2x^2+xy-6y^2$ and $3x^2+8xy+4y^2$.

Their H. C. F. $=x+2y$.

Hence, their L. C. M. $=(2x-3y)(3x+2y)(x+2y)$

$=6x^3+7x^2y-16xy^2-12y^3$.

Next, to find the L. C. M. of this expression and $6x^2-5xy-6y^2$.

Their H. C. F. $=6x^2-5xy-6y^2$.

$\therefore$ L. C. M. required $=6x^3+7x^2y-16xy^2-12y^3$.

This question may also be easily worked out by resolving the expressions into factors. Thus,

$$2x^2+xy-6y^2=(2x-3y)(x+2y),$$
$$3x^2+8xy+4y^2=(3x+2y)(x+2y),$$
$$\text{and } 6x^2-5xy-6y^2=(3x+2y)(2x-3y).$$

Hence, the L. C. M. required $=(2x-3y)(3x+2y)(x+2y)$.

Exercise LXXXV.

Find the L. C. M. of :—

1. $3x^4-x^3+3x-1$ and $4x^4-x^3+4x-1$.
2. $x^3+9x^2+23x+15$ and $x^3+10x^2+33x+36$.
3. $a^3+2a^2b-ab^2-2b^3$ and $a^3-2a^2b-ab^2+2b^3$.
4. $3x^2-10ax+7a^2$ and $x^3-5ax^2+7a^2x-3a^3$. (C. E. 1880).
5. $6x^3+7x^2-9x+2$ and $8x^4+6x^3-15x^2+9x-2$. (M. M. 1890).
6. $6x^4+7x^3-27x^2+17x-3$ and $3x^4+17x^3+27x^2+7x-6$. (M. M. 1888).
7. $x^5-5x^3+x^2+4x-4$ and $x^4+x^3-6x^2-4x+8$. (C. E. 1868).
8. $x^3-6x^2+11x-6$, $x^3-9x^2+26x-24$ and $x^3-8x^2+19x-12$.
9. x^4+7x^2+16, x^3+3x+4 and x^3+3x-4. (B. M. 1892).
10. $x^3-7x^2-80x+576$, $3x^2-14x-80$ and $3x^2+17x-90$. (C. E. 1882).
11. $x^3+7x^2+8x-16$, $x^3-13x+12$ and $3x^3+13x^2-16$.

12. $x^3-2x^2-19x+20$, $x^3+2x^2-23x-60$ and $x^4+7x^3-4x^2-52x+48$. (B. M. 1891).

13. $x^6+x^4-4x^3+2x^2+6x-9$, x^4-x^2+6x-9 and $x^4+2x^3-5x^2-6x+9$. (B. M. 1886).

14. The H. C. F. of two expressions is $x-1$, and their L. C. M. is $x^6+4x^4+6x^3+x^2-6x-6$. One of the expressions is x^3+x^2-2. Find the other.

15. The H. C. F. of two expressions is $a-1$ and their L. C. M. is $a^8-4a^6+6a^4-4a^3+2a-1$. One of the expressions is $a^4-3a^3+2a^2+a-1$. Find the other.

217. Every common multiple of two or more expressions is a multiple of their L. C. M.

Let m be any common multiple of the expressions A and B, and M their L. C. M.; and let m contain M (if possible) r times with remainder s, which will of course be less than the divisor M; hence we should have

$$m=r\mathrm{M}+s, \text{ and, } \therefore s=m-r\mathrm{M}:$$

but since A and B measure both m and M, they would also measure $m-r\mathrm{M}$, or s (Art. 200); *i. e.* s, which is less than M, would be a common multiple of A and B, contrary to our supposition that M was their L. C. M. Hence m will contain M with *no* remainder, and will therefore be a *multiple* of M.

III. REMAINDER THEOREM.

218. *If px^2+qx+r is divided by $x-a$ until the remainder is independent of x, that remainder will be pa^2+qa+r.*

First Method. Performing the actual division,

$$\begin{array}{l|l|l} x-a & px^2+qx+r & px+(pa+q) \\ & \underline{px^2-pax} & \\ & \quad (pa+q)x+r & \\ & \quad \underline{(pa+q)x-(pa+q)a} & \\ & \qquad\qquad pa^2+qa+r & \end{array}$$

This proves the theorem.

Here, we observe that the remainder is of the same form as the dividend with a in the place of x.

Second Method. Let Q denote the quotient, and R the remainder, which is independent of x, when px^2+qx+r is divided by $x-a$.

Then $px^2+qx+r=(x-a)\times Q+R$..................(1)

Since the above is *identically* true, and R does not contain x, it remains the same, whatever value we assign to x.

Let $x=a$. Then the equation (1) becomes

$$pa^2+qa+r=R, \text{ for } (x-a)Q=(a-a)Q=0.$$

This proves the theorem.

(i) When $3x^2-5x+7$ is divided by $x-2$,

the remainder $=3\times 2^2-5\times 2+7$

$=12-10+7=9.$

(ii) When $5x^2-9x+6$ is divided by $x+4$,

the remainder $=5(-4)^2-9(-4)+6$,

$=80+36+6=122.$

219. *If px^3+qx^2+rx+s is divided by $x-a$ until the remainder is independent of x, that remainder will be*

$$pa^3+qa^2+ra+s.$$

First Method. Performing the actual division,

$$\begin{array}{l|l|l}
x-a & px^3+qx^2+rx+s & px^2+(pa+q)x+(pa^2+qa+r) \\
 & \underline{px^3-pax^2} & \\
 & (pa+q)x^2+rx & \\
 & \underline{(pa+q)x^2-(pa+q)ax} & \\
 & \quad (pa^2+qa+r)x+s & \\
 & \quad \underline{(pa^2+qa+r)x-(pa^2+qa+r)a} & \\
 & \qquad\qquad pa^3+qa^2+ra+s &
\end{array}$$

This proves the theorem.

• As before, the remainder is of the same form as the dividend with a in the place of x.

Second Method. Let Q denote the quotient and R the remainder, which is independent of x, when px^3+qx^2+rx+s is divided by $x-a$.

Then $px^3+qx^2+rx+s=(x-a)\times Q+R$..................(1)

Since the above is *identically* true, and R does not contain x, it remains the same, whatever value we assign to x.

Let $x=a$. Then the equation (1) becomes

$$pa^3+qa^2+ra+s=R, \text{ for } (x-a)Q=(a-a)Q=0.$$

This proves the theorem.

(i) When $2x^3+7x^2-9x+2$ is divided by $x-2$,

the remainder $=2\times 2^3+7\times 2^2-9\times 2+2$

$=16+28-18+2=28.$

220. *If any expression in x vanishes identically when $x=a$, then will the expression be exactly divisible by $x-a$.*

Thus, when $3x^3+7x^2-2x+12$ is divided by $x+3$,

the remainder $=3(-3)^3+7(-3)^2-2(-3)+12$

$=-81+63+6+12=0.$

$\therefore$ $3x^3+7x^2-2x+12$ is exactly divisible by $x+3$.

Ex. 1. For what value of a is $x^2-(a+5)x+14$ divisible by $x-2$ without remainder?

When the division is performed the remainder, (Art. 218)

$=2^2-(a+5)\times 2+14=18-2(a+5).$

$\therefore$ the required value of a is obtained by equating this remainder to zero, in which case

$18-2(a+5)=0$; $\therefore$ $2(a+5)=18$;

$\therefore$ $a+5=9$; and $\therefore$ $a=4$.

Ex. 2. For what value of a is $3x^3-7x^2-ax+9$ divisible by $x-3$ without remainder?

We have $3\times 3^3-7\times 3^2-3a+9=0$, (Art. 219)

$\therefore$ $81-63-3a+9=0$; $\therefore$ $27-3a=0$;

$\therefore$ $3a=27$; and $\therefore$ $a=9$.

Exercise LXXXVI.

Without actual division, find the remainder when

1. $x^2-7x+12$ is divided by $x-5$.
2. $4x^2+7x+15$ is divided by $x+3$.
3. $8x^2+13x-5$ is divided by $x-2$.
4. $2x^3-13x^2+17x-16$ is divided by $x-5$.
5. $4x^3-5x^2+11x-19$ is divided by $x+9$.
6. $2x^3-x^2+3x-11$ is divided by $2x-3$.
7. $7x^3-24x^2+58x-25$ is divided by $7x-3$.
8. $4x^4-33x^2+8x-5$ is divided by $x+3$.
9. $4x^4-3x^2+8$ is divided by x^2-3.

10. For what value of a is $x^2-ax+120$ divisible by $x-15$ without remainder?

11. For what value of d is x^6+24x^3+d divisible by x^3+12 without remainder?

12. For what value of a is $x^3-(a+6)x^2+(6a+5)x+30$ divisible by $x-a$ without remainder?

Without actual division show that the following expressions are exactly divisible.—

13. $2x^2+3x-2$ by $x+2$.

14. $5x^2-7x-6$ by $x-2$

15. $3a^2-8a+4$ by $3a-2$.

16. x^3-3x-2 by $x+1$.

17. $3a^3-2a^2b-13ab^2+10b^3$ by $a-2b$.

18. $12x^4-17x^3+21x^2-31x+14$ by $3x-2$.

Find for what value of a the following are exactly divisible :—

19. $x^3+3x^2-ax+35$ by $x+7$.

20. x^3-4x^2+6x-a by $x-2$.

21 $x^2+px+40$ and x^2-x-20 have a common factor, find the possible values of p.

22 If $x+a$ be the H.C.F. of x^2+px+q and $x^2+p'x+q'$, shew that $(p-p')a=q-q'$.

23. If x^2+qx+1 and x^3+px^2+qx+1 have a common measure of the form $x+a$, shew that

$$(p-1)^2-q(p-1)+1=0.$$

24. If $x+b$ be the H.C.F. of $x^2+ax+ab$ and $x^2+cx+bc$, then L.C.M will be $x^3+(a+c)x^2+acx$.

25. Find the relation between b and c, so that x^3+bx+c and x^3+cx+b may have a common divisor. (P.E. 1891).

26. If h be the H.C.D. and l the L.C.M. of two quantities x and y, and if $x+y=h+l$, prove that $x^3+y^3=h^3+l^3$. (P.E. 1891).

27. Find the condition that $x^3+(p+q)x+a$ may be divisible by $x+p+q$. (P.E. 1895).

88. If $x+p$ be the H.C.F. of ax^2+bx+c and cx^2+bx+a, prove that either $a+c=b$ or $a+b+c=0$

REVISION PAPERS II.

Paper I.

1. Divide the product of $y^3-12y+16$ and $y^3-12y-16$ by y^2-16. (B. M. 1872).

2. Resolve into elementary factors :—

(i) $8x^3+729y^9$. (B. M. 1890). (ii) x^4+324. (B. M. 1895).

(iii) $56x^2+5xy-99y^2$. (B. M. 1895).

3. Simplify

$$(b+c-a)(c+a-b)(a+b-c)+(a+b+c)(a^2+b^2+c^2-2bc-2ca-2ab)$$

4. If $a+b=1$, prove that $(a^2-b^2)^2=a^3+b^3-ab$. (B. M. 1889).

5. Simplify $24\{x-\frac{1}{2}(x-1)\}\{x-\frac{2}{3}(x-2)\}\{x-\frac{3}{4}(x-1\frac{1}{3})\}$; and subtract the result from $\dfrac{(x^2+7x+12)(x^2-x-6)}{x-3}$. (B. M. 1886).

6. Shew that $(4x^2-5x+7)^2-(5x^2+14x+2)^2$ is divisible by x^2+x+1, and find the quotient. (M. M. 1893).

7. Find the G. C. M. of

$6x^6-4x^4-11x^3-3x^2-3x-1$ and $4x^4+2x^3-18x^2+3x-5$.

(M. M. 1893).

8. Extract the square root of

$x^{4m+2}+6x^{3m+1}-10x^{2m+1}y^{m-2}+9x^{2m}-30x^my^{m-2}+25y^{2m-4}$.

(M. M. 1893).

9. Solve the following equations :—

(i) $\dfrac{x+10}{3}-\frac{3}{4}(3x-4)+\dfrac{(3x-2)(2x-3)}{6}=x^2-\frac{8}{15}$.

(ii) $\frac{1}{3}(x-3)+\frac{1}{4}(x-8)+\frac{1}{5}(x-4)=2\frac{7}{15}$. (C. E. 1901).

10. A boy bought a number of oranges for 2 Rupees. Had he bought 8 more for the same money, he would have paid 4 pies less for each. How many did he buy? (M. M. 1893).

Paper II.

1. Divide the product of $ab(x^2+1)+(a^2+b^2)x$ and x^3+1 by that of $x+1$ and $ax+b$. (M. M. 1894).

2. Multiply $x^2+(3a-2b)x-6ab$ by $x^2+(2a-3b)x-6ab$.

(M. M. 1899).

3. Find the H. C. F. of $3x^4-2x^3+2x^2+8$ and $x^3-7x^2+12x-10$. (M. M. 1898).

4. Extract the square root of

$b^2(a+4b)^2+3(3a^2-2ab+b^2)(a^2+3b^2)$. (M. M. 1898).

5. Solve the equations :—

(i) $\frac{1}{4}(x-4)+\frac{3}{7}(2x-7)-\frac{1}{5}(1+5x)=4(1-x)$. (M. M. 1897).

(ii) $\frac{1}{5}(2x-3)+\frac{9}{10}(3x+8)=5x+\frac{1}{3}(4x-19)$. (M. M. 1899).

6. Simplify $2(z^3+x^3)-[(x+y)(xy-x^2-y^2)-\{2(x+y+z)\times(yz+zx+xy-x^2-y^2-z^2)-(x-y)(x^2+xy+y^2)\}]$.

(M. M. 1894).

7. Find the L. C. M of $x-a$, x^2-a^2, x^3-a^3, $(x^3+a^3)^2$.

(M. M. 1894).

8. Shew that $(4x^2-8x-1)^2-(2x^2-5x+7)^2$ is divisible by $2x^2-3x-8$; and express the quotient as the product of two factors. (M. M. 1895).

9. If a, b, c be three quantities whose sum is zero, show that $a^4+b^4+c^4=2(a^2b^2+b^2c^2+c^2a^2)$ (M. M. 1892).

10. The number of months in the age of a man, on his birth day in the year 1875, was exactly half of the number denoting the year in which he was born. In what year was he born? (A. E. 1898).

Paper III.

1. Divide $(x^2-1)^4-3(x^2-1)^2+1$ by x^4-3x^2+1. (M. M. 1898).

2. Resolve into factors :—

(i) $16x^8-1$. (M. M. 1897). (ii) x^3+4x^2+4x. (A. E. 1895).

(iii) $(2x^2-5x+3)(2x^2-5x+4)-2$.

3. Find the H. C. F. of

$6x^4+2x^3+19x^2+8x+21$ and $4x^4-2x^3+10x^2+x+15$.

(M. M. 1896).

4. Find the L. C. M. of

x^3+a^3, x^3-a^3, $x^4+a^2x^2+a^4$ and x^2-ax+a^2. (M. M. 1896).

5. Extract the square root of

$x^4(x-a)^2+4a^2x^4+a^4(2x+a)^2-2a^3x^2(x+a)$. (M. M. 1894).

6. If $x^2=ab+bc+ca$, show that $(a^2+x^2)(b^2+x^2)(c^2+x^2)$ is a perfect square. (M. M. 1889).

7. Divide the difference of $(x^2 - bx + b^2)(x + a - b)$ and $(x^2 - ax + a^2)(x - a + b)$ by $a - b$. (M. M. 1872).

8. Find what term is wanting to make the following expression a complete square :—

$a^2x^4 + 64b^2 - 4(ax^2 + 8b)(a - b)x$ (M. M. 1875).

9 Solve the equations :—

(i) $\dfrac{3x+1}{4} - 2(6 - x) = \dfrac{5x-4}{7} - \dfrac{x-2}{3}$

(ii) $\dfrac{2x+a}{b} - \dfrac{x-b}{a} = \dfrac{3ax + (a-b)^2}{ab}$

10. The gross income of a certain man was £40 more in the second of two particular years than in the first, but in consequence of the income-tax rising from 4*d.* in the £ to 6*d.* in the £ in the second year his net income after paying the tax was unaltered Find his income in each year (B. M 1891).

Paper IV.

1 From $a(b+c)^2 + b(c+a)^2 + c(a+b)^2$, subtract
$(a+b)(a-c)(b-c) + (b+c)(a-b)(a-c) - (a+c)(a-b)(b-c)$.
(M M. 1893)

2 Multiply together the expressions $1 + ax + \frac{1}{2}a(a-1)x^2 + \frac{1}{6}a(a-1)(a-2)x^3$ and $1 + bx + \frac{1}{2}b(b-1)x^2 + \frac{1}{6}b(b-1)(b-2)x^3$ as far as the term involving x^3 and resolve into factors the coefficient of x^2 in the product. (B. M 1897).

3. Find what quantity not involving higher powers of x beyond the second should be added to $x^8 - 3x^7 - 5x^6 + 2x^4 + 5x^3 + 4x^2 + 1$ to make it exactly divisible by $x^3 + 2x - 1$ (B. M 1897).

4. Resolve into factors :—

(i) $x^4 - 11x^2y^2 + y^4$. (B. M. 1897).

(ii) $4(ac + bd)^2 - (a^2 - b^2 + c^2 - d^2)^2$. (M. M 1897).

5. Find the expression of lowest dimensions which is exactly divisible by $a^2b - b(b-c)^2$, $ac^2 - a(a-b)^2$ and $(a+c)^2c - b^2c$. (B. M. 1890)

6. Find k when $(x-a)(x-3a)(x+a)(x+3a) + k$ is a perfect square. (B. M. 1890).

7. Given $ax + by = m$, $bx - ay = n$, $a^2 + b^2 = 1$;
shew that $x^2 + y^2 = m^2 + n^2$. (B.M. 1890).

8 Extract the square root of

(i) $(2x+1)(2x+3)(2x+5)(2x+7)+16$. (B.M. 1890).

(ii) $x^6+2x^4+4x^3+x^2+4x-\frac{4}{x^2}-\frac{8}{x^3}+\frac{4}{x^6}$. (M.M. 1895).

9. Solve the equation

$\cdot 15x+1\cdot 2-\cdot 875x+\cdot 375-\cdot 0625x=0$ (P.E. 1897).

10. Two men leave two places **A** and **B**, distant d miles from each other, and travel a and b miles a day respectively in the same straight line **AB**. What is their distance apart at the end of t days, and after what time will they come together? (P.E. 1895).

Paper V

1 Find the difference between $(1+x)^3+(1+x)^2y+(1+x)y^2+y^3$ and $3x(x+1)+y(y+1)+2xy+1$, and show by what expression this difference must be multiplied that the product may be y^4-x^4

(A.E. 1899).

2 Simplify $(a-b+c)^3+(a+b-c)^3+6a\{a^2-(b-c)^2\}$ (A.E. 1901).

3 Find the H. C. F. of $x^5-4x^3-x^2+2x+2$ and x^3-x^2-2x+2 and find such a value of x as will make both the expressions vanish. (A.E. 1899)

4 Find the L. C. M. of $4x^2-6yz-(9y^2+z^2)$,

$9y^2+4xz-(4x^2+z^2)$ and $z^2-12xy-(4x^2+9y^2)$ (B.M. 1896).

5. Resolve into factors :—

(i) $4x^2-9y^2-6x-9y$ (B.M. 1902). (ii) $x^4-x^2y^2+16y^4$

(iii) x^4-10x^2+9 (iv) $512(x-\frac{1}{8})^3-(8ax-a)^3$

6 Find the square root of

(i) $a^2x^6+6abx^4-2acx^3+9b^2x^2-6bcx+c^2$. (B.M. 1902).

(ii) $a^3\left(\frac{a^3}{9}-\frac{10}{3b^3}\right)+b^3\left(2+\frac{9b^3}{a^6}\right)-\frac{30}{a^3}+\frac{25}{b^6}$ (B.M. 1895).

7 Find the cube root of

(i) $\frac{a^3}{b^3}-\frac{b^3}{a^3}-3\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)+5$. (B.M. 1892.)

(ii) $x^3+\frac{8}{x^3}-12x^2-\frac{48}{x^2}+54x+\frac{108}{x}-112$. (B.M. 1893).

8. Solve the equation :—

$$\frac{1\cdot05x+10}{50}+\frac{1\cdot35x-2}{20}-\frac{1\cdot5x-18}{10}+\frac{1\cdot5x-3}{15}=1\cdot854.$$ (B.M. 1902).

9. Shew that if a number of two digits is four times the sum of its digits, the number formed by interchanging the digits is seven times their sum. (B.M. 1889).

10. A number has three digits which increase by 1 from left to right. The quotient of the number divided by the sum of the digits is 26. What is the number? (A.F. 1901).

Paper VI.

1. If a number is equal to the sum of two perfect squares, shew by an algebraical relation that the square of the number is itself the sum of two other perfect squares. (B.M. 1896).

(a) Express $(34)^2$ as the sum of two perfect squares. (B.M. 1896).

2. Resolve into factors :—

(i) $x^2+6x-187$. (B.M. 1901). (ii) x^4-5x^2+4.

(iii) $a^2(a+b-c)^2-c^2(b+c-a)^2$. (A.F. 1899).

3. Find the L. C. M. of $4x^3-20x^2+17x-4$, $2x^3-15x^2+31x-12$ and $4x^3-16x^2+13x-3$. (B.M. 1902).

4. Divide $(a+1)^2x^3+(a+1)x^2+a^4(a-1)x-a^6$ by $(a+1)x-a^2$, and find the value of the quotient when $a=-\frac{1}{2}$ and $x=-21$

(M.M 1899)

5. Extract the square root of

(i) $(a^2+3b^2)^2+10ab(a+b)(a-3b)+33a^2b^2$. (B.M 1899).

(ii) $a^2(x^6+2x^3+1)+2ab(x^5+x^4+x^2+x)+b^2(x^4+2x^3+x^2)$.

(P.E. 1895)

6. Find an expression containing no higher power of x than the first, which added to $x^4+6x^3+13x^2+6x+1$ will make it a complete square. (B.M. 1896 & P.E. 1899).

7. Exhibit $(x^2+y^2)(a^2+b^2)$ as the sum of two squares.

(P.E. 1894)

8. Find the G. C. M. of

$x^4+5x^3-36x^2+50x+48$ and $x^4+x^3-12x^2-2x+80$.

(B.M. 1900)

9 Solve the equation :—

$$\tfrac{1}{5}x-\{3x+6-\tfrac{4}{5}(x+10)\}=2\tfrac{1}{4}-11(9-\tfrac{1}{12}x).$$ (B. M. 1901).

10. A certain number of two digits is equal to seven times the sum of the digits. If the digit in the units' place be decreased by 2 and that in the tens' place by 1, and the number thus found be divided by the sum of its digits, the quotient is 10. Find the number. (B. M. 1896).

Paper VII.

1. Substitute $y+3$ for x in $x^4-x^3+2x^2-3$ and arrange the result. (B. M. 1868).

2. Divide $a^3(1-x)+ab(a-b)(x+y)+b^3(1+y)$ by $a(1-x)+b(1+y)$. (M. M. 1898).

3. Find the H. C. F. of $6x^5+35x^4+59x^3+19x^2-17x-6$ and $6x^5-5x^4-41x^3+71x^2-37x+6$. (B. M. 1897).

4. Extract the square root of

(i) $a^2x^2+6ac+\dfrac{12bc}{x}+4b(ax+b)+\dfrac{9c^2}{x^2}$. (B. M. 1890).

(ii) $(a+b)(a+b+c)(a+b+2c)(a+b+3c)+c^4$. (B.M. 1897).

5. Find the cube root of

(i) $x^6+\dfrac{27}{x^6}-6\left(x^4+\dfrac{9}{x^4}\right)+21\left(x^2+\dfrac{3}{x^2}\right)-44$. (B.M. 1902)

(ii) $8+36x+42x^2-9x^3-21x^4+9x^5-x^6$. (B.M. 1901).

6. Resolve into factors :—

(i) $12x^2+x-35$. (B. M. 1900). (ii) $8x^2+6x-27$. (A. E. 1896).

(iii) $81a^4+64b^4$. (C. E. 1898). (iv) $x^3-3a^2x+2a^3$.

7. Transform $(x^2+y^2+z^2+2xy)^2-2(x+y)^2z^2$ into the sum of two perfect squares. (M. M. 1879).

8. Find the value of $(ma-nb)(mb-nc)(mc-na)+(na-mb)\times(nb-mc)(nc-ma)$, when $a-b=0$. (M. M. 1883).

9. Solve the equation :—

$$\frac{21}{4}\left(\frac{2x}{3}-\frac{5}{18}\right)+\frac{7x-3\frac{3}{4}}{12}=2\frac{19}{144}-\frac{14-15x}{3}.$$ (B. M. 1900).

10. A person bought a certain number of eggs, half of them at 2 a penny and half at 3 a penny. He sold them again at 5 for $2d.$ and lost a penny by the transaction. What was the number of eggs? (C. E. 1900).

Paper VIII.

1. Simplify $(x-y+z)(x+y-z)-(x+y+z)(x-y-z)-4xyz$.
(A. E. 1891).

2. Find the H. C. F. of $6x^4-2x^3+9x^2+9x-4$ and $9x^4+80x^2-9$; and find such a value of x as will make both these expressions vanish. (B. M. 1895).

3. Find the square root of

(i) $x^6-\frac{3}{2}x^4+\frac{2}{3}x^3+\frac{9}{16}x^2-\frac{1}{2}x+\frac{1}{9}$. (B. M. 1901).

(ii) $\dfrac{4x^2}{9a^2}+\dfrac{4x^2}{b^2}+\dfrac{9a^2}{4x^2}+\dfrac{9a^2x^2}{b^4}+\dfrac{9a^2}{b^2}+2$. (M. M. 1892).

4. Find the G. C. M. and L. C. M. of $3x^4+17x^3+27x^2+7x-6$ and $6x^4+7x^3-27x^2+17x-3$. (M. M. 1888).

5. Find the cube root of $8x^9-12x^8+6x^7-37x^6+36x^5-9x^4+54x^3-27x^2-27$. (B. M. 1876-96).

6. A ladder with its foot at a horizontal distance of 20 ft. from a vertical wall, just reaches a point on the wall 30 ft. from the ground; find, by means of squared paper, to the nearest tenth of a foot, the length of the ladder.

7. Divide $(a^2-bc)^3+8b^3c^3$ by a^2+bc. (C. E. 1899).

8. Show by means of a formula that $(ax+by+cz)^3+(cx-by+az)^3$ is divisible by $(a+c)(x+z)$. (B. M. 1887).

9. Solve (i) $\dfrac{3x+\frac{1}{2}}{3}+\dfrac{x+1}{5}=\dfrac{7}{6}$.

(ii) $\dfrac{x-5}{7}+\dfrac{x^2+6}{3}+\dfrac{x^2-x+1}{6}-\dfrac{x^2-2}{2}=x+26$.

10. Find the value of x which will make the expression $x^6-8x^5+11x^3+7x-1789$ exactly divisible by x^2+7x-1. (B. P. E. 1887).

CHAPTER VIII.

ELEMENTARY FRACTIONS.

221. Algebraical Fractions are for the most part precisely similar both in their nature and treatment to common Arithmetical Fractions. We shall have therefore to repeat much of what has been said in Arithmetic; but the Rules which were there *shewn* to be true only in the particular examples given, will here, by the use of letters, which stand for *any* quantities, be proved to be true in *all* cases.

222. A **Fraction** is a quantity which represents a part or parts of a unit or whole.

It consists of two members, the **numerator** and **denominator**, the former placed over the latter with a line between them. Now we have already agreed (Art. 13) that such an expression shall denote that the upper quantity is divided by the lower; and, in accordance with this, it will be seen presently that a fraction does also express the quotient of the numerator divided by the denominator.

223. The numerator and denominator are the **terms** of the fraction. The denominator shews into how many equal parts the unit is divided, and the numerator the number taken of such parts.

Thus, $\frac{a}{b}$ means that the unit is divided into b equal parts, a of which are taken.

224. Every integral quantity may be considered as a fraction whose denominator is 1; thus a is $\frac{a}{1}$.

225. To **multiply** a fraction by an integer, we may either multiply the numerator or divide the denominator by it; and *conversely*, to **divide** a fraction by any integer, we may either divide the numerator or multiply the denominator by it

Thus, $\frac{a}{b}\times x=\frac{ax}{b}$; for in each of the fractions $\frac{a}{b}$, $\frac{ax}{b}$, the unit is divided into b equal parts, and x times as many of them are taken in the latter as in the former; hence the latter fraction is x times the former, that is, $\frac{ax}{b}=\frac{a}{b}\times x$: and, by similar reasoning, $\frac{ax}{b}\div x=\frac{a}{b}$.

Again, $\frac{a}{b}\div x=\frac{a}{bx}$; for in each of the fractions $\frac{a}{b}$, $\frac{a}{bx}$, the same number of parts is taken, but each of the parts in the latter is $\frac{1}{x}$th of each in the former, since the unit in the latter case is divided into x times as many parts as in the former; hence the latter fraction is $\frac{1}{x}$th of the former, that is, $\frac{a}{bx}=\frac{a}{b}\div x$: and, similarly, $\frac{a}{bx}\times x=\frac{a}{b}$.

226. If any quantity be **both** multiplied and divided by the same quantity, its value will, of course, remain unaltered. Hence, if the numerator and denominator of a fraction be **both** multiplied or divided by the same quantity, its value will remain unaltered.

Thus, $\frac{a}{b}=\frac{ax}{bx}=\frac{a^2}{ab}=$&c. and $\frac{a^2b}{a^2bc}=\frac{a}{c}=\frac{ac}{c^2}=$&c.

227. Since $a=\frac{a}{1}$ (Art. 224), and, therefore, a divided by $b=\frac{a}{1}\div b=\frac{a}{b}$ (Art. 225), it follows, as stated in (Art. 222), that a fraction represents the quotient of the numerator by the denominator.

In fact, we may get $\frac{1}{b}$th of a units, (or $a\div b$), by taking $\frac{1}{b}$th part of *each* of the a units, and this is the same as a such parts of one unit, which is expressed by $\frac{a}{b}$ (Art. 223). Hence it is that, in Arithmetic, $\frac{1}{4}$ of Rs. 3 is the same as $\frac{3}{4}$ of Re. 1, &c.

228. To reduce an integer to a fraction with a given denominator, multiply it by the given denominator, and the product will be the numerator of the required fraction.

Thus, a, expressed as a fraction with denominator x, is $\frac{ax}{x}$; or, with denominator $b-c$, is $\frac{ab-ac}{b-c}$.

The truth of this is evident from Art. 226.

229. The signs of all the terms in both the numerator and denominator of a fraction may be changed without altering its value

Thus, $\frac{b^2-2ab-a^2}{3ab-a^2}$ is identical with $\frac{a^2+2ab-b^2}{a^2-3ab}$.

This follows also from Art. 226, as the process is equivalent to that of multiplying both numerator and denominator by -1.

Thus, $\frac{1}{3-x}=\frac{-1}{x-3}=-\frac{1}{x-3}$; $\frac{a-b}{a-c}=\frac{b-a}{c-a}$; and $\frac{a-b}{c-a}=-\frac{a-b}{a-c}$

I. REDUCTION OF FRACTIONS.

230. To reduce a fraction to its lowest terms.

Rule. Divide the numerator and denominator by their H. C. F.

Ex. 1. Reduce to their lowest terms $\frac{15a^3b^2c^4}{25a^2b^3c^2}$ and $\frac{a^2x^2y^2}{a^2xy+axy^2}$.

(i) The H. C. F. of the numerator and denominator is $5a^2b^2c^2$.

$$\therefore \frac{15a^3b^2c^4}{25a^2b^3c^2}=\frac{15a^3b^2c^4\div 5a^2b^2c^2}{25a^2b^3c^2\div 5a^2b^2c^2}=\frac{3ac^2}{5b}.$$

(ii) The denominator $= axy(a+y)$.

The H. C. F. of this and the numerator clearly is axy.

$$\therefore \frac{a^2x^2y^2}{a^2xy+axy^2} = \frac{a^2x^2y^2 \div axy}{axy(a+y) \div axy} = \frac{axy}{a+y}.$$

Note. The operation of dividing out any common factor is called the **cancelling** of that factor.

231. When the numerator and denominator can, on inspection, be resolved into factors, then any common factors can be cancelled out and the fraction will thus be reduced to its lowest terms.

Ex. 2. Reduce to its lowest terms $\frac{x^2+4x+3}{x^2+5x+6}$.

The fraction $= \frac{(x+3)(x+1)}{(x+3)(x+2)} = \frac{x+1}{x+2}$, on dividing both numerator and denominator by the H. C. F. $x+3$.

Ex. 3. Reduce to its lowest terms $\frac{a^3+x^3}{a^2-x^2}$.

The fraction $= \frac{(a+x)(a^2-ax+x^2)}{(a+x)(a-x)} = \frac{a^2-ax+x^2}{a-x}$, on dividing both numerator and denominator by the H. C. F. $a+x$.

Exercise LXXXVII.

Reduce the following fractions to their lowest terms :—

1. $\frac{15a^3b^2cd}{25a^4b^4c^2d}$.
2. $\frac{25a^3b^2c^4d}{35a^2b^3c^6d^2}$.
3. $\frac{143a^2x^3y^6z^7}{91a^7x^2y^8z^3}$.
4. $\frac{18a^4b^2x}{21a^2b^2y}$.
5. $\frac{28a^4b^5c^7}{49a^6b^3c^2}$.
6. $\frac{21a^3b^2c^4x^6y^3}{33a^4b^6c^2x^3y^5}$.
7. $\frac{axy+xy^2}{axy}$.
8. $\frac{cx+x^2}{a^2c+a^2x}$.
9. $\frac{11m^2+22mx}{33(m^2-4x^2)}$.
10. $\frac{14x^2-7xy}{10ax-5ay}$.
11. $\frac{5a^3b-15a^2b^2}{20ab^3+10a^2b^2}$.
12. $\frac{6x^3-18xy^2}{6x^2y-12xy^2}$.
13. $\frac{4m^2n^2}{2m^2n+2mn^2}$.
14. $\frac{3a^2b^2c^2}{a^2bc+ab^2c+abc^2}$.
15. $\frac{9x^2y^3-15xy^4}{12x^2y^2-21xy^3}$.
16. $\frac{abc+9bc-5c^2}{20abdf+18bdf-10cdf}$.
17. $\frac{ac+by+ay+bc}{af+2bx+2ax+bf}$.

18. $\dfrac{x^2-1}{ax+a}$. 19. $\dfrac{x^4-a^4}{x^5-a^2x^3}$. 20. $\dfrac{a^6-b^6}{a^4-b^4}$.

21. $\dfrac{x^3-b^2x}{x^2+2bx+b^2}$. 22. $\dfrac{x^4-1}{x^6-1}$. 23. $\dfrac{a^2-ab+ax-bx}{a^2+ab+ax+bx}$.

24. $\dfrac{x^2-(a-b)x-ab}{x^2-(a+c)x+ac}$. 25. $\dfrac{a^2+2ab+b^2-c^2}{a^2-b^2-2bc-c^2}$.

26. $\dfrac{x^2-4x+3}{x^2-2x-3}$. 27. $\dfrac{x^2+2x-3}{x^2+5x+6}$. 28. $\dfrac{a^2-ab-2b^2}{a^2-3ab+2b^2}$.

29. $\dfrac{6a^2-13ax+6x^2}{10a^2-9ax-9x^2}$. 30. $\dfrac{6a^2+7ax-3x^2}{6a^2+11ax+3x^2}$. 31. $\dfrac{(2a+b)^2-c^2}{4a^2-(b+c)^2}$.

32. $\dfrac{acx^2+(ad-bc)x-bd}{a^2x^2-b^2}$. 33. $\dfrac{x^2-7x+10}{2x^2-x-6}$. 34. $\dfrac{a^3+27}{3a+9}$.

35. $\dfrac{x^4-x^3-x+1}{x^4+x^3-x-1}$ (C.E. 1867). 36. $\dfrac{8xy-3x^2-4y^2}{7x^2y^2-2x^4+4y^4}$ (M. M. 1886).

37. $\dfrac{(a+b)^2-(c+d)^2}{(a+c)^2-(b+d)^2}$. 38. $\dfrac{x^4-9a^2}{x^4-6ax^2+9a^2}$. 39. $\dfrac{(x-y)^2-1}{(x+1)^2-y^2}$.

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232. If no common factor can be found by inspection, the H. C. F. must be obtained by the method of Art. 196.

Ex. 1. Reduce $\dfrac{x^3+x^2+3x-5}{x^2-4x+3}$ to its lowest terms.

The H. C. F. of the numerator and denominator is $x-1$.

Dividing both numerator and denominator by $x-1$,

$$\frac{x^3+x^2+3x-5}{x^2-4x+3}=\frac{(x^3+x^2+3x-5)\div(x-1)}{(x^2-4x+3)\div(x-1)}=\frac{x^2+2x+5}{x-3}.$$

Ex. 2. Reduce $\dfrac{3x^4-14x^3-9x+2}{2x^4-9x^3-14x+3}$ to its lowest terms.

Here, the H. C. F. of the numerator and denominator is x^2-5x+1.

$$\therefore \text{ the fraction}=\frac{(3x^4-14x^3-9x+2)\div(x^2-5x+1)}{(2x^4-9x^3-14x+3)\div(x^2-5x+1)}=\frac{3x^2+x+2}{2x^2+x+3}.$$

Exercise LXXXVIII.

Reduce to their lowest terms :—

1. $\dfrac{x^2+x-12}{x^3-5x^2+7x-3}$.

2. $\dfrac{7x^2-23xy+6y^2}{5x^3-18x^2y+11xy^2-6y^3}$.

3. $\dfrac{x^3-3x+2}{x^3+4x^2-5}$.

4. $\dfrac{5a^5+10a^4x+5a^3x^2}{a^3x+2a^2x^2+2ax^3+x^4}$.

5. $\dfrac{x^4+a^2x^2+a^4}{x^4+ax^3-a^3x-a^4}$.

6. $\dfrac{x^3+3x^2-4}{x^3-1}$.

7. $\dfrac{3a^2x^4-2ax^2-1}{4a^3x^6-2a^2x^4-3ax^2+1}$.

8. $\dfrac{3x^3-2x^2-x}{4x^3-2x^2-3x+1}$ (C.E. 1869).

9. $\dfrac{x^4-4x+3}{2x^5-11x^2-9}$ (M.M. 1885).

10. $\dfrac{x^3-6x^2-37x+210}{x^3+4x^2-47x-210}$ (C.E. 1865).

11. $\dfrac{2x^3+ax^2+4a^2x-7a^3}{x^3-7ax^2+8a^2x-2a^3}$ (C. F. A. 1862).

12. $\dfrac{10x^3+19x^2-9}{25x^3-19x+6}$ (C.E. 1871).

13. $\dfrac{x^3+x^2+x-3}{x^3+3x^2+5x+3}$.

14. $\dfrac{3x^3-27ax^2+78a^2x-72a^3}{2x^3+10ax^2-4a^2x-48a^3}$ (C.E. 1889).

15. $\dfrac{2x^4-x^3-9x^2+13x-5}{7x^3-19x^2+17x-5}$ (C.E. 1870).

16. $\dfrac{2x^4+11x^3+41x^2+99x-45}{2x^3-7x^2-52x+21}$.

17. $\dfrac{x^4-2x^3-25x^2+26x+120}{x^4-4x^3-19x^2+46x+120}$.

18. $\dfrac{x^3-4x^2+9x-10}{x^3+2x^2-3x+20}$.

19. $\dfrac{a^6+11a+12}{a^6+11a^3-54}$ (M.M. 1884).

233. If the numerator be of lower dimensions than the denominator, the fraction may be considered in the light of a **proper fraction** in Arithmetic; if greater, in that of an **improper fraction**.

234. To reduce an improper fraction to a mixed fraction.

Rule. *Divide the numerator by the denominator, as far as the division is possible, and annex to the quotient the remainder for numerator and the divisor for denominator in the form of a fraction.*

Ex. 1. $\frac{25a}{8} = 3a + \frac{a}{8}$.   **Ex. 2.** $\frac{a^2 - 3ab}{a - b} = a - \frac{2ab}{a - b}$.

Ex. 3. $\frac{3x^3 + 2x^2 + 1}{x^2 - x + 4} = 3x + 5 + \frac{-7x - 19}{x^2 - x + 4} = 3x + 5 - \frac{7x + 19}{x^2 - x + 4}$.

235. To reduce a mixed fraction to an improper fraction.

Rule. *Multiply the integral part by the denominator; to this product add the numerator and under this result write the denominator.*

Ex. 1. $3a - \frac{4ab + 3}{2b} = \frac{6ab - (4ab + 3)}{2b} = \frac{2ab - 3}{2b}$

Ex. 2. $x^2 + x + 1 + \frac{2}{x - 1} = \frac{(x^2 + x + 1)(x - 1) + 2}{x - 1}$

$$= \frac{x^3 - 1 + 2}{x - 1} = \frac{x^3 + 1}{x - 1}$$

236. Sometimes it is convenient to express a single fraction as a group of fractions.

Thus, $\frac{4a^2b + 6a^3b^2 - 15b^3}{12a^2b^2} = \frac{4a^2b}{12a^2b^2} + \frac{6a^3b^2}{12a^2b^2} - \frac{15b^3}{12a^2b^2}$

$$= \frac{1}{3b} + \frac{a}{2} - \frac{5b}{4a^2}.$$

Exercise LXXXIX.

Reduce the following to mixed fractions :—

1. $\frac{26x}{7}$.   **2.** $\frac{8a^2 + 5b}{4a}$.   **3.** $\frac{a^2 - b^2}{a}$.   **4.** $\frac{3x^2 + 6x + 5}{x + 4}$.

5. $\frac{a^2 - ax + x^2}{a + x}$.   **6.** $\frac{2x^2 + 5}{x - 3}$.   **7.** $\frac{10a^2 - 17ax + 10x^2}{5a - x}$.

8. $\frac{16(3x^2 + 1)}{4x - 1}$.   **9.** $\frac{x^3 - 2x^2}{x^2 - x + 1}$.   **10.** $\frac{x^3 - 6x + 14}{x^2 - 3x + 4}$.

Reduce the following to improper fractions :—

11. $4x - \frac{5xy - 2}{3y}$.   **12.** $1 + \frac{a - b}{a + b}$.   **13.** $x^2 - 3x - \frac{3x(3 - x)}{x - 2}$.

14. $x - 5 - \frac{2x - 15}{x - 3}$.   **15.** $a^2 - 2ax + 4x^2 - \frac{6x^3}{a + 2x}$.

16. $x - a + y + \frac{a^2 - ay + y^2}{x + a}$.   **17.** $x + y - \frac{\quad}{x^2 - xy + y^2}$.

237. To reduce fractions to a common denominator.

Rule. *Multiply the numerator of each fraction by all the denominators except its own, for the new numerator corresponding to that fraction, and all the denominators together for the common denominator.*

The truth of this Rule is obvious; since, the numerator and denominator of each fraction being *both* multiplied by the same quantities, *viz.*, the denominators of the other fractions, its value will not be altered, though all the fractions will now appear with the same denominator.

Ex. Reduce $\frac{a}{b}$, $\frac{b}{c}$, $\frac{c}{d}$ to a common denominator.

For the numerators, $a \times c \times d = acd$
$b \times b \times d = b^2d$
$c \times b \times c = bc^2$
For the denominator, $b \times c \times d = bcd$;

and the required fractions are $\frac{acd}{bcd}$, $\frac{b^2d}{bcd}$, $\frac{bc^2}{bcd}$.

238. If, however, the original denominators of the fractions have, any of them, common factors, this process will not give them their *lowest* common denominator, which, as in Arithmetic, will be found by forming the L. C. M. of the given denominators; and the numerator corresponding to any one of the given fractions will be obtained, by multiplying its numerator by that quotient, which is obtained by dividing the L. C. M. by its denominator.

Ex. 1 Reduce $\frac{a}{2bx}$, $\frac{c}{6abxy}$, $\frac{b}{3acx}$ to their lowest common denominator.

Here, the L. C. M. of the denominators is $6abcxy$.

Then, $6abcxy \div 2bx = 3acy$; $\therefore \frac{a}{2bx} = \frac{3a^2cy}{6abcxy}$.

$6abcxy \div 6abxy = c$; $\therefore \frac{c}{6abxy} = \frac{c^2}{6abcxy}$.

$6abcxy \div 3acx = 2by$; $\therefore \frac{b}{3acx} = \frac{2b^2y}{6abcxy}$.

Hence the required fractions are $\frac{3a^2cy}{6abcxy}$, $\frac{c^2}{6abcxy}$, $\frac{2b^2y}{6abcxy}$.

Ex. 2. Reduce $\frac{3x^2}{2(a+b)}$, $\frac{xy}{3(a-b)}$, $\frac{2y^2}{4(a^2-b^2)}$ to their least common denominator.

The L. C. M. of the denominators $= 12(a^2 - b^2)$.

Then, $12(a^2 - b^2) \div 2(a+b) = 6(a - b)$;

$$\therefore \frac{3x^2}{2(a+b)} = \frac{3x^2}{2(a+b)} \times \frac{6(a-b)}{6(a-b)} = \frac{18x^2(a-b)}{12(a^2-b^2)}.$$

$$12(a^2 - b^2) \div 3(a-b) = 4(a+b);$$

$$\therefore \frac{xy}{3(a-b)} = \frac{xy}{3(a-b)} \times \frac{4(a+b)}{4(a+b)} = \frac{4xy(a+b)}{12(a^2-b^2)}.$$

$$12(a^2 - b^2) \div 4(a^2 - b^2) = 3;$$

$$\therefore \frac{2y^2}{4(a^2-b^2)} = \frac{2y^2}{4(a^2-b^2)} \times \frac{3}{3} = \frac{6y^2}{12(a^2-b^2)}.$$

Exercise XC.

Reduce the following fractions to their lowest common denominator.

1. $\frac{x}{a}, \frac{y}{b}, \frac{z}{c}$.
2. $\frac{\ }{2ab}, \frac{y}{3ac}, \frac{\ }{4bc}$.
3. $\frac{3}{5a}, \frac{b}{3x}, \frac{3c}{ax}$.
4. $\frac{2x^2y}{3a^3}, \frac{3x^3}{4a^2b}, \frac{4y^3}{5ab^2}, \frac{5xy^2}{6b^3}$.
5. $\frac{a+x}{a-x}, \frac{a-x}{a+x}$.
6. $\frac{3a-b}{4x^2}, \frac{5a-b}{2x^3}, \frac{2a-b}{8x}$.
7. $\frac{x^2}{a^2+b^2}, \frac{y^2}{a^2-b^2}$.
8. $\frac{4x^2}{3(a+b)}, \frac{xy}{6(a^2-b^2)}$.
9. $\frac{\ }{4a^3(a+x)}, \frac{\ }{4a^3(a-x)}, \frac{\ }{2a^2(a^2-x^2)}$.
10. $\frac{1}{x-1}, \frac{ax}{(x-1)^2}, \frac{3a}{x+1}, \frac{4b}{(x+1)^2}, \frac{5}{x^2-1}$.

II. ADDITION OR SUBTRACTION OF FRACTIONS.

239. To add or subtract fractions.

Rule. *Reduce the fractions to a common denominator, and add or subtract the numerators for a new numerator, retaining the common denominator.*

Ex. 1. Add together $\frac{x}{a}$, $\frac{y}{b}$ and $\frac{z}{c}$.

Here, the common denominator is abc.

$$\therefore \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \frac{bcx + acy + abz}{abc}.$$

Ex. 2. Add together $\frac{3a-4}{7}$ and $\frac{5-7a}{8}$.

$$\frac{3a-4}{7}+\frac{5-7a}{8}=\frac{24a-32}{56}+\frac{35-49a}{56}$$

$$=\frac{24a-32+35-49a}{56}=\frac{3-25a}{56}.$$

Ex. 3. From $\frac{3a-5}{6}$ take $\frac{a+2}{5}$.

$$\frac{3a-5}{6}-\frac{a+2}{5}=\frac{15a-25}{30}-\frac{6a+12}{30}=\frac{15a-25-(6a+12)}{30}$$

$$=\frac{15a-25-6a-12}{30}=\frac{9a-37}{30}.$$

Ex. 4. Add $\frac{1+x}{1+x+x^2}$ to $\frac{1-x}{1-x+x^2}$.

Here, the L. C. D. $=(1+x+x^2)(1-x+x^2)=1+x^2+x^4$.

Hence the Exp. $=\frac{(1+x)(1-x+x^2)+(1-x)(1+x+x^2)}{1+x^2+x^4}$

$$=\frac{(1+x^3)+(1-x^3)}{1+x^2+x^4}=\frac{2}{1+x^2+x^4}.$$

Ex. 5. From $\frac{1+x}{1+x+x^2}$ take $\frac{1-x}{1-x+x^2}$.

Here, the L. C. D. $=(1+x+x^2)(1-x+x^2)=1+x^2+x^4$

Hence the Exp. $=\frac{(1+x)(1-x+x^2)-(1-x)(1+x+x^2)}{1+x^2+x^4}$

$$=\frac{(1+x^3)-(1-x^3)}{1+x^2+x^4}=\frac{1+x^3-1+x^3}{1+x^2+x^4}=\frac{2x^3}{1+x^2+x^4}.$$

Ex. 6. Find the value of $\frac{a^2+b^2}{a^2-b^2}-\frac{a-b}{a+b}+2$.

Here, the L. C. D. $=a^2-b^2$.

Hence the Exp. $=\frac{(a^2+b^2)-(a-b)^2+2(a^2-b^2)}{a^2-b^2}$

$$=\frac{a^2+b^2-(a^2-2ab+b^2)+2a^2-2b^2}{a^2-b^2}=\frac{2(a^2+ab-b^2)}{a^2-b^2}$$

Exercise XCI.

Reduce the following to a single fraction :—

1. $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}$.

2. $\frac{a}{2x}+\frac{a}{3x}+\frac{a}{4x}$.

3. $\frac{2a}{3x}+\frac{5b}{12y}+\frac{3c}{4z}$.

4. $\frac{x+1}{3}-\frac{2x-1}{4}$.

5. $\frac{x-3}{5x}-\frac{x-5}{6x}$.

6. $\frac{5x^3-2}{8x^2}-\frac{3x^2-x}{8x}$.

7. $\frac{3a-2b}{5a}+\frac{5a-7b}{10a}+\frac{8a+2b}{25a}$.

8. $\frac{3a-4b}{2}-\frac{2a-b-c}{3}+\frac{15a-4c}{12}$.

9. $\frac{3x-4y}{2}+\frac{15x-4z}{12}-\frac{2x-y-z}{3}$.

10. $\frac{a-x}{ax}-\frac{a+x}{x^2}+\frac{4a^2+x^2}{4ax^2}$.

11. $\frac{2a^2+5ab}{a^2}+\frac{5a^2b-2ab^2}{ab^2}-\frac{4a^4-3a^3b}{a^4b}$.

12. $\frac{a}{2b}-\frac{a-b}{2(a+b)}$.

13. $\frac{a}{2b}+\frac{a+b}{3(a-b)}$.

14. $\frac{a}{a+b}+\frac{b}{a-b}$.

15. $\frac{a}{a-b}-\frac{b}{a+b}$.

16. $\frac{a^2}{a-b}-a$.

17. $\frac{a^2+b^2}{a^2-b^2}+\frac{a-b}{a+b}$.

18. $\frac{a^2+b^2}{a^2-b^2}-\frac{a-b}{a+b}$.

19. $\frac{a-b}{a+b}+\frac{ab}{a^2-b^2}$.

20. $\frac{2x^2-2xy+y^2}{x^2-xy}-\frac{x}{x-y}$.

21. $\frac{1}{a-7}-\frac{1}{a-}$

22. $\frac{1+5x}{1-5x}-\frac{1-5x}{1+5x}$.

23. $\frac{a}{c}-\frac{(ad-bc)x}{c(c+dx)}$.

24. $\frac{x}{a^2}+\frac{a-x}{a(a+x)}$.

25. $\frac{x}{a^2}-\frac{a+x}{a(a-x)}$.

26. $\frac{1}{x}+\frac{2}{x-1}-\frac{3}{x+2}$.

27. $\frac{1}{x-1}-\frac{2}{x}+\frac{1}{x+1}$.

28. $\frac{1}{2(a-x)}+\frac{1}{2(a+x)}+\frac{a}{a^2-x^2}$.

29. $\frac{a-b}{ab}+\frac{c-a}{ca}+\frac{b-c}{bc}$. (B.M. 1864.)

30. $\frac{1}{2(x-1)}-\frac{1}{2(x+1)}-\frac{1}{x^2}$.

31. $\frac{1}{2a+b}+\frac{1}{2a-b}-\frac{3a}{4a^2-b^2}$.

32. $\frac{1}{x^4}-\frac{1}{(x^2+1)^2}+\frac{x-1}{x^2+1}$.

33. $\frac{x^2+y^2}{x^2-y^2}-\frac{y}{x-y}+\frac{x}{x+y}$.

34. $\frac{a}{b}-\frac{(a^2-b^2)x}{b^2}+\frac{a(a^2-b^2)x^2}{b^2(b+ax)}$. (B.M. '64).

35. $2-\frac{x^2-y^2}{x^2+y^2}+\frac{x^2+y^2}{x^2-y^2}$.

36. $\frac{x}{1-x}-\frac{x^2}{(1-x)^2}+\frac{x^3}{(1-x)^3}$.

37. $\frac{3a}{a+x}+\frac{a}{a-x}-\frac{2ax}{a^2-x^2}$.   **38.** $\frac{a}{a-4}-\frac{2a}{a+4}+\frac{a^2+5a}{a^2-16}$.

39. $\frac{x^2+y^2}{xy}-\frac{x^2}{xy+y^2}-\frac{y^2}{x^2+xy}$.   **40.** $\frac{a^2}{a^3+b^3}+\frac{a-b}{a^2-ab+b^2}+\frac{1}{a+b}$.

41. $\frac{x-1}{(x+2)(x+5)}-\frac{2(x+2)}{(x+5)(x-1)}+\frac{x+5}{(x-1)(x+2)}$.

240. Sometimes the denominators of the fractions have to be resolved into factors either by inspection or by the method of finding the L. C. M.

Ex. 1. Simplify $\frac{3x-2}{x^2-5x+6}+\frac{2x+3}{x^2-3x+2}-\frac{5x+2}{x^2-4x+3}$.

The first Denr. $=(x-2)(x-3)$; the second $=(x-1)(x-2)$; and the third $=(x-1)(x-3)$; of these the L. C. D. $=(x-1)(x-2)(x-3)$.

$$\text{Hence the Exp.}=\frac{3x-2}{(x-2)(x-3)}+\frac{2x+3}{(x-1)(x-2)}-\frac{5x+2}{(x-1)(x-3)}$$

$$=\frac{(3x-2)(x-1)+(2x+3)(x-3)-(5x+2)(x-2)}{(x-1)(x-2)(x-3)}$$

$$=\frac{(3x^2-5x+2)+(2x^2-3x-9)-(5x^2-8x-4)}{(x-1)(x-2)(x-3)}$$

$$=\frac{-3}{(x-1)(x-2)(x-3)}.$$

241. Sometimes it would be convenient to combine two of the fractions together, instead of finding the L. C. M. of all the denominators at a time.

Ex. 2. Simplify $\frac{5}{3x-2}+\frac{2}{3x+2}-\frac{21x+6}{9x^2+4}$.

The L. C. D. of the first two $=9x^2-4$.

Hence the first two terms $=\frac{5(3x+2)+2(3x-2)}{9x^2-4}=\frac{21x+6}{9x^2-4}$.

Therefore the whole expression

$$=\frac{21x+6}{9x^2-4}-\frac{21x+6}{9x^2+4}=(21x+6)\left\{\frac{1}{9x^2-4}-\frac{1}{9x^2+4}\right\}$$

$$=(21x+6)\times\frac{(9x^2+4)-(9x^2-4)}{81x^4-16}=\frac{8(21x+6)}{81x^4-16}=\frac{24(7x+2)}{81x^4-16}.$$

Ex. 3. Simplify $\frac{1}{a^2-2}-\frac{2}{a^2-1}+\frac{2}{a^2+1}-\frac{1}{a^2+2}$.

The Exp. $=\left(\frac{1}{a^2-2}-\frac{1}{a^2+2}\right)+2\left(\frac{1}{a^2+1}-\frac{1}{a^2-1}\right)$,

(re-arranging the terms)

$$=\frac{a^2+2-a^2+2}{a^4-4}+2\left(\frac{a^2-1-a^2-1}{a^4-1}\right)=\frac{4}{a^4-4}-\frac{4}{a^4-1}$$

$$=4\left(\frac{1}{a^4-4}-\frac{1}{a^4-1}\right)=\frac{4(a^4-1-a^4+4)}{(a^4-1)(a^4-4)}=\frac{12}{a^8-5a^4+4}.$$

242. Sometimes the work may be simplified by first reducing the fractions to their lowest terms.

Ex. 4. Simplify $\frac{a^2-3ab+2b^2}{a-2b}+\frac{6a^2-5ab-6b^2}{2a-3b}-\frac{6a^2+ab-2b^2}{3a+2b}$.

The first fraction $=\frac{(a-b)(a-2b)}{a-2b}=a-b$

The second....... $=\frac{(3a+2b)(2a-3b)}{2a-3b}=3a+2b.$

The third.......... $=\frac{(2a-b)(3a+2b)}{3a+2b}=2a-b.$

Hence the expression $=(a-b)+(3a+2b)-(2a-b)=2(a+b)$.

243. By applying the principle of Art. 229, fractions may often be changed to simpler form for addition or subtraction.

Ex. 5. Simplify $\frac{a}{a-b}+\frac{b}{b-a}$.

Since $b-a=-(a-b)$, the above fractions may be written thus :

$$\frac{a}{a-b}-\frac{b}{a-b}=\frac{a-b}{a-b}=1.$$

Ex. 6. Simplify $\frac{2b-a}{x-b}+\frac{b-2a}{x+b}-\frac{3x(a-b)}{b^2-x^2}$.

Since $b^2-x^2=-(x^2-b^2)$, therefore we have

the Exp. $=\frac{2b-a}{x-b}+\frac{b-2a}{x+b}+\frac{3x(a-b)}{x^2-b^2}$

$$=\frac{(2b-a)(x+b)+(b-2a)(x-b)+3x(a-b)}{x^2-b^2}$$

$$= \frac{(2bx - ax + 2b^2 - ab) + (bx - 2ax - b^2 + 2ab) + 3ax - 3bx}{x^2 - b^2}$$

$$= \frac{ab + b^2}{x^2 - b^2} = \frac{b(a+b)}{x^2 - b^2}.$$

Exercise XCII.

Find the values of :—

1. $\frac{1}{3x+2} + \frac{2x-5}{(3x+2)^2} + \frac{x^2-6x-7}{(3x+2)^3}$.

2. $\frac{1}{x-y} + \frac{x-y}{x^2+xy+y^2} + \frac{xy-2x^2}{x^3-y^3}$.

3. $\frac{1}{x^2-7x+12} - \frac{1}{x^2-x-12}$.

4. $\frac{4}{x^2+x} + \frac{2x-5}{x^2-x+1} - \frac{2x^2-11}{x^3+1}$.

5. $\frac{1}{x-1} - \frac{2}{x-2} + \frac{x}{x^2-3x+2}$.

6. $\frac{x+3}{x-1} - \frac{x+1}{x+3} + \frac{8}{x^2+2x-3}$.

7. $\frac{2a}{(x-2a)^2} - \frac{x-a}{x^2-5ax+6a^2} + \frac{2}{x-3a}$.

8. $\frac{1}{9a^2-3ab+b^2} - \frac{3a}{27a^3+b^3}$.

9. $\frac{x+4}{x^2-3x-28} - \frac{x-5}{x^2+2x-35}$.

10. $\frac{(2a-5b)^2-4a^2}{4a-5b} + \frac{(3a-2b)^2-4b^2}{3a-4b}$.

11. $\frac{1}{2(a-x)} + \frac{1}{2(a+x)} + \frac{a}{a^2+x^2}$.

12. $1 - \frac{x-y}{x+y} + \frac{y^2}{x^2-y^2} + \frac{2xy}{x^2+y^2}$.

13. $\frac{2}{a-2} - \frac{1}{2+a} - \frac{a+6}{a^2+4}$.

14. $\frac{3}{8(1-x)} + \frac{1}{8(1+x)} - \frac{1-x}{4(1+x^2)}$

15. $\frac{1+x}{1-x} + \frac{4x}{1+x^2} + \frac{8x}{1-x^4} - \frac{1-x}{1+x}$.

16. $\frac{4a^2b^2}{a^4-b^4} + \frac{2a^2}{a^2+b^2} + \frac{a}{a+b} - \frac{a}{b-a}$.

17. $\frac{x-3a}{x-4a} + \frac{x+4a}{x+3a} - \frac{(x+4a)(x-3a)}{x^2-ax-12a^2}$.

18. $\frac{a^2}{a^2-1} + \frac{a}{a+1} - \frac{a}{1-a}$.

19. $\frac{1}{x+3} + \frac{x+1}{x^2-3x+9} - \frac{2x^2+x+12}{x^3+27}$. (C. E. 1860).

20. $\frac{10x-11}{3(x-1)} - \frac{10x-1}{3(x^2+x+1)} + \frac{x^2-2x+5}{(x^3-1)(x+1)}$.

21. $\frac{1}{(x-1)(x-2)} - \frac{2}{(x-2)(3-x)} + \frac{3}{(x-3)(1-x)}$.

22. $\dfrac{x-2}{(x-1)(x-3)}+\dfrac{x-4}{(x-3)(x-5)}+\dfrac{2(x-3)}{(x-1)(5-x)}.$

23. $\dfrac{y-x}{a-y}-\dfrac{x-2y}{y+a}-\dfrac{3a(x-y)}{y^2-a^2}.$ **24.** $\dfrac{1}{x-1}+\dfrac{1}{2+2x}-\dfrac{x-3}{2x^2+2}.$

25. $\dfrac{5}{x^2-3x-28}+\dfrac{3}{x^2+x-12}+\dfrac{9}{x^2-10x+21}.$

26. $\dfrac{1}{a-b}-\dfrac{1}{2(a+b)}-\dfrac{a+3b}{2(a^2+b^2)}-\dfrac{4b^3}{a^4-b^4}.$

27. $\dfrac{1}{a^2+2ab-3b^2}+\dfrac{1}{b^2+2ab-3a^2}-\dfrac{2}{3a^2+3b^2+10ab}.$ (A. E. 1897).

28. $\dfrac{a^2+ac}{a^2c-c^3}-\dfrac{a-c}{(a+c)c}-\dfrac{2c}{a^2-c^2}.$ (C. E. 1869).

29. $\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}-\dfrac{x^2+y^2}{x^2-y^2}.$ (C. E. 1862).

30. $\dfrac{1}{x}+\dfrac{1}{x-1}+\dfrac{1}{x+1}-\dfrac{x}{x^2-1}+\dfrac{3}{x(x^2-1)}.$ (B. M. 1893)

31. $\dfrac{x+y}{y}-\dfrac{x}{x+y}-\dfrac{x^3-x^2y}{x^2y-y^3}.$ (C. E. 1863).

32. $\dfrac{b}{a+b}-\dfrac{a+b}{2b}+\dfrac{a^2+b^2}{2b(a-b)}.$ (C. E. 1874).

33. $1+\dfrac{a}{b}-\dfrac{b}{a+b}-\dfrac{a^2}{ab-b^2}+\dfrac{2a^2}{a^2-b^2}.$ (C. E. 1871).

34. $\dfrac{x-y}{x-z}+\dfrac{x-z}{x-y}-\dfrac{(y-z)^2}{(x-y)(x-z)}.$ (C. E. 1879).

35. $\dfrac{a+c}{(x-a)(b-a)}+\dfrac{b+c}{(x-b)(a-b)}.$ (C. E. 1860).

36. $\dfrac{x+3y}{4(x+y)(x+2y)}+\dfrac{x+2y}{(x+y)(x+3y)}-\dfrac{x+y}{4(x+2y)(x+3y)}.$ (C. E. 1866)

37. $\dfrac{1}{2}\cdot\dfrac{1}{x-1}-\dfrac{x-5}{x^2-7x+10}+\dfrac{1}{2}\cdot\dfrac{x-6}{x^2-9x+18}.$ (C. E. 1864).

38. $\dfrac{a+x}{a^2+ax+x^2}+\dfrac{a-x}{a^2-ax+x^2}+\dfrac{2x^3}{a^4+a^2x^2+x^4}.$ (B. M. 1892).

39. $\dfrac{1}{4a^3(a+x)}+\dfrac{1}{4a^3(a-x)}+\dfrac{1}{2a^2(a^2+x^2)}.$ (C. E. 1861).

40. $\dfrac{2+x}{2(x+1)}+\dfrac{2-x}{2(x-1)}+\dfrac{x}{x^2+1}$. (C. E. 1873).

41. $\dfrac{2}{a+x}-\dfrac{1}{a-x}+\dfrac{3x}{a^2-x^2}+\dfrac{ax}{a^3+x^3}$. (C. E. 1883).

42. $\dfrac{x+2}{1+x+x^2}-\dfrac{x-2}{1-x+x^2}-\dfrac{2x^2-4}{1-x^2+x^4}$. (C. E. 1877).

43. $\dfrac{1}{1+x}-\dfrac{1}{1-x}+\dfrac{2x}{1+x^2}+\dfrac{4x^3}{1+x^4}-\dfrac{8x^7}{1+x^8}$.

44. $\dfrac{x^2-(y-z)^2}{(z+x)^2-y^2}+\dfrac{y^2-(z-x)^2}{(x+y)^2-z^2}+\dfrac{z^2-(x-y)^2}{(y+z)^2-x^2}$. (C. E. 1866).

45. $\dfrac{(2k-3l)^2-k^2}{4k^2-(3l+k)^2}+\dfrac{4k^2-(3l-k)^2}{9(k^2-l^2)}+\dfrac{9l^2-k^2}{(2k+3l)^2-k^2}$. (C. E. 1894).

46. $\dfrac{9y^2-(4z-2x)^2}{(2x+3y)^2-16z^2}+\dfrac{16z^2-(2x-3y)^2}{(3y+4z)^2-4x^2}+\dfrac{4x^2-(3y-4z)^2}{(4z+2x)^2-9y^2}$. (B.M. 1890).

47. $\dfrac{1}{a^2-b^2-c^2+2bc}+\dfrac{1}{b^2-c^2-a^2+2ca}+\dfrac{1}{c^2-a^2-b^2+2ab}$.

48. $\dfrac{1}{a-2b}-\dfrac{4}{a-b}+\dfrac{6}{a}-\dfrac{4}{a+b}+\dfrac{1}{a+2b}$.

49. $\dfrac{2x+9}{x^2+7x+12}-\dfrac{x}{x^2+5x+6}-\dfrac{x}{x^2+3x+2}$.

50. $\dfrac{a-2b}{2a^2-11ab+12b^2}+\dfrac{2(2a-b)}{4a^2-4ab-3b^2}-\dfrac{3(a-b)}{2a^2-7ab-4b^2}$.

III. MULTIPLICATION OF FRACTIONS.

244. To multiply one fraction by another.

Rule. *Multiply the numerators together for a new numerator, and the denominators for a new denominator.*

Suppose that we have to multiply $\dfrac{a}{b}$ by $\dfrac{c}{d}$.

Let $\dfrac{a}{b}=x$ and $\dfrac{c}{d}=y$; $\therefore$ $a=bx$, $c=dy$ and $ac=bdxy$;

hence, (dividing each of these equals by bd), we get

$\dfrac{ac}{bd}=xy$; but $xy=\dfrac{a}{b}\times\dfrac{c}{d}$;

$$\therefore \frac{ac}{bd} = \frac{a \times c}{b \times d} = \frac{\text{product of numerators}}{\text{product of denominators}},$$

whence the truth of the Rule is manifest.

Similarly, we may proceed for any number of fractions.

245. It is always advisable, before multiplying out the factors for the new numerator and denominator, to see if some of them do exist in *both* the numerator and denominator, in which case they may be struck out, and the result will be more simple.

Ex. 1. $\frac{a}{bx} \times \frac{cx}{d} = \frac{a \times cx}{bx \times d} = \frac{ac \times x}{bd \times x} = \frac{ac}{bd}.$

Ex. 2. $\frac{5ax}{3cy} \times \frac{xy+y^2}{x^2+xy} = \frac{5ax \times y(x+y)}{3cy \times x(x+y)} = \frac{5a}{3c}.$

Ex. 3. $3by \quad {}^2 - x^2 \times a^2 - ax \quad 3by \times (c+x)(c-x) \times a(a-x)$

$$\frac{4x(a+x)}{3y(c-x)}.$$

Exercise XCIII

Find the value of :—

1. $\frac{5ab}{6b^2c} \times \frac{12bc^2}{15ab}$. 2. $\frac{16bc}{25c^2} \times \frac{5bc^2}{[illegible]}$. 3. $\frac{65a^4b^2c^2}{68x^3y^4z^3} \times \frac{85x^4y^3z^2}{91a^3b^4c^2}$.

4. $\frac{2x}{a} \times \frac{3ab}{c} \times \frac{3ac}{2b}$. 5. $\frac{3ab}{10b^2} \times \frac{4ab^2}{9b^2c} \times \frac{5ac}{8a^2b}$.

6. $\frac{ax}{(a-x)^2} \times \frac{a^2-x^2}{ab}$. 7. $\frac{9a^3yz}{15bcx^3} \times \frac{4b^3xz}{3abz^3} \times \frac{10c^2xy}{8acy^3}$.

8. $\frac{a^2-4}{a^2+2a} \times \frac{a^2-9}{a^2+3a}$. 9. $\frac{2a(x^2-y^2)^2}{cx} \times \frac{x^3}{(x-y)(x+y)^2}$.

10. $\frac{a^4-b^4}{a^2-2ab+b^2} \times \frac{a-b}{a(a+b)}$. (C. E. 1860).

11. $\frac{a^2+2ab}{a^2+4b^2} \times \frac{ab-2b^2}{a^2-4b^2}$. 12. $\frac{x^2+xy}{x-y} \times \frac{(x-y)^2}{x^4-y^4}$.

13. $\frac{a^4-x^4}{a^3-x^3} \times \frac{a^2+ax+x^2}{a+x}$. 14. $\frac{a^2-11a+30}{a^2-6a+9} \times \frac{a^2-3a}{a^2-5a}$.

15. $\dfrac{a^2+3a+2}{a^2+2a+1}\times\dfrac{a^2+5a+4}{a^2+7a+10}$ (C.E. 1866).

16. $\dfrac{x^2+4x-5}{x^2-2x-3}\times\dfrac{x^2-x-6}{x^2-4x-5}\times\dfrac{x^2+2x+1}{x^2+x-2}$.

17. $\dfrac{x^2-y^2}{x^2-3xy+2y^2}\times\dfrac{xy-2y^2}{x^2+xy}\times\dfrac{x^2-xy}{(x-y)^2}$.

18. $\dfrac{x^2-2ax+a^2}{x^2+4ax-5a^2}\times\dfrac{x^2-9a^2}{ax+2a^2}\times\dfrac{x^2+5ax}{x^2-4ax+3a^2}$.

IV. DIVISION OF FRACTIONS.

246. To divide one fraction by another.

Rule. *Invert the divisor and proceed as in Multiplication.*

Suppose that we have to divide $\dfrac{a}{b}$ by $\dfrac{c}{d}$.

Let $\dfrac{a}{b}=x$ and $\dfrac{c}{d}=y$; $\therefore$ $a=bx$, and $c=dy$.

Hence $ad=bdx$, $bc=bdy$, and $\dfrac{ad}{bc}=\dfrac{bdx}{bdy}=\dfrac{x}{y}$;

But $\dfrac{x}{y}=x\div y=\dfrac{a}{b}\div\dfrac{c}{d}$, and $\dfrac{ad}{bc}=\dfrac{a}{b}\times\dfrac{d}{c}$,

whence the truth of the Rule is manifest.

247. In division of fractions, as in multiplication, cancel factors common to both numerator and denominator before multiplying.

Ex. 1. $\dfrac{8a^4b}{15xy^3}\div\dfrac{2a^3}{3xy^2}=\dfrac{8a^4b}{15xy^3}\times\dfrac{3xy^2}{2a^3}=\dfrac{4ab\times6a^3xy^2}{5y\times6a^3xy^2}=\dfrac{4ab}{5y}$.

Ex. 2. $\dfrac{x^2+xy}{x-y}\div\dfrac{x^4-y^4}{(x-y)^2}=\dfrac{x(x+y)}{x-y}\times\dfrac{(x-y)^2}{x^4-y^4}$

$$=\frac{x(x+y)}{x-y}\times\frac{(x-y)(x-y)}{(x^2+y^2)(x+y)(x-y)}=\frac{x}{x^2+y^2}.$$

Exercise XCIV.

Find the value of :—

1. $\dfrac{4xy^2}{7yz^2}\div\dfrac{14xy}{20yz}$.　　2. $\dfrac{2a^5b^4}{5x^4y^2}\div\dfrac{8ab}{15xy}$.　　3. $\dfrac{a^2+b^2}{a^2-b^2}\div\dfrac{a-b}{a+b}$.

4. $\dfrac{a^3+b^3}{a^2-b^2} \div \dfrac{a^2-ab+b^2}{a-b}$.

5. $\dfrac{a^4-b^4}{a^2+b^2-2ab} \div \dfrac{a-b}{a^2+ab}$ (A.E. '89).

6 $\dfrac{a^4-b^4}{a^2-2ab+b^2} \div \dfrac{a^2+b^2}{b(a-b)}$.

7. $\dfrac{x^2-11x+30}{x^2-6x+9} \div \dfrac{x^2-5x}{x^2-3x}$.

$\dfrac{x^2-xy+y^2}{6x} \times \dfrac{4x^2y^2}{x-y} \div \dfrac{x^3+y^3}{x^2-y^2}$.

9. $\dfrac{x^4-2x^2y^2+y^4}{x^3y+xy^3} \div \dfrac{x^2-y^2}{x^2+y^2}$.

10. $\dfrac{a^3-3a^2b+3ab^2-b^3}{a^2-b^2} \div 2ab-2b^2 \times \dfrac{a^2+ab}{a-b}$.

11. $\dfrac{x^2+2x-15}{x^2+8x-33} \div \dfrac{x^2+9x+20}{x^2+7x-44}$.

12. $\dfrac{a^2+b^2-c^2+2ab}{c^2-a^2-b^2+2ab} \div \dfrac{a+b+c}{b+c-a}$.

13. $\dfrac{x^2-7x+6}{x^2+3x-4} \times \dfrac{x^2+10x+24}{x^2-14x+48} \div \dfrac{x^2+6x}{x^3-8x^2}$.

14. $\dfrac{x^4-b^4}{x^2-2bx+b^2} \div \dfrac{x^2+bx}{x-b} \times \dfrac{x^6-b^2x^3}{x^3+b^3} \div \dfrac{x^4-2bx^3+b^2x^2}{x^2-bx+b^2}$.

15. $\dfrac{x^4-a^4}{x^6+a^6} \times \dfrac{x^3+a^3}{x^4-2a^2x^2+a^4} \div \dfrac{x^3-a^3}{(x^4-a^2x^2+a^4)(x^2-2ax+a^2)}$.

248. Before applying the Rules for multiplication or division, it is necessary to change mixed expressions to a fractional form.

Ex. 1. Multiply $x+\dfrac{x^2}{a-x}$ by $\dfrac{a}{x}-\dfrac{x}{a}$.

Here, $x+\dfrac{x^2}{a-x}=\dfrac{ax-x^2+x^2}{a-x}=\dfrac{ax}{a-x}$; $\dfrac{a}{x}-\dfrac{x}{a}=\dfrac{a^2-x^2}{ax}$.

$\therefore$ Product $=\dfrac{ax}{a-x}\times\dfrac{a^2-x^2}{ax}=\dfrac{ax}{a-x}\times\dfrac{(a-x)(a+x)}{ax}=a+x$.

Ex. 2. Divide $1-\dfrac{a^2x^2}{y^2}$ by $\dfrac{y+ax}{y}$.

$$\text{Quotient}=\left(1-\frac{a^2x^2}{y^2}\right)\div\frac{y+ax}{y}=\frac{y^2-a^2x^2}{y^2}\times\frac{y}{y+ax}$$

$$=\frac{(y+ax)(y-ax)y}{y^2(y+ax)}=\frac{y-ax}{y}.$$

Exercise XCV.

Find the value of :—

1. $\left(a-\frac{x^2}{a}\right)\times\left(\frac{a}{x}+\frac{x}{a}\right)$.

2. $\left(1-\frac{a^2+1}{2a}\right)\times\left(\frac{1+a}{1-a}-1\right)$.

3. $\frac{x+y}{x-y}\times\left(1-\frac{y}{x}\right)\div\left(1+\frac{x}{y}\right)$.

4. $\frac{a}{bx}\times\left(b+\frac{bx}{a}\right)\times\left(1-\frac{a}{a+x}\right)$.

5. $\left(a^4-\frac{a^2}{x^2}\right)\times\frac{a^2x^2+abx^2}{ax+1}\times\frac{ax}{a^2-b^2}$.

6. $\left(1+\frac{1}{x}\right)\div\left(x-\frac{1}{x}\right)\times\left(1-\frac{1}{x}\right)^2$.

7. $\left(1-\frac{b^4}{a^4}\right)\div\left(\frac{a}{b}+\frac{b}{a}\right)$.

8. $\left(1-\frac{2xy}{x^2+y^2}\right)\div\left(\frac{x^3-y^3}{x-y}-3xy\right)$. (A. E. 1891).

9. $\left\{\frac{x}{a}+\frac{2x^2}{a(b-x)}\right\}\left\{\frac{a}{x}-\frac{2ax}{x(b+x)}\right\}$ (C. E. 1880).

10. $\left(\frac{a^2}{x}-\frac{x^3}{a^2}\right)\div\left(\frac{a}{x}+\frac{x}{a}\right)$. (B. M. 1863).

11. $\frac{(x+y)^2+(x-y)^2}{(x+y)^2-(x-y)^2}\div\frac{x^4-y^4}{2xy(x-y)}$. (M. M. 1887.)

12. $\left(x+\frac{16x-27}{x^2-16}\right)\div\left(x-1+\frac{13}{x-4}\right)$. (B. M. 1885).

Multiply

13. x^2-x+1 by $\frac{1}{x^2}+\frac{1}{x}+1$. (C. E. 1876).

14. $\frac{x^2}{a^2}+\frac{x}{a}+1$ by $\frac{x^2}{a^2}-\frac{x}{a}+1$. (B. M. 1865).

15. $\frac{x^2}{y^2}+1+\frac{y^2}{x^2}$ by $\frac{x}{y}-\frac{y}{x}$. (C. E. 1875).

16. $x+\frac{1}{x}+1$ by $x-1+\frac{1}{x}$.

Divide

17. $\frac{a^3}{b^3}+\frac{b^3}{a^3}$ by $\frac{a}{b}+\frac{b}{a}$. (C. E. 1870).

18. $\frac{a^3}{b^3}-1$ by $\frac{a^2}{b^2}+\frac{a}{b}+1$.

19. $\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}-2\right)^2$ by $\frac{y}{x}-\frac{x}{y}$. (B. M. 1895).

20. $x^5-y^5+\frac{y^{10}}{x^5}$ by $x-y+\frac{y^2}{x}$. (M. M. 1897).

V. COMPLEX AND CONTINUED FRACTIONS.

249. A **Complex fraction** is one which has a fraction in its numerator or in its denominator, or in both.

Thus, $\dfrac{\frac{a}{b}}{c}$, $\dfrac{a}{\frac{b}{c}}$, $\dfrac{\frac{a}{b}}{\frac{c}{d}}$ are *complex fractions.*

In the last of these forms, the upper and lower quantities a and d are called the **extremes**, and the two middle ones, b and c are called the **means**.

Sometimes the above are conveniently written thus :—

$$\frac{a}{b}\Big/c,\quad a\Big/\frac{b}{c},\quad \frac{a}{b}\Big/\frac{c}{d}.$$

250. Simplification of Complex Fractions. We have, by definition of fraction,

$$\frac{a}{b}\Big/\frac{c}{d}=\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}.$$

Similarly, $\dfrac{a}{b}\Big/c=\dfrac{a}{bc}$ and $a\Big/\dfrac{b}{c}=\dfrac{ac}{b}$.

Hence, we observe that when a complex fraction is put into the form of a $\dfrac{\text{fraction}}{\text{fraction}}$, the simple expression for it will be found by taking the product of the *extremes*, for the numerator, and that of the *means*, for the denominator ; and that any factor may be struck out from either of the extremes, if it be struck out also from one or other of the means.

$$\text{Thus, } \frac{\frac{a}{x}-\frac{x}{a}}{\frac{1}{x}-\frac{1}{a}}=\frac{\frac{a^2-x^2}{ax}}{\frac{a-x}{ax}}=\frac{(a^2-x^2)\times ax}{(a-x)\times ax}=\frac{(a+x)(a-x)}{a-x}=a+x.$$

Hence the following Rule :—

251. To simplify a complex fraction.

Rule. *Multiply both terms of the fraction by the* L. C. M. *of the denominators in each.*

Ex. 1. Simplify $\dfrac{1-\frac{1}{2}x}{4x}$ and $\dfrac{5-\frac{1}{4}x}{x+1\frac{1}{3}}$.

Here, the L. C. M. of the denrs. in (i) is 2, and in (ii) is 12.

Hence, (i) The Exp. $=\dfrac{1-\frac{1}{2}x}{4x}\times\dfrac{2}{2}=\dfrac{2-x}{8x}$.

(ii) The Exp. $=\dfrac{5-\frac{1}{4}x}{x+1\frac{1}{3}}\times\dfrac{12}{12}=\dfrac{60-3x}{12x+16}=\dfrac{3(20-x)}{4(3x+4)}$.

Ex. 2. Simplify $\dfrac{\dfrac{a}{a-1}-\dfrac{a+1}{a}}{\dfrac{a}{a+1}-\dfrac{a-1}{a}}$.

Here, the L. C. M. of the denrs. $=a(a^2-1)$.

$$\text{Hence, the Exp.}=\frac{\dfrac{a}{a-1}-\dfrac{a+1}{a}}{\dfrac{a}{a+1}-\dfrac{a-1}{a}}\times\frac{a(a^2-1)}{a(a^2-1)}$$

$$=\frac{a^2(a+1)-(a+1)(a^2-1)}{a^2(a-1)-(a-1)(a^2-1)}=\frac{(a+1)(a^2-a^2+1)}{(a-1)(a^2-a^2+1)}=\frac{a+1}{a-1}.$$

252. The following is an example of a **Continued fraction.** The best way to simplify it is to begin from the bottom and proceed upwards step by step.

Ex. Simplify $\dfrac{1}{1+\dfrac{x}{1+x+\dfrac{2x^2}{1-x}}}$.

$$\text{The Exp.}=\frac{1}{1+\dfrac{x}{\dfrac{1-x^2+2x^2}{1-x}}}=\frac{1}{1+\dfrac{x(1-x)}{1+x^2}}=\frac{1}{\dfrac{1+x^2+x-x^2}{1+x^2}}=\frac{1+x^2}{1+x}$$

Exercise XCVI.

Simplify the following :—

1. $\dfrac{2-\frac{3}{2}x}{5}$. 2. $\dfrac{x}{5-\frac{2}{3}x}$. 3. $\dfrac{3-x}{x+2\frac{1}{2}}$. 4. $\dfrac{3x+2\frac{1}{3}}{3\frac{1}{2}}$. 5 $\dfrac{2x}{x-}$

6. $\dfrac{2\frac{1}{4}-\frac{1}{3}x}{\frac{2}{3}x-1\frac{1}{2}}$. 7. $\dfrac{x-3\frac{1}{3}}{2\frac{3}{4}-\frac{1}{6}x}$. 8. $\dfrac{x-\frac{1}{2}(3x-2)}{3}$. 9. $\dfrac{6x-\frac{3}{4}(5x+3)}{2\frac{1}{4}}$

10. $\dfrac{2\frac{1}{3}-\frac{1}{2}(x-2)}{\frac{1}{3}(x+1)-4\frac{1}{2}}$. 11. $\dfrac{1\frac{4}{5}-\frac{2}{3}(x+2)}{1\frac{4}{10}(x+1)}$. 12. $1+\dfrac{\cdot}{1+a+\dfrac{2a^2}{1-a}}$

13. $\dfrac{\dfrac{1}{1+x}}{1-\dfrac{1}{1+x}}$.

14. $\dfrac{\dfrac{1}{1+x}+\dfrac{x}{1-x}}{\dfrac{1}{1-x}-\dfrac{x}{1+x}}$.

15. $\dfrac{\dfrac{a^2+b^2}{2a^2}-\dfrac{2b^2}{a^2+b^2}}{\dfrac{a^2+b^2}{2b^2}-\dfrac{2a^2}{a^2+b^2}}$.

16. $\dfrac{\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}}{\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}}$.

17. $\dfrac{x-\dfrac{x-y}{1+xy}}{1+\dfrac{x(x-y)}{1+xy}}$.

18. $\dfrac{a-1+\dfrac{6}{a-6}}{a-2+\dfrac{3}{a-6}}$.

19. $\dfrac{x+2}{1-x+\dfrac{1}{2-x+\dfrac{3}{x}}}$.

20. $\dfrac{1}{x-1+\dfrac{1}{1+\dfrac{x}{4-x}}}$.

21. $\dfrac{x}{1-\dfrac{x}{1+x+\dfrac{x}{1-x+x}}}$

22. $\dfrac{\dfrac{x^2+y^2}{y}-x}{\dfrac{1}{y}-\dfrac{1}{x}}\times\dfrac{x^2-y^2}{x^3+y^3}$ (M.M. 1887).

23. $\dfrac{x+\dfrac{1}{y}}{x+\dfrac{1}{y+\frac{1}{z}}}-\dfrac{1}{y(xyz+x+z)}$.

24. $\dfrac{1}{a-1+\dfrac{1}{1+\dfrac{a}{4-a}}}$.

25. $\dfrac{1}{1+\dfrac{1}{a\div x}}+\dfrac{1}{1-\dfrac{1}{a\div x}}+\dfrac{2}{1+\dfrac{1}{a^2\div x^2}}$ (C.F. 1870).

VI. SIMPLIFICATION OF FRACTIONS.

253. The following are illustrative Examples.

Ex. 1. Simplify $\left(\dfrac{a-b}{a+b}+\dfrac{a+b}{a-b}\right)\div\left(\dfrac{a^2+b^2}{a^2-b^2}+\dfrac{a^2-b^2}{a^2+b^2}\right)$.

Here, Numr. $=\dfrac{(a-b)^2+(a+b)^2}{a^2-b^2}=\dfrac{a^2-2ab+b^2+a^2+2ab+b^2}{a^2-b^2}$

$$=\frac{2(a^2+b^2)}{a^2-b^2}.$$

Denr. $=\dfrac{(a^2+b^2)^2+(a^2-b^2)^2}{(a^2-b^2)(a^2+b^2)}=\dfrac{a^4+2a^2b^2+b^4+a^4-2a^2b^2+b^4}{(a^2-b^2)(a^2+b^2)}$

$$=\frac{2(a^4+b^4)}{(a^2-b^2)(a^2+b^2)}.$$

Hence, the Exp. $=\dfrac{2(a^2+b^2)}{a^2-b^2}\div\dfrac{2(a^4+b^4)}{(a^2-b^2)(a^2+b^2)}$

$$=\frac{2(a^2+b^2)}{a^2-b^2}\times\frac{(a^2-b^2)(a^2+b^2)}{2(a^4+b^4)}=\frac{(a^2+b^2)^2}{a^4+b^4}.$$

Ex. 2. Simplify $\left(\frac{2x}{x+y}+\frac{y}{x-y}-\frac{y^2}{x^2-y^2}\right)\div\left(\frac{1}{x+y}+\frac{x}{x^2-y^2}\right)$.

Here, Numr. $=\frac{2x(x-y)+y(x+y)-y^2}{x^2-y^2}=\frac{2x^2-xy}{x^2-y^2}=\frac{x(2x-y)}{x^2-y^2}$.

Denr. $=\frac{x-y+x}{x^2-y^2}=\frac{2x-y}{x^2-y^2}$.

Hence, the Exp. $=\frac{x(2x-y)}{x^2-y^2}\div\frac{2x-y}{x^2-y^2}=\frac{x(2x-y)}{x^2-y^2}\times\frac{x^2-y^2}{2x-y}=x$.

Ex. 3. Simplify $\left(1+\frac{45}{x-8}-\frac{26}{x-6}\right)\left(3-\frac{65}{x+7}+\frac{8}{x-2}\right)$.

Here, 1st fraction $=1+\frac{45(x-6)-26(x-8)}{(x-8)(x-6)}=1+\frac{19x-62}{x^2-14x+48}$

$$=\frac{x^2-14x+48+19x-62}{(x-8)(x-6)}=\frac{x^2+5x-14}{(x-8)(x-6)}=\frac{(x+7)(x-2)}{(x-8)(x-6)}.$$

2nd fraction $=3-\left(\frac{65}{x+7}-\frac{8}{x-2}\right)=3-\frac{65(x-2)-8(x+7)}{(x+7)(x-2)}$

$$=3-\frac{57x-186}{x^2+5x-14}=\frac{3(x^2+5x-14)-(57x-186)}{x^2+5x-14}$$

$$=\frac{3x^2-42x+144}{(x+7)(x-2)}=\frac{3(x^2-14x+48)}{(x+7)(x-2)}=\frac{3(x-8)(x-6)}{(x+7)(x-2)}.$$

Hence, the Exp. $=\frac{(x+7)(x-2)}{(x-8)(x-6)}\times\frac{3(x-8)(x-6)}{(x+7)(x-2)}=3$.

Exercise XCVII.

Simplify the following expressions :—

1. $\left(\frac{2a}{x^2-a^2}-\frac{1}{x-a}+\frac{2}{x+a}\right)\times\frac{x^2}{x-a+\frac{a^2}{x}}$. (C. E. 1872)

2. $\left(1-\frac{1}{1+x}\right)\left(x+\frac{1}{2+x}\right)\times\frac{\frac{1}{x^2}-x}{1+\frac{1}{x}}\div\left(1+x+\frac{1}{x}\right)$. (C. E. 1871 & B. M. 1893).

3. $\left\{\frac{x}{1-\frac{1}{x}}-x-\frac{1}{1-x}\right\}\div\left\{\frac{x}{1+\frac{1}{x}}+x-\frac{1}{1+x}\right\}$. (C. E. 1886).

4. $\left(\dfrac{a+b}{a-b}+\dfrac{a^2+b^2}{a^2-b^2}\right) \div \left(\dfrac{a-b}{a+b}-\dfrac{a^3-b^3}{a^3+b^3}\right)$. (C. E. 1868).

5. $\dfrac{\dfrac{1}{a}+\dfrac{1}{b+c}}{\dfrac{1}{a}-\dfrac{1}{b+c}}\left\{1+\dfrac{b^2+c^2-a^2}{2bc}\right\}$. 6. $\dfrac{\dfrac{1}{m-n}-\dfrac{1}{m-s}}{\dfrac{1}{(m-n)^2}-\dfrac{1}{(m-s)^2}}$. (C. E. 1886).

7. $\dfrac{1+x+x^2}{1-x^3}+\dfrac{1-x+x^2}{1+x^3}-\left(\dfrac{x}{1+x}+\dfrac{1-x}{x}\div\dfrac{1+x}{x}\right)\times\dfrac{1}{1-x}$.
(C. E. 1875).

8. $\left(\dfrac{x-y}{x+y}-\dfrac{x^3-y^3}{x^3+y^3}\right)\left(\dfrac{x+y}{x-y}+\dfrac{x^3+y^3}{x^3-y^3}\right)$. (A. E. 1893).

9. $\left(y-\dfrac{a^2-xy}{y-x}\right)\left(x+\dfrac{a^2-xy}{y-x}\right)+\left(\dfrac{a^2-xy}{y-x}\right)^2$. (A. E. 1890).

10. $\dfrac{6x^2y^2}{m+n}\div\left[\dfrac{3(m-n)x}{7(r+s)}\div\left\{\dfrac{4(r-s)}{21xy^2}\div\dfrac{r^2-s^2}{4(m^2-n^2)}\right\}\right]$. (B. M. 1892).

11. $\dfrac{1+\dfrac{a-b}{a+b}}{1-\dfrac{a-b}{a+b}}\div\dfrac{1+\dfrac{a^2-b^2}{a^2+b^2}}{1-\dfrac{a^2-b^2}{a^2+b^2}}$. (C. E. 1859).

12. $\dfrac{x}{x-a}-\dfrac{x}{x+a}-\dfrac{\dfrac{x+a}{x-a}-\dfrac{x-a}{x+a}}{\dfrac{x+a}{x-a}+\dfrac{x-a}{x+a}}$. (C. E. 1869.)

13. $\left(\dfrac{x^2+y^2}{x^2-y^2}-\dfrac{x^2-y^2}{x^2+y^2}\right)\div\left(\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}\right)$. (C. E. 1867).

14. $\left(\dfrac{x+1}{x-1}+\dfrac{5}{x-7}\right)\left(\dfrac{x-2}{3x-5}-\dfrac{2}{x+3}\right)$.

15. $\left(\dfrac{x+2a}{a-2x}-\dfrac{a+2x}{x-2a}\right)\times\left(\dfrac{3}{2a-x}-\dfrac{1}{a-x}\right)$.

CHAPTER IX.

SIMULTANEOUS EQUATIONS.

254. If *one* equation contain *two* unknown quantities, there are an infinite number of pairs of values of these by which it may be satisfied.

Thus, consider the equation $3x+2y=13$, which contains *two* unknown quantities.

By transposition, we get $3x=13-2y$;

$$\therefore x=\frac{13-2y}{3}.$$

From this it appears that if we give *any value* to y, we shall get a corresponding value for x, by which pair of values the equation will of course be satisfied.

If, for example, we take $y=1$, we shall get $x=\frac{11}{3}$; if $y=2$, $x=3$; if $y=3$, $x=\frac{7}{3}$; and so on.

255. *One* equation then between *two* unknown quantities admits of an infinite number of solutions; but if we have as many different equations, as there are unknown quantities, the number of solutions will be limited.

Thus, while each of the equations $3x+2y=13$, $4x+3y=18$, separately considered, is satisfied by an infinite number of pairs of values of x and y, there will only be found *one* pair common to both.

Thus, being given the two equations, we must have

$x=\frac{13-2y}{3}$, from equation (1) and $x=\frac{18-3y}{4}$, from equation (2).

Now, if x is to have the *same* value in the two given equations (1) and (2), we must have

$$\frac{13-2y}{3}=\frac{18-3y}{4},$$

Multiplying up, $52-8y=54-9y$;

$$\therefore 9y-8y=54-52, \text{ or } y=2.$$

Substituting this value of y in either of the two equations, we shall obtain the corresponding value of x. Thus from (1) we obtain

$$3x+4=13;\ \therefore 3x=9;\ \therefore x=3.$$

Hence the *only* values of x and y which satisfy *both* the equations (1) and (2), are

$$x=3 \text{ and } y=2.$$

256. Equations of this kind, which are to be satisfied by the *same* pair or pairs of values of x and y are called **Simultaneous Equations**.

257 If there be *three* unknowns, there must be *three* equations, and so on; and, moreover, these equations must all be *different* from one another; *i. e.* must *all* express *different* relations between the unknown quantities.

Thus, if we had the equation $3x+2y=13$, it would be of no use to join with it the equation $6x+4y=26$, (which is obtained by merely doubling it), or any other, derived, like this, immediately from the former; since this expresses no new relation between x and y, but repeats in another form the same as before.

258. It may be observed, that if any two or more equations be given, any equations formed by adding or subtracting any multiples of these equations, will be also *true*, though expressing, in reality, no *new* relations between the quantities.

Thus, if $x+2y+3z=10$, and $2x-3y+z=1$; then, subtracting the first from three times the second, we have $5x-11y=-7$, which expresses no new relation.

I. EQUATIONS INVOLVING TWO UNKNOWNS.

259. There are generally given three methods for solving Simultaneous Equations of two unknowns; but the object aimed at is the same in each, *viz.*, to combine the two equations in such a manner as to expel, or, as the phrase is, **eliminate** from the result one quantity, and so get an equation of *one* unknown only.

260. **First Method**. Multiply, when possible, one equation by some number, that may make the coefficient of x or y in it the same as in the other; then, adding or subtracting the two equations, according as these equal quantities have different or same signs, these terms will destroy each other, and the **elimination** will be effected.

Ex. 1. Solve
$$\left.\begin{aligned}4x+9y&=51\\8x-13y&=9\end{aligned}\right\}\begin{aligned}&\ldots\ldots\ldots(1)\\&\ldots\ldots\ldots(2)\end{aligned}$$

Here, it will be convenient to eliminate x.

Multiply (1) by 2,	$8x+18y=102$
but	$8x-13y=9$, from (2)
subtracting,	$31y=93$, and

To find x, substitute this value of y in *either* of the given equations.

Thus, from (1), $4x+27=51$; $\therefore\ 4x=51-27=24$;

$\therefore\ x=6$, and $y=3$.

Verification. When $x=6$, and $y=3$.

$$4x+9y=4\times6+9\times3=24+27=51.$$

$\therefore$ equation (1) is satisfied.

Again, when $x=6$, and $y=3$,

$$8x-13y=8\times6-13\times3=48-39=9.$$

$\therefore$ equation (2) is satisfied. Q. E. D.

Ex. 2. Solve
$$\left.\begin{aligned}4x-y&=7\\3x+4y&=29\end{aligned}\right\}\begin{aligned}&\text{...............(1)}\\&\text{...............(2)}\end{aligned}$$

Here, it will be convenient to eliminate y.

Multiply (1) by 4, $16x-4y=28$.

but $3x+4y=29$, from (2)

$\therefore$ adding, $19x\quad=57$, and $\therefore\ x=3$.

Substitute this value in (1),

$\therefore\ 12-y=7$; $\therefore\ y=12-7=5$ and $x=3$.

261. Sometimes we cannot make the coefficients equal by multiplying only one of the equations; but shall have to multiply both by some numbers, which it will be easy to perceive in any case.

Ex. 3. Solve
$$\left.\begin{aligned}7x-16y&=42\\5x+17y&=30\end{aligned}\right\}\begin{aligned}&\text{...............(1)}\\&\text{...............(2)}\end{aligned}$$

Here, to eliminate x we should proceed thus :—

Multiply (1) by 5, $35x-80y=210$

„ (2) by 7, $35x+119y=210$

$\therefore$ subtracting, $-199y=\ 0$, and $\therefore\ y=0$.

Substitute this value of y in (1),

$\therefore\ 7x=42$ and $\therefore\ x=6$, and $y=0$.

262. Second Method. Express one of the unknown quantities in terms of the other by means of one of the equations, and put this value for it in the other equation.

Ex. 4. Solve
$$\left.\begin{aligned}2x+3y&=4\\3x-2y&=-7\end{aligned}\right\}\begin{aligned}&\text{...............(1)}\\&\text{...............(2)}\end{aligned}$$

Here, from (2), $3x=2y-7$ and $\therefore\ x=\frac{1}{3}(2y-7)$.......(3)

Substitute this value of x in (1),

$\therefore\ \frac{2}{3}(2y-7)+3y=4,$

$\therefore\ 4y-14+9y=12,\qquad \therefore\ 13y=26$ and $\therefore\ y=2$

Hence, from (3), by substitution, $x=\frac{1}{3}(4-7)=-1.$

$\therefore$ we have $x=-1$ and $y=2$.

263. Third Method. Express the *same* quantity in terms of the other in both equations, and put these values equal.

Ex. 5. Solve
$$\left.\begin{aligned}4x+7y&=62\\3y-2x&=8\end{aligned}\right\}.$$

From (1) $4x=62-7y$ and $\therefore\ x=\dfrac{62-7y}{4}$

From (2) $2x=3y-8$ and $\therefore\ x=\dfrac{3y-8}{2}$........(4),

$\therefore$ from (3) and (4), $\dfrac{62-7y}{4}=\dfrac{3y-8}{2}$.

clearing of fractions, $62-7y=6y-16,$

$\therefore\ -13y=-78$, and $\therefore\ y=6.$

Hence, from (4), by substitution $x=\dfrac{3\times6-8}{2}=\dfrac{10}{2}=5.$

$\therefore$ we have $x=5$ and $y=6$.

264. The first of these methods is generally used: but the second may be used with advantage, whenever either x or y has a coefficient *unity* in one of the equations.

Ex. 6. Solve
$$\left.\begin{aligned}3x-y&=3 &&\text{(1)}\\9x-5y&=-45 &&\text{..(2)}\end{aligned}\right\}$$

From (1) $y=3x-3$......(3).

Substitute this value of y in (2),

$\therefore\ 9x-5(3x-3)=-45$, or $9x-15x+15=-45,$

$\therefore -6x=-60,\ \therefore\ x=10.$

Hence, from (3) $y=3\times10-3=27.$

$\therefore$ we have $x=10$ and $y=27$.

265. In some cases, the work of solution may be shortened by certain arithmetical artifices.

Ex. 7. Solve
$$\left.\begin{aligned}14x+13y&=35\\21x+19y&=56\end{aligned}\right\}\begin{aligned}&\ldots\ldots\ldots\ldots(1)\\&\ldots\ldots\ldots\ldots(2)\end{aligned}$$

Observing that 14 and 21 have a common factor 7, we shall make the coefficients of x in the two equations equal to the L. C. M of 14 and 21 if we multiply (1) by 3 and (2) by 2.

Thus,
$$42x+39y=105$$
$$42x+38y=112$$

Subtracting, $y=-7$, and $\therefore x=9$.

266. In some cases, it will be necessary to simplify the equations before proceeding to solve them.

Ex. 8. Solve
$$\left.\begin{aligned}7(2x-y)+5(3y-4x)+30&=0\\5(y-x+3)&=6(y-2x)\end{aligned}\right\}\begin{aligned}&\ldots\ldots\ldots(1)\\&\ldots\ldots\ldots(2)\end{aligned}$$

From (1), $14x-7y+15y-20x+30=0$;

$\therefore -6x+8y=-30$ or $-3x+4y=-15$.........(3)

From (2), $5y-5x+15=6y-12x$;

$\therefore -y+7x=-15$. ....(4)

Multiply (4) by 4, $-4y+28x=-60$

From (3) $4y-3x=-15$

adding, $25x=-75$, and $\therefore x=-3$.

From (4), $y=7x+15=-21+15=-6$.

Ex. 9. Solve
$$\left.\begin{aligned}(x+5)(y+7)&=(x+1)(y-9)+112\\2x+10&=3y+1\end{aligned}\right\}\begin{aligned}&\ldots\ldots(1)\\&\ldots\ldots(2)\end{aligned}$$

From (1), multiplying out, we have

$$xy+7x+5y+35=xy-9x+y-9+112;$$

Reducing and transposing, we have

$$16x+4y=68, \text{ or } 4x+y=17\ldots\ldots\ldots\ldots(3)$$

From (2), $2x-3y=-9$..(4)

Multiplying (4) by 2, $4x-6y=-18$.

Subtracting, $7y=35$ and $\therefore y=5$.

From (3) $4x+5=17$; $\therefore 4x=12$ and $\therefore x=3$.

Exercise XCVIII.

Solve the following equations :-

1. $\left.\begin{array}{l} 3x+2y=9 \\ x+3y=10 \end{array}\right\}$ (C. E. 1861.)   2. $\left.\begin{array}{l} x+y=37 \\ x-y=1 \end{array}\right\}$ 3. $\left.\begin{array}{l} 2x+9y=11 \\ 4x+y=5 \end{array}\right\}$

4. $\left.\begin{array}{l} 2x-y=8 \\ 2y+x=9 \end{array}\right\}$ 5. $\left.\begin{array}{l} 2x-9y=11 \\ 3x-12y=15 \end{array}\right\}$ 6. $\left.\begin{array}{l} 2x+3y=8 \\ 7x-y=5 \end{array}\right\}$

7. $\left.\begin{array}{l} 5x+4y=58 \\ 3x+7y=67 \end{array}\right\}$ 8. $\left.\begin{array}{l} 4x+y=34 \\ 4y+x=16 \end{array}\right\}$ 9. $\left.\begin{array}{l} 5x+6y=137 \\ 13x-4y=23 \end{array}\right\}$

10. $\left.\begin{array}{l} 4x+7y=62 \\ 3y-2x=8 \end{array}\right\}$ 11. $\left.\begin{array}{l} 7x-8y=-22 \\ 11x-10y=4 \end{array}\right\}$

12. $\left.\begin{array}{l} 4x+6y=11 \\ 17x-5y=1 \end{array}\right\}$ 13. $\left.\begin{array}{l} 12x+13y=37 \\ 17x-19y=15 \end{array}\right\}$ 14. $\left.\begin{array}{l} 8x-21y=33 \\ 6x+35y=177 \end{array}\right\}$

15. $\left.\begin{array}{l} 3x-11y=4 \\ 5x-12y=13 \end{array}\right\}$ 16. $\left.\begin{array}{l} 38x+17y=127 \\ 133x+71y=479 \end{array}\right\}$ 17. $\left.\begin{array}{l} 4x+3y=43 \\ 3x-2y=11 \end{array}\right\}$

18. $\left.\begin{array}{l} 5x-4y=8\frac{1}{6} \\ 2x+3y=14 \end{array}\right\}$ 19. $8x-4y=9x-3y=6.$

20. $5x-2y=7x+2y=x+y+11.$

21. $8(2x-3y)-(2x+3y)=1,\ 7(2x-3y)+(2x+3y)=14.$

22. $2(3x-4y)=5(x-y-3),\ 9(x-y+7)=4(4x-3y).$

23. $4(x+4)-7(x-y-1)=56(x-2),\ 6(y-2)-(3x-2y)=9x.$

24. $3x-2y+2=5x-3y+1\frac{5}{6}=6x-y-4\frac{1}{2}.$ (B. M. 1892).

25. $(x+7)(y-3)+7=(x-1)(y+3)+5,\ 5x-11y+35=0.$ (C.E. 1888).

26. $\left.\begin{array}{l} x(y+7)=y(x+1) \\ 2x+20=3y+1 \end{array}\right\}$ 27. $\left.\begin{array}{l} (x-1\frac{1}{2})(y-1\frac{1}{3})=xy-5 \\ 20(x-1)-5(y+5)=x+2 \end{array}\right\}$

28. $\left.\begin{array}{l} 12(x-2)-20(10-x)=15(y-10) \\ 16(y+2)-3(2x+y)=6(x+13) \end{array}\right\}.$

267. If the equations are given in fractional form, we should first remove the fractions before proceeding to find the solution.

Ex. 1. Solve

$$7x+\frac{2y+4}{5}=16 \quad \ldots(1)$$

$$3y-\frac{x+2}{4}=8 \quad \ldots(2)$$

To clear of fractions,

Multiply (1) by 5, $35x+2y+4=80$,

$\therefore\ 35x+2y=76$............(3)

Multiply (2) by 4, $12y-x-2=32$,

$\therefore\ 12y-x=34$..................(4)

Here, from (4) $x=12y-34$..............................(5)

Substituting x in (3), we have

$35(12y-34)+2y=76$, or $420y-1190+2y=76$;

$\therefore\ 422y=1266$, and $\therefore\ y=3$.

Hence, from (5), $x=12\times 3-34=2$.

Ex. 2 Solve $\left.\begin{array}{l}\dfrac{2x-3}{4}-\dfrac{y-8}{5}=\dfrac{y+3}{4} \\ \dfrac{x-7}{3}+\dfrac{4y+1}{11}=3\end{array}\right|\begin{array}{l}\text{.............(1)} \\ \text{.............(2)}\end{array}$

Multiplying (1) by 20, $5(2x-3)-4(y-8)=5(y+3)$,

$\therefore\ 10x-15-4y+32=5y+15$, or $10x-9y=-2$....(3)

Multiplying (2) by 33, $11(x-7)+3(4y+1)=99$;

$\therefore\ 11x-77+12y+3=99$, or $11x+12y=173$.........(4)

Multiply (3) by 4 and (4) by 3; thus

$$\begin{aligned} 40x-36y&=-8 \\ 33x+36y&=519 \end{aligned}$$

$\therefore$ adding $73x=511$, $\therefore\ x=7$

Hence, from (3) $9y=10x+2=72$, $\therefore\ y=8$.

Exercise XCIX.

Solve the following equations :—

1. $\left.\begin{array}{l}\frac{1}{2}x+\frac{1}{3}y=13 \\ \frac{1}{5}x+\frac{1}{8}y=5\end{array}\right\}$

2. $\left.\begin{array}{l}\frac{1}{2}x+\frac{2}{3}y+\frac{1}{6}=0 \\ \frac{1}{2}y+\frac{2}{3}x-\frac{3}{72}=0\end{array}\right\}$ (M. M. 1886).

3. $\left.\begin{array}{l}2x-\dfrac{y-3}{5}=4 \\ 3y+\dfrac{x-2}{3}=9\end{array}\right\}$ (C.E. 1863)

4. $\left.\begin{array}{l}5x-\dfrac{5y+2}{4}=32 \\ 3y+\dfrac{x+2}{3}=9\end{array}\right|$

5. $\left.\begin{aligned}\frac{x+y}{2}+\frac{3x-5y}{4}=2\\ \frac{x}{14}+\frac{y}{18}=1\end{aligned}\right\}$ (C. E. 1876).

6. $\left.\begin{aligned}x-\frac{y-2}{7}=5\\ 4y-\frac{x+10}{3}=3\end{aligned}\right\}$

7. $\left.\begin{aligned}4x-\tfrac{1}{3}(5y-4)=1\\ \tfrac{1}{4}(3y-2x)+\tfrac{1}{3}x=\tfrac{1}{2}\end{aligned}\right\}$ (C. E. 1875).

8. $\left.\begin{aligned}\tfrac{1}{6}x+\tfrac{1}{16}y=6\\ \tfrac{1}{12}y-\tfrac{1}{9}x=2\end{aligned}\right\}$

9. $\left.\begin{aligned}4x-\tfrac{3}{5}(2y-3)=6\tfrac{1}{5}\\ 3y-\tfrac{2}{3}(3x-1)=7\end{aligned}\right\}$ (C. E. 1871).

10. $\left.\begin{aligned}\tfrac{1}{6}(2x+3y)+\tfrac{1}{3}x=8\\ \tfrac{1}{2}(7y-3x)-y=11\end{aligned}\right\}$

11. $\left.\begin{aligned}\frac{x-y}{3}=\frac{y-1}{4}\\ \frac{4x-5y}{7}=x-7\end{aligned}\right\}$ (C. E. 1872).

12. $\left.\begin{aligned}\frac{x-7}{5}-\frac{y-11}{3}=2\\ \frac{x+y}{7}+\frac{x-y}{3}=6\end{aligned}\right\}$

13. $\frac{x}{4}+\frac{y}{5}+1=\frac{x}{5}+\frac{y}{4}=23.$ (P. E. 1893).

14. $\frac{x}{3}+\frac{y}{2}-3=\frac{3x-2y}{5}-\frac{x-y}{2}=1.$

15. $\left.\begin{aligned}\frac{x+2}{7}+\frac{y-x}{4}=2x-8\\ \frac{2y-3x}{3}+2y=3x+4\end{aligned}\right\}$ (P. E. 1892).

16. $\left.\begin{aligned}\frac{7+x}{5}-\frac{2x-y}{4}=3y-5\\ \frac{5y-7}{2}+\frac{4x-3}{6}=18-5x\end{aligned}\right\}$ (C. E. 1880).

17. $\left.\begin{aligned}\frac{2x-y}{4}+1=\frac{7+x}{5}\\ \frac{3-4x}{6}+3=\frac{5y-7}{2}\end{aligned}\right\}$

18. $\left.\begin{aligned}2x-\frac{y+3}{4}=7+\frac{3y-2x}{5}\\ 4y-\frac{8-x}{3}=24\tfrac{1}{2}-\frac{2y+1}{2}\end{aligned}\right\}$

19. $\frac{2x+y}{5}-3=\frac{3x-5y}{2},\quad \frac{x}{2}+\frac{y}{3}=\frac{y}{2}-\frac{x}{4}+8$ (P. E. 1888).

20. $\frac{x-y}{13}-\frac{x-3y}{5}=y-3,\quad \frac{3(x-y)}{4}+\frac{5(x+y)}{6}=18.$

21. $\left.\begin{array}{l}\dfrac{106}{63}y+\dfrac{x-6y+1}{7}=\dfrac{x-3}{9}\\[2ex] \dfrac{x-5y+8}{9}=\dfrac{3x-13y}{7}+\dfrac{55}{63}\end{array}\right\}$

22. $\left.\begin{array}{l}\dfrac{2x-3}{4}-\dfrac{y-8}{5}=\dfrac{y+3}{4}\\[2ex] \dfrac{x-7}{3}+\dfrac{4y+1}{11}=3\end{array}\right\}$

23. $\left.\begin{array}{l}\dfrac{3x+4y+3}{10}-\dfrac{3x-y}{15}=5+\dfrac{y-8}{5}\\[2ex] \dfrac{9y+5x-8}{12}-\dfrac{x+y}{4}=\dfrac{7x+6}{11}\end{array}\right\}$

24. $\left.\begin{array}{l}2x+\cdot 4y=1\cdot 2\\ 3\cdot 4x-\cdot 02y=\cdot 01\end{array}\right\}$ (P. E. 1891).

25. $\left.\begin{array}{l}\cdot 3x+\cdot 125y=x-6\\ 3x-\cdot 5y=28-\cdot 25y\end{array}\right\}$

26. $\left.\begin{array}{l}\dfrac{x-2}{2}-\dfrac{x+y}{14}=\dfrac{x-y-1}{8}-\dfrac{y+12}{4}\\[2ex] \dfrac{x+7}{3}+\dfrac{y-5}{10}=1-x-\dfrac{5(y+1)}{7}\end{array}\right\}$ (C. E. 1882).

27. $\cdot 6x+\cdot 7y+3\cdot 95=0,\ \dfrac{x}{\cdot 5}+\dfrac{y}{\cdot 7}+10=0.$

28. $\cdot 5x+\cdot 6y=1\cdot 38,\ \dfrac{x}{3}+\dfrac{y}{\cdot 5}=2.$

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268. Fractional Simultaneous Equations, involving the **reciprocals** of x and y, may be solved as they stand without clearing them of fractions.

Ex. Solve $\dfrac{2}{x}+\dfrac{7}{y}=29$ .... (1), $\dfrac{5}{x}-\dfrac{6}{y}=2$ ....(2)

Multiplying (1) by 5, and (2) by 2, we obtain

$\left.\begin{array}{l}\dfrac{10}{x}+\dfrac{35}{y}=145\\[2ex] \dfrac{10}{x}-\dfrac{12}{y}=\ \ 4\end{array}\right\}$ Subtracting, $\dfrac{47}{y}=141$;

$\therefore\ 141y=47$, and $\therefore\ y=\frac{1}{3}$.

Substitute y in (1), $\dfrac{2}{x}+21=29$; $\therefore\ \dfrac{2}{x}=8$; $\therefore\ x=\frac{1}{4}$.

269. When the coefficients of x and y are interchanged in the two equations, it is advantageous to employ the method of **addition** and **subtraction**.

Ex. Solve $\left.\begin{array}{l}11x+13y=118\\ 13x+11y=122\end{array}\right\}\begin{array}{l}.......\ (1)\\ ....(2)\end{array}$

Adding (1) and (2), $24x+24y=240$.

Dividing by 24, $x+y=10$..........................(3)

Subtracting (1) from (2), $2x-2y=4$.

Dividing by 2, $x-y=2$ ....(4)

Adding (3) and (4), $2x=12$, and $\therefore x=6$.

Subtracting (4) from (3), $2y=8$, and $\therefore y=4$.

Exercise C.

Solve the following equations :—

1. $\left.\begin{aligned} x+\frac{3}{y}&=\frac{7}{2} \\ 3x-\frac{2}{y}&=8\frac{2}{3} \end{aligned}\right\}$

2. $\left.\begin{aligned} \frac{3}{x}-\frac{4}{y}&=5 \\ \frac{4}{x}-\frac{5}{y}&=6 \end{aligned}\right\}$

3. $\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}&=\frac{5}{6} \\ \frac{3}{y}+\frac{4}{x}&=5 \end{aligned}\right\}$

4. $\left.\begin{aligned} \frac{1}{5x}+\frac{y}{9}&=5 \\ \frac{1}{3x}+\frac{y}{2}&=14 \end{aligned}\right\}$ (C. E. 1870)

5. $\left.\begin{aligned} \frac{4}{x}+\frac{10}{y}&=2 \\ \frac{3}{x}+\frac{2}{y}&=\frac{19}{20} \end{aligned}\right\}$ (C. E. 1879).

6. $\left.\begin{aligned} \frac{2}{x}+\frac{3}{y}&=2 \\ \frac{5}{x}+\frac{10}{y}&=5\frac{5}{6} \end{aligned}\right\}$ (C. E. 1887)

7. $\left.\begin{aligned} \frac{6}{x}+\frac{4}{y}&=3 \\ \frac{9}{x}-\frac{1}{y}&=2\frac{3}{4} \end{aligned}\right\}$ (C. E. 1893)

8. $\left.\begin{aligned} \frac{1}{3x}-\frac{1}{7y}&=\frac{2}{3} \\ \frac{1}{2x}-\frac{1}{3y}&=\frac{1}{6} \end{aligned}\right\}$ (C. E. 1890).

9. $\left.\begin{aligned} \frac{3}{2x}+\frac{2}{3y}&=5 \\ \frac{2}{x}+\frac{3}{y}&=13 \end{aligned}\right\}$ (M. M. 1881

10. $\left.\begin{aligned} \frac{2}{x}-\frac{3}{2y}&=\frac{41}{35} \\ \frac{2\frac{1}{2}}{2x}+\frac{3\frac{1}{2}}{y}&=-\frac{73}{70} \end{aligned}\right\}$ (B. M. 1893).

11. $\left.\begin{aligned} \frac{15}{x}-\frac{1}{y}&=4\frac{1}{2} \\ \frac{9}{x}+\frac{2}{y}&=4 \end{aligned}\right\}$ (B. M. 1886).

12. $\left.\begin{aligned} x-y&=3 \\ \frac{1}{x}+\frac{1}{y}&=\frac{11}{3}\left(\frac{1}{y}-\frac{1}{x}\right) \end{aligned}\right\}$

13. $\left.\begin{aligned} 5x+11y&=146 \\ 11x+5y&=110 \end{aligned}\right\}$ (C. E. 1864

14. $\left.\begin{aligned} 13x+21y&=79 \\ 21x+13y&=23 \end{aligned}\right\}$

15. $\left.\begin{aligned} 7x-9y&=17 \\ 9x-7y&=31 \end{aligned}\right\}$

16. $\left.\begin{aligned} 49x-57y&=172 \\ 57x-49y&=252 \end{aligned}\right\}$ (M. M. 1891).

17. $\left.\begin{aligned} 11x+13y&=23 \\ 13x+11y&=25 \end{aligned}\right\}$

18. $\left.\begin{aligned} 3x+20&=4y-10 \\ 4(x-1)&=3(y-3) \end{aligned}\right\}$ (C. E. 1895).

19. $\left.\begin{aligned} \tfrac{1}{5}x+\tfrac{1}{8}y&=43 \\ \tfrac{1}{8}x+\tfrac{1}{5}y&=42 \end{aligned}\right\}$

20. $\dfrac{x}{7}+7y=65,\ \dfrac{y}{7}+7x=99\tfrac{3}{7}.$

21. $\dfrac{2x}{3}+3y=31,\ \dfrac{2y}{3}+3x=24$

270. In Simultaneous Equations the known quantities may be denoted by letters.

Ex. Solve $\left.\begin{aligned} ax+by&=m \\ cx+dy&=n \end{aligned}\right\}\begin{aligned} &\ldots\ldots\ldots(1) \\ &\ldots\ldots\ldots(2) \end{aligned}$

Multiply (1) by c, $acx+bcy=cm$ ....(3)

" (2) by a, $acx+ady=an$..........(4)

Subtract (4) from (3), $(bc-ad)y=cm-an$,

Divide by coefficient of y, $y=\dfrac{cm-an}{bc-ad}$.

To find the value of x proceed thus :-

Multiply (1) by d, $adx+bdy=dm$............(5)

" (2) by b, $bcx+bdy=bn$............(6)

Subtract (6) from (5), $(ad-bc)x=dm-bn$,

Divide by coefficient of x, $x=\dfrac{dm-bn}{ad-bc}$.

Otherwise thus : Substitute the value of y in (1).

$$ax+\frac{b(cm-an)}{bc-ad}=m,$$

$$\therefore\ ax=m-\frac{b(cm-an)}{bc-ad}=\frac{a(bn-dm)}{bc-ad}.$$

$$\therefore\ x=\frac{bn-dm}{bc-ad}=\frac{dm-bn}{ad-bc}\text{, as before.}$$

Exercise CI.

Solve the following equations :—

1. $\left.\begin{aligned} x+y&=a \\ ax+by&=b^2 \end{aligned}\right\}$ **2.** $\left.\begin{aligned} ax+y&=b \\ x+by&=a \end{aligned}\right\}$ **3.** $\left.\begin{aligned} bx+ay&=b \\ ax-by&=a \end{aligned}\right\}$.

4. $\left.\begin{aligned} ax&=by \\ x+y&=c \end{aligned}\right\}$ **5.** $\left.\begin{aligned} ax+by+c&=0 \\ a_1x+b_1y+c_1&=0 \end{aligned}\right\}$ (C. E. 1867).

6. $\left.\begin{array}{l} ax+by=c \\ a^2x+b^2y=c^2 \end{array}\right\}$ (C. E. 1881.)

7. $\left.\begin{array}{l} nx+my=c \\ px+qy=d \end{array}\right\}$

8. $\left.\begin{array}{l} x+y=a+b \\ bx+ay=2ab \end{array}\right\}$

9. $\left.\begin{array}{l} ax+by=c \\ bx-ay=d \end{array}\right\}$

10. $\left.\begin{array}{l} \dfrac{x}{a}-\dfrac{y}{b}=m \\ \dfrac{x}{c}+\dfrac{y}{d}=n \end{array}\right\}$

11. $\left.\begin{array}{l} \dfrac{x}{b}+\dfrac{y}{c}=1 \\ \dfrac{ax}{c}-\dfrac{by}{a}=0 \end{array}\right\}$

12. $\left.\begin{array}{l} \dfrac{x}{a}+\dfrac{y}{b}=1 \\ \dfrac{x}{b}-\dfrac{y}{a}=1 \end{array}\right\}$

13. $\left.\begin{array}{l} \dfrac{x}{a}+\dfrac{y}{b}=2 \\ ax-by=a^2-b^2 \end{array}\right\}$ (M. M. 1884).

14. $\left.\begin{array}{l} \dfrac{m}{x}-\dfrac{n}{y}=a \\ px=qy \end{array}\right\}$ (C. E. 1885).

15. $\left.\begin{array}{l} \dfrac{x}{a}+\dfrac{y}{b-a}=5m \\ \dfrac{x}{b}+\dfrac{y}{a-b}=7m \end{array}\right\}$ (M. M. 1891).

16. $ax+by=1=bx-\dfrac{b}{a}+ay-\dfrac{a}{b}$. (M. M. 1883).

17. $\dfrac{p}{x}+\dfrac{q}{y}=0,\ px+qy=r$. (B. M. 1885).

18. $\left.\begin{array}{l} \dfrac{b}{x}+\dfrac{a+c}{y}=m \\ \dfrac{a-c}{x}+\dfrac{b}{y}=n \end{array}\right\}$ (M. M. 1887.)

19. $\left.\begin{array}{l} \dfrac{a+b}{x}-5b=\dfrac{a-b}{y}-a \\ \dfrac{a}{x}-2a=\dfrac{b}{y}-3b \end{array}\right\}$ (A. E. 1894).

20. $\left.\begin{array}{l} (a-b)x+(a+b)y=2(a^2-b^2) \\ ax-by=a^2+b^2 \end{array}\right\}$ (P. E. 1889).

21. $\left.\begin{array}{l} (a+b)x+(a-b)y=2a \\ (a-b)x+(a+b)y=2b \end{array}\right\}$ (A. E. 1891).

22. $\left.\begin{array}{l} ax+by=c^2 \\ \dfrac{a}{b+y}-\dfrac{b}{a+x}=0 \end{array}\right\}$ (B. M. 1879).

23. $\left.\begin{array}{l} \dfrac{x-a}{c-a}+\dfrac{y-b}{c-b}=1 \\ \dfrac{x+a}{c}+\dfrac{y-a}{a-b}=\dfrac{a}{c} \end{array}\right\}$ (A.E. 1893).

24. $\left.\begin{array}{l} \dfrac{a}{x}+\dfrac{b}{y}=m \\ \dfrac{b}{x}+\dfrac{a}{y}=n \end{array}\right\}$ (C. E. 1869).

25. $\left.\begin{array}{l} \dfrac{m}{x}+\dfrac{n}{y}=a \\ \dfrac{n}{x}+\dfrac{m}{y}=b \end{array}\right\}$ (M.M. 1885 and A. E. 1892).

26. $\left.\begin{array}{l}\dfrac{2}{ax}+\dfrac{3}{by}=5\\[2mm]\dfrac{5}{ax}-\dfrac{2}{by}=3\end{array}\right\}$

27. $\left.\begin{array}{l}\dfrac{x}{a}+\dfrac{y}{b}=1-\dfrac{x}{c}\\[2mm]\dfrac{y}{a}+\dfrac{x}{b}=1+\dfrac{y}{c}\end{array}\right\}$ (M. M. 1893).

28. $\left.\begin{array}{l}(a+b)x-(a-b)y=3ab\\(a+b)y-(a-b)x=ab\end{array}\right\}$ (A. E. 1889).

29. $\left.\begin{array}{l}(a-b)x+(a+b)y=2a(a^2-b^2)\\2ab(x+y)=(a^2+b^2)(x-y)\end{array}\right\}$

30. $a(2x-y)+b(2x+y)=c(2x-y)+d(2x+y)=1.$

II. EQUATIONS INVOLVING THREE UNKNOWNS.

271. Simultaneous Equations of three unknown quantities are solved by eliminating one of them by means of any pair of the equations, and then the *same* one by means of another pair ; we shall thus have two equations involving the same two unknown quantities, which may now be solved by the preceding Rules.

Ex. 1. Solve $\left.\begin{array}{r}2x-3y+2z=5\\x+2y-3z=4\\3x+y-4z=7\end{array}\right\}\begin{array}{r}\ldots\ldots(1)\\\ldots\ldots(2)\\\ldots\ldots(3)\end{array}$

Take any two of the equations, say (1) and (2)

Bring down (1), $2x-3y+2z=5 \ldots\ldots(4)$

Multiply (2) by 2, $2x+4y-6z=8 \ldots\ldots(5)$

Subtract (4) from (5), $7y-8z=3 \ldots\ldots(6)$

Again, take any pair, say (2) and (3),

Multiply (2) by 3, $3x+6y-9z=12 \ldots\ldots(7)$

Bring down (3), $3x+y-4z=7 \ldots\ldots(8)$

Subtract (8) from (7), $5y-5z=5 \ldots\ldots(9)$

Divide by 5, $y-z=1 \ldots\ldots(10)$.

Now, the equations (6) and (10) contain only y and z.

$$\left.\begin{array}{r}7y-8z=3\\y-z=1\end{array}\right\}\begin{array}{r}\ldots\ldots(\alpha)\\\ldots\ldots(\beta)\end{array}$$

Multiply (β) by 7 and subtract it from (α) ; thus

$$\left.\begin{array}{r}7y-8z=3\\7y-7z=7\end{array}\right\}$$

$$\therefore \quad -z=-4 \text{ and } \therefore z=4.$$

From (10) $y=z+1=5$ and by substituting these values for y and z in any one of the given equations, $x=6$.

Ex. 2. Solve
$$\left.\begin{aligned} x+y&=a \\ y+z&=b \\ z+x&=c \end{aligned}\right\}\begin{aligned} &\ldots\ldots(1) \\ &\ldots\ldots(2) \\ &\ldots\ldots(3) \end{aligned}$$

In this case it would be convenient to add the three given equations, and thus deduce the value of $x+y+z$.

Thus, adding, we have $2x+2y+2z=a+b+c$

Dividing by 2, $x+y+z=\frac{1}{2}(a+b+c)$.........(4)

Subtracting (2) from (4), $x=\frac{1}{2}(a+b+c)-b=\frac{1}{2}(a-b+c)$.

Similarly, $y=\frac{1}{2}(a+b-c)$ and $z=\frac{1}{2}(b+c-a)$.

Ex. 3. Solve
$$\left.\begin{aligned} x+y&=axy \\ y+z&=byz \\ z+x&=czx \end{aligned}\right\}\begin{aligned} &\ldots\ldots\ldots(1) \\ &\ldots\ldots\ldots(2) \\ &\ldots\ldots\ldots(3) \end{aligned}$$

Here, dividing (1) by xy, $\frac{1}{x}+\frac{1}{y}=a$,

Similarly, from (2) and (3), $\frac{1}{y}+\frac{1}{z}=b$, and $\frac{1}{z}+\frac{1}{x}=c$.

Now, solving these equations in $\frac{1}{x}$, $\frac{1}{y}$ and $\frac{1}{z}$ as in the last example, we obtain the values of x, y and z.

Thus, $x=\frac{2}{a-b+c}$, $y=\frac{2}{a+b-c}$ and $z=\frac{2}{b+c-a}$.

Exercise CII.

Solve the following equations.

1. $\left.\begin{aligned} x-2y+3z&=2 \\ 2x-3y+z&=1 \\ 3x-y+2z&=9 \end{aligned}\right\}$

2. $\left.\begin{aligned} 2x+3y+4z&=20 \\ 3x+4y+5z&=26 \\ 3x+5y+6z&=31 \end{aligned}\right\}$

3. $\left.\begin{aligned} x+5y-4z&=5 \\ 3x-2y+2z&=14 \\ -10x+8y+z&=6 \end{aligned}\right\}$ (C. E. 1867).

4. $\left.\begin{aligned} x-y-z&=-15 \\ y+x+2z&=40 \\ 4z-5x-6y&=-150 \end{aligned}\right\}$ (C. E. 1886)

5. $\left.\begin{aligned} 5x+3y&=65 \\ 2y-z&=11 \\ 3x+4z&=57 \end{aligned}\right\}$

6. $\left.\begin{aligned} 3x+2y-z&=20 \\ 2x+3y+6z&=70 \\ x-y+6z&=41 \end{aligned}\right\}$

7. $\left.\begin{aligned} x-y+z&=1 \\ x-2y+4z&=8 \\ x-3y+9z&=27 \end{aligned}\right\}$ (M. M. 1890)

8. $\left.\begin{aligned} 3y+x-2&=0 \\ 3z-4y&=x+15 \\ 2x+7z&=7 \end{aligned}\right\}$ (C. E. 1883).

9 $\left.\begin{aligned} 2x-3y+z+1&=0 \\ 5x-3z&=6 \\ 3x+2y&=4z \end{aligned}\right\}$ (M. M. 1888).

10. $\left.\begin{aligned} x+y+z&=5 \\ x+y&=z-7 \\ x-3&=y+z \end{aligned}\right\}$

11. $\left.\begin{aligned} x+2y&=7 \\ y+2z&=2 \\ 3x+2y&=z-1 \end{aligned}\right\}$

12. $\left.\begin{aligned} 3x+4y-11&=0 \\ 5y-6z+\ 8&=0 \\ 7z-8x-13&=0 \end{aligned}\right\}$ (C. E. 1877).

13. $\left.\begin{aligned} x-2y&=5 \\ 3y+4z&=6 \\ 5z+6x&=21 \end{aligned}\right\}$ (M. M. 1889)

14. $\left.\begin{aligned} 2(x-y)&=3z-2 \\ x+1&=3(y+z) \\ 2x+3z&=4(1-y) \end{aligned}\right\}$

15 $\left.\begin{aligned} \tfrac{1}{2}x+\tfrac{1}{3}y&=12-\tfrac{1}{6}z \\ \tfrac{1}{2}y+\tfrac{1}{3}z-\tfrac{1}{6}x&=8 \\ \tfrac{1}{2}x+\tfrac{1}{3}z&=10 \end{aligned}\right\}$ (C. E. 1868)

16. $\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}&=6 \\ \frac{2}{x}-\frac{3}{y}-\frac{4}{z}&=8 \\ \frac{3}{x}-\frac{4}{y}+\frac{5}{z}&=10 \end{aligned}\right\}$ (B. M. 1890).

17. $\left.\begin{aligned} \frac{x}{2}-\frac{y}{3}+z&=7 \\ x+\frac{y}{2}+\frac{z}{3}&=11. \\ \frac{x}{3}+y-\frac{z}{2}&=5 \end{aligned}\right\}$

18. $\left.\begin{aligned} \frac{a}{x}+\frac{b}{y}+\frac{c}{z}&=3 \\ \frac{a}{x}+\frac{b}{y}-\frac{c}{z}&=1 \\ \frac{2a}{x}-\frac{b}{y}-\frac{c}{z}&=0 \end{aligned}\right\}$

19. $\left.\begin{aligned} x+y-3z&=-a \\ z+x-3y&=-b \\ y+z-3x&=-c \end{aligned}\right\}$ (M. M. 1881).

20. $\left.\begin{aligned} x+y+z&=1 \\ ax+by+cz&=0 \\ a^3x+b^3y+c^3z&=abc \end{aligned}\right\}$ (M. M. 1883).

21. $\left.\begin{aligned} xy&=x+y \\ xz&=2(x+z) \\ yz&=3(y+z) \end{aligned}\right\}$

22. $\dfrac{x+y}{xy}=\dfrac{y+z}{yz}=\dfrac{z+x}{zx}=\dfrac{2}{3}$. (P. E. 1887).

23. $\dfrac{a}{x}+\dfrac{b}{y}=\dfrac{1}{r}, \; \dfrac{a}{x}+\dfrac{c}{z}=\dfrac{1}{q}, \; \dfrac{b}{y}+\dfrac{c}{z}=\dfrac{1}{p}$.

24. $\left.\begin{array}{r} y+\frac{1}{3}z=\frac{1}{5}x+5 \\ \frac{1}{4}(x-1)-\frac{1}{5}(y-2)=\frac{1}{10}(z+3) \\ x-\frac{1}{7}(2y-5)=1\frac{3}{4}-\frac{1}{12}z \end{array}\right\}$

25. $\left.\begin{array}{r} x-ay+a^2z=a^3 \\ x-by+b^2z=b^3 \\ x-cy+c^2z=c^3 \end{array}\right\}$

III. EASY PROBLEMS.

272. We shall now solve some problems which lead to Simultaneous Equations of one dimension with more than one unknown quantity.

Ex. 1. The sum of two numbers is 50, and their difference is one-third part of the greater number. Find them.

Let x = the greater number and y = the less.

By question, $x+y=50$ ....(1)
and $x-y=\frac{1}{3}x$ ....(2)

From (2), we obtain $3x-3y=x$; $\therefore$ $2x=3y$............(3)

Substitute (3) in (1) ; thus $\frac{3}{2}y+y=50$,

$\therefore$ $3y+2y=100$; $\therefore$ $5y=100$ and $\therefore$ $y=20$

Then, from (3) $x=\frac{3}{2}y=30$.

Hence the numbers are 30 and 20.

Ex. 2. A farmer sold to one person 9 horses and 7 cows for *Rs.* 3000, and to another, at the same prices, 6 horses and 13 cows for the same sum. What was the price of each? (B.M. 1871).

Let x = the number of rupees in the price of a horse,

and y = ... a cow.

Then 9 horses cost $9x$ *Rs.* and 7 cows cost $7y$ *Rs.*

Hence, $9x+7y=3000$ ....(1)
Similarly, $6x+13y=3000$ ....(2)

Multiply (1) by 2 and (2) by 3; thus

$18x+14y=6000$ ....(3)
$18x+39y=9000$ ....(4)

Subtract (3) from (4), $25y=3000$, $\therefore$ $y=120$.

Substituting in (1), $9x+840=3000$,

$\therefore 9x=2160$, and $\therefore x=240$.

Hence a horse costs *Rs.* 240 and a cow *Rs.* 120.

Ex. 3. Find the fraction which becomes equal to $\frac{1}{2}$ when the numerator is increased by 4, and equal to $\frac{1}{5}$ when the denominator is increased by 7.

Let x = the numerator and y = the denominator.

Then $\frac{x}{y}$ = the fraction.

By question, $\frac{x+4}{y}=\frac{1}{2}$ and $\frac{x}{y+7}=\frac{1}{5}$.

Clear the equations of fractions; thus we obtain

$$\left.\begin{aligned} 2x+8&=y \\ 5x&=y+7 \end{aligned}\right\}\begin{aligned}&\text{............(1)}\\&\text{............(2)}\end{aligned}$$

Subtract (1) from (2), $3x-8=7$; $\therefore 3x=15$ and $x=5$.

Substitute the value of x in (1):

thus, $10+8=y$, or $y=18$.

Hence the required fraction is $\frac{5}{18}$.

273. To represent algebraically numbers of more than one digit, we must remember that **45** means $\mathbf{4\times10+5}$, and not $\mathbf{4\times5}$. Hence the number, whose tens' digit is $\mathbf{x}$ and units' digit $\mathbf{y}$, is $\mathbf{10x+y}$, and not $\mathbf{xy}$, for $\mathbf{xy}$ denotes $\mathbf{x\times y}$.

Ex. 4. When a number consisting of two digits is divided by the sum of its digits the quotient is 4, and if 27 is added to the number the number is inverted. Find the number.

Let x = the digit in the tens' place,

and y = ...units'.

Then $10x+y$ = the number required, and

$10y+x$ = the number with the same digits inverted.

$$\left.\begin{aligned} &\text{By question, } \frac{10x+y}{x+y}=4 \\ &\text{and } 10x+y+27=10y+x \end{aligned}\right\}\begin{aligned}&\text{............(1)}\\&\text{............(2)}\end{aligned}$$

From (1), $10x+y=4x+4y$, or $6x=3y$; $\therefore 2x=y$............(3)

From (2), $9x-9y=-27$, or $x-y=-3$............(4)

Substitute (3) in (4), $x-2x=-3$; $\therefore x=3$.

From (3) $y=2x=6$.

Hence, the required number $=10\times3+6=36$.

Verification. The sum of the digits is $3+6$ or 9

Thus, $\frac{36}{9}=4$ and $36+27=63$. Q. E. D.

Ex. 5. Ten years ago a father was seven times as old as his son, two years hence twice his age will be equal to five times his son's. What are their present ages?

Let $x=$ the present age of the father in years,

and $y=$ .... son in years.

Ten years ago, the father's age was $(x-10)$ years and the son's age was $(y-10)$ years.

By question, $x-10=7(y-10)$... ......(1)

Again, two years hence, the father's age will be $(x+2)$ years and the son's $(y+2)$ years.

By question, $2(x+2)=5(y+2)$ ... (2)

$$\left.\begin{array}{ll}\text{From (1), we obtain} & x-7y=-60\\ \text{From (2),} & 2x-5y=6\end{array}\right\}\begin{array}{l}\ldots\ldots(3)\\ \ldots\ldots(4)\end{array}$$

Subtract twice (3) from (4), $9y=126$, and $\therefore y=14$.

From (3) $x=7y-60=98-60=38$.

Hence, father's present age is 38, and son's 14 years.

Verification. Ten years ago, the father was 28 and son 4 years; and $28=7\times4$.

Again, two years hence, the father will be 40 and son 16 years; and $2\times40=80=5\times16$. Q. E. D.

Ex. 6. A person spent 9*s.* in buying apples at the rate of 7*d.* per dozen, and oranges at 20 a shilling. If he had bought two-thirds as many apples and twice as many oranges, he would have had to pay 13*s.* 4*d.* How many of each did he buy?

Let $x=$ the number of apples,

and $y=$ ... . oranges.

The cost of each apple is $\frac{7}{12}d.$, and the cost of each orange is $\frac{12}{20}d.$ or $\frac{3}{5}d.$ Also $9s.=108d.$ and $13s.\ 4d.=160d.$

$$\left.\begin{array}{r}\text{By question, } \frac{7}{12}x+\frac{3}{5}y=108\\ \text{and } \frac{7}{12}\times\frac{2}{3}x+\frac{3}{5}\times2y=160\end{array}\right\}\begin{array}{l}\ldots\ldots(1)\\ \ldots\ldots(2)\end{array}$$

Multiply (1) by 2 and subtract (2) from it; we thus get $\frac{14}{12}x-\frac{14}{18}x=2\times108-160=56$; $\therefore x=72$.

Substituting x in (1), we have

$$42+\frac{3}{5}y=108;\ \therefore \frac{3}{5}y=66 \text{ and } \therefore y=110.$$

Hence, he bought 72 apples and 110 oranges.

Ex. 7. A sum of money was divided equally among a certain number of persons, had there been six more, each would have received a shilling less, and had there been four fewer, each would have received a shilling more than he did; find the sum of money and the number of men.

Let x = the number of persons,

and y = ... shillings which each received.

Then xy = the number of shillings in the sum of money which was divided.

$$\text{By question, } \left.\begin{aligned}(x+6)(y-1)&=xy\\ \text{and } (x-4)(y+1)&=xy\end{aligned}\right\}\begin{aligned}&\ldots\ldots(1)\\ &\ldots\ldots(2)\end{aligned}$$

From (1), we have $xy+6y-x-6=xy$;

$$\therefore\ 6y-x=6\ \ldots\ldots\ldots\ldots(3)$$

From (2), we have $xy-4y+x-4=xy$;

$$\therefore\ -4y+x=4\ \ldots\ldots\ldots\ldots(4)$$

From (3) and (4), by addition, $2y=10$; $\therefore\ y=5$.

Substitute the value of y in (3); thus $30-x=6$; $\therefore\ x=24$.

Hence, there were 24 men,

and sum divided was (24×5) shillings $=\pounds 6$.

Ex. 8 Two passengers have together 5 cwt. of luggage, and are charged for the excess above the weight allowed 5*s.* 2*d.* and 9*s.* 10*d.* respectively; if the luggage had all belonged to one of them, he would have been charged 19*s.* 2*d.* How much luggage is each passenger allowed to carry free of charge? And how much luggage had each passenger? (C. E. 1877).

Let x = the no. of cwt. of luggage the first passenger had:

then $5-x$ = ... second ... .

Also let y = no. of cwt of luggage each passenger is allowed to carry free of charge;

and let z = the charge per cwt. for excess luggage (in shillings).

Then, the excess luggage carried by the first passenger is $(x-y)$ cwt., for which he has to pay $(x-y)z$ shillings.

The excess luggage carried by the second passenger is $(5-x-y)$ cwt., for which he has to pay $(5-x-y)z$ shillings.

If the luggage had all belonged to one passenger, the excess luggage carried by him would have been $(5-y)$ cwt., for which he would have had to pay $(5-y)z$ shillings.

$$\text{By question, } \left.\begin{aligned}(x-y)z&=5\tfrac{1}{6}\\ (5-x-y)z&=9\tfrac{5}{6}\\ \text{and } (5-y)z&=19\tfrac{1}{6}\end{aligned}\right\}\begin{aligned}&\ldots\ldots(1)\text{, for } 5s.\ 2d.=5\tfrac{1}{6}s.\\ &\ldots\ldots(2)\text{, for } 9s.\ 10d.=9\tfrac{5}{6}s.\\ &\ldots\ldots(3)\text{, for } 19s.\ 2d.=19\tfrac{1}{6}s.\end{aligned}$$

Dividing (2) by (1), $\dfrac{5-x-y}{x-y}=\dfrac{9\frac{5}{6}}{5\frac{1}{6}}=\dfrac{59}{31}$;

$\therefore\ 31(5-x-y)=59(x-y) \ldots\ldots\ldots\ldots (4)$

Again, dividing (3) by (1), $\dfrac{5-y}{x-y}=\dfrac{19\frac{1}{6}}{5\frac{1}{6}}=\dfrac{115}{31}$;

$\therefore\ 31(5-y)=115(x-y) \ldots\ldots\ldots\ldots\ldots\ldots (5)$

Subtracting (4) from (5), $31x=56(x-y)$,

$\therefore\ 25x-56y=0 \ldots\ldots\ldots (6)$

From (4), $90x-28y=155$;

Multiplying by 2, $180x-56y=310 \ldots\ldots (7)$

Subtract (6) from (7); $155x=310$, and $\therefore\ x=2$.

Therefore from (6), $y=\frac{25}{56}x=\frac{50}{56}=\frac{25}{28}$.

Hence, the first passenger had 2 cwt. and the second 3 cwt.; and each passenger is allowed to carry free of charge $\frac{25}{28}$ cwt. or 100 lbs.

Exercise CIII.

1. The sum of two numbers is 124, and their difference 20. Find them.

2. The sum of two numbers is 100, and twice the less exceeds the greater by 5. Find the numbers.

3. If 7 yards of stuff and 17 yards of silk together cost *Rs.*133. 8*a.* and 12 yards of stuff and 7 yards of silk cost *Rs.*96, what are the prices per yard?

4. What fraction is that, to the numerator of which if 7 be added, its value is $\frac{2}{3}$; but if 7 be taken from the denominator, its value is $\frac{3}{8}$?

5. A bill of 25 guineas was paid with crowns and half-guineas; and twice the number of half-guineas exceeded three times that of the crowns by 17; how many were there of each?

6. Find a fraction such that if 1 be subtracted from its numerator, the value will be $\frac{3}{1}$ and if 6 be added to the denominator, the value will be $\frac{1}{2}$. (C. E. 1858).

7. A person has two horses and a saddle worth *Rs.*75; if the saddle be put on the *first* horse, his value becomes *double* that of the second; but if the saddle be put on the *second* horse, *his* value will not amount to that of the *first* horse by *Rs* 350. What is the value of each? (C. E. 1859).

8. *Rs.*1100 are so divided among **A**, **B** and **C**, that if **A** were to give **B** *Rs.*200, **B** would then have twice as much as **A**, and three times as much as **C**. How many rupees did **A**, **B** and **C** each receive originally? (C. E. 1872).

9. **A** says to **B** : Two-fifths of my salary is $\frac{4}{25}$ of yours, and the difference between our salaries is *Rs.*600. What is **A**'s salary? (A. E. 1894).

10. What fraction is that which, if 1 be added to the numerator, becomes 1, and if 1 be added to the denominator, becomes $\frac{1}{2}$? (C. E. 1862).

11. **A** and **B** received £5. 17*s.* for their wages, **A** having been employed 15, and **B** 14 days; and **A** received for working four days 11*s.* more than **B** did for three days: what were their daily wages?

12. A draper bought two pieces of cloth for *Rs.*126. 8*a.*, one being *Rs.*4 and the other *Rs.*4. 8*a.* per yard. He sold them each at an advanced price of *Re.*1 per yard, and gained by the whole *Rs.*30. What were the lengths of the pieces?

13. Find three numbers **A**, **B**, **C**, such that **A** with half of **B**, **B** with a third of **C**, and **C** with a fourth of **A**, may each be 1000.

14. A rectangular bowling-green having been measured, it was observed that, if it were 5 feet broader and 4 feet longer, it would contain 116 square feet more; but, if it were 4 feet broader and 5 feet longer, it would contain 113 square feet more. Find its present area.

15. The sum of three numbers is 24. The greatest exceeds the least by 4, and the other number is half the sum of the greatest and least. Find the numbers.

16. A certain resolution was carried in a debating society by a majority which was equal to one-third of the number of votes given on the losing side; but if with the same number of votes, 10 more votes had been given to the losing side, the resolution would only have been carried by a majority of one; find the number of votes given on each side. (B. M. 1889).

17. A shop keeper sold to one person 30 maunds of rice and 40 maunds of oats for *Rs.*135; to another person he sold 50 maunds of rice and 30 maunds of oats for *Rs.*170; find the price of rice and oats per maund.

18. Seven years ago **A** was three times as old as **B** was, and seven years hence **A** will be twice as old as **B** will be: find their present ages.

19. Four times **A**'s age exceeds **B**'s age by 16, and one-fifth of **A**'s age is equal to one-sixteenth of **B**'s age. Find their ages.

20. **A**'s age is twice **B**'s. Four years hence **B**'s will be twice **C**'s, and 12 years after that **A**'s will be twice **C**'s. Find their present ages.

21. Divide the numbers 80 and 90 each into two parts, so that the sum of one out of each pair may be 100, and the difference of the others 30.

22. The sum of the two digits of a certain number is six times their difference, and the number itself exceeds six times their sum by 3 : find it.

23. There is a number of two digits, which, when divided by their sum, gives the quotient 4 ; but if the digits be inverted, and the number thus formed be increased by 12, and then divided by their sum, the quotient is 8. Find the number. (M. M. 1858).

24. The united ages of a man and his wife are six times the united ages of their children. Two years ago, their united ages were ten times the united ages of the children, and six years hence their united ages will be three times the united ages of the children. How many children have they? (B. M. 1891).

25. The dimensions of a rectangular court are such that if the length were increased by 3 yards, and the breadth diminished by the same, its area would be diminished by 18 square yards, and if its length were increased by 3 yards, and its breadth increased by the same, its area would be increased by 60 square yards ; find the dimensions. (C. E. 1888).

26. A person spent *Rs.*19 in buying mangoes at *Rs.*6 per hundred and apples at *Re.*1. 8*a.* per dozen ; if he had bought three times as many mangoes and a quarter of the number of apples, he would have spent *Rs.*29. 8*a.* How many of each did he buy?

27. There is a number, the sum of whose digits is 5, and if 10 times the digit in the place of tens be added to four times the digit in the place of units, the number will be inverted. What is the number? (C. E. 1868).

28. A certain number consisting of two digits becomes 110 when the number obtained by reversing the digits is added to it ; also the first number exceeds unity by five times the excess of the second number over unity. What is the number? (B. M. 1884).

29. A number consists of two digits. When the number is divided by the sum of the digits, the quotient is 7. The sum of the reciprocals of the digits is nine times the reciprocal of the product of the digits. Find the number. (M. M. 1887).

30. Reverse the digits of a number and it will become five-sixths of what it was before ; also the difference between the two digits is one. Find the number. (C. E. 1883).

31. There are three numbers, such that the *sum* of the first and second divided by their *product* is $\frac{1}{2}$; the sum of the second and

third divided by their product is $\frac{1}{3}$; and the sum of the first and third divided by their product is $\frac{1}{4}$. Find the numbers. (C. E. 1859).

32. A certain company in a tavern found, when they came to pay their bill, that if there had been three more persons to pay the same bill, they would have paid one shilling each less than they did; and if there had been two fewer persons they would have paid one shilling each more than they did; find the number of persons and the number of shillings each paid.

33. A grocer bought tea at 10*s.* per lb. and coffee at 2*s.* 6*d.* per lb., to the amount altogether of £31. 5*s.*: he sold the tea at 8*s* per lb. and the coffee at 4*s.* 6*d.* per lb., and gained £5 by the bargain: how many lbs. of each did he buy?

34. A train left Calcutta for Allahabad with a certain number of passengers, 40 more second-class than first-class; and 7 of the former would pay together *Re.*1 less than 4 of the latter. The fare of the whole was *Rs.*550. But they took up, half-way, 35 more second-class and 5 first-class passengers, and the whole fare now received was $\frac{1}{4}$ as much again as before. What was the first-class fare, and the whole number of passengers at first?

35. A number consists of three digits whose sum is 10. The middle digit is equal to the sum of the other two; and the number will be increased by 99, if its digits be reversed. Find the number. (B. M. 1888).

36 Find that number of three digits which is the same when reversed, and the sum of whose digits is 16 and the difference 2. (C. E. 1883).

37. Two men start from two places 48 miles apart. When they travel in opposite directions they meet in 4 hrs. 48 min.; when they travel in the same direction, one overtakes the other in 9 hrs. 36 min. Find their rates of travelling.

38. If £2. 11*s.* 6*d.* is paid in florins and half-crowns, the number of coins being 24, how many are there of each?

39. What fraction is that which becomes equal to one-half or one-third, according as its numerator and denominator are both increased by 2 or both diminished by 2?

40. A sum of £12. 18*s.* might be distributed to the poor of a parish by giving $\frac{1}{2}$ a crown to each man and 1*s.* to each woman and each child, or $\frac{1}{3}$ a crown to each woman and 1*s.* to each man and each child, or $\frac{1}{4}$ a crown to each child and 1*s.* to each man and each woman; how many were there in all?

41. A person spends *Re.*1. 14*a.* in apples and pears, buying the apples at four, and the pears at five an anna; and afterwards accommodates a friend with half his apples and a third of his pears for 13 annas. How many of each did he buy?

42. A party was composed of a certain number of men and women, and, when four of the women were gone, it was observed that there were left just half as many men again as women: they came back, however, with their husbands, and now there were only a third as many men again as women. What were the original numbers of each?

43. Having *Rs.*45 to give away among a certain number of persons, I find that if I give *Rs.*3 to each man and *Re.*1 to each woman, I shall have *Re.*1 too little, but that, by giving *Rs.*2. 8*a.* to each man and *Re.*1. 8*a.* to each woman, I may distribute the sum exactly. How many were there of men and women?

44. A party at a tavern, having to pay their reckoning, and being a third as many men again as women, agree that each man shall pay half as much again as each woman, but, a man and his wife having gone off without paying their share, 10*d.*, the rest had each to pay 2*d.* more. What was the reckoning?

45. A and B lay a wager of *Rs.*10; if A loses he will have as much as B will then have, if B loses he will have half of what A will then have: find the money of each.

46. A man receives *Re.*1. 12*a.* for every day that he works, but is fined 8*a.* for every day that he is absent. After 20 days he receives the same wages that he would have earned by steadily working for 11 days. How many days was he absent from work?

47. A traveller walks a certain distance. Had he gone half-a-mile an hour faster, he would have walked it in four-fifths of the time; had he gone half-a-mile an hour slower, he would have been $2\frac{1}{2}$ hours longer on the road. Find the distance, and his rate of walking.

48. In going the shortest way from A to B, a man had to go back one mile to pick up something he had dropped, and took $3\frac{1}{2}$ hours over the walk. He went back by a route which was half-a-mile longer and took 3 hours over the return walk. Find his rate of walking, and the shortest distance from A to B.

49. The sum of the digits of a certain number, less than 100, is 11 and if the digits are reversed, the number is diminished by 9. Find the number.

50. A man does a journey in a motor car at a uniform speed in 6 hours. On his return he is delayed at half-way for half-an-hour, but quickening his pace by 3 miles an hour does the journey in the same time. Find his original speed and the length of the journey.

CHAPTER X.

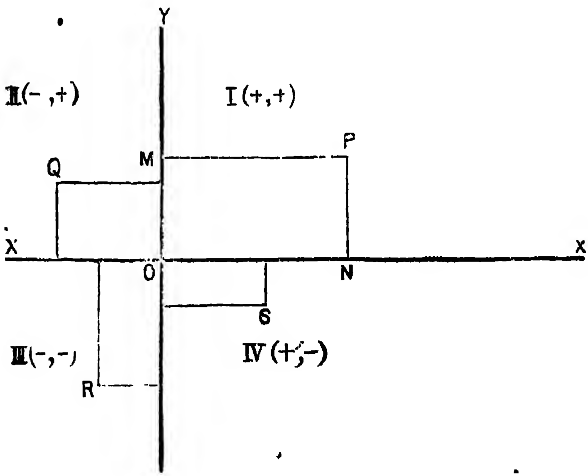
CO-ORDINATES AND GRAPHS.

I. AXES OF CO-ORDINATES.

274. Let **XOX′**, **YOY′** be two straight lines cutting at right angles to each other in **O**, thus dividing the plane in which they are into four spaces **XOY**, **YOX′**, **X′OY′**, **Y′OX**, which are called the first, second, third and fourth **quadrants** respectively. Take a point **P** in their plane, and draw **PN**, **PM** perpendicular to **XOX′** and **YOY′** respectively.

Let **PM** or **ON** $= x$, and **PN** or **OM** $= y$.

Then, for this point **P**, x and y are definitely fixed; and *conversely*, when x and y are given, the position of the point **P** is definitely determined as the point of intersection of the perpendiculars **NP** and **MP**.



The numbers x and y are called the **co-ordinates** of the point **P**. The lines **XOX′**, **YOY′** are the **axes of co-ordinates**, or more briefly, the **axes**; they are taken as lines of reference and are called the **axis of x and y** respectively. The point **O** is called the **origin**, and is the point (o, o).

x is called the **abscissa**, and y the **ordinate** of the point **P**, and **P** is briefly described as "the point (x, y)." In thus describing

the position of a point the first co-ordinate is the abscissa and the second the ordinate.

275. The values of x are measured from **O** along the axis of x, according to some convenient scale of measurement These values are *positive* when drawn to the *right* of **O** along **OX** and *negative* when drawn to the *left* of **O** along **OX'**. The values of y are *positive* when drawn *above* **XX'** and *negative* when drawn *below* **XX'**.

Thus. if the co-ordinates of a point be given by $x=6$, and $y=4$; mark off **ON** $=6$ units of length along **OX**, and **OM** $=4$ units of length along **OY** and draw through **N** and **M** straight lines parallel to the axes meeting at a point **P**. Then **PM** $=$ **ON** $=6$ and **PN** $=$ **OM** $=4$, and therefore the position of the point **P** is determined Similarly, any pair of corresponding values of x and y will determine a point relatively to the axes.

276. The process of marking the position of a point on the diagram by means of its co-ordinates is called **plotting the point**. This process is made very easy by using *squared paper*, as shewn in the following Examples.

Ex. 1. Plot the points

(i) **A** (5,4); (ii) **B** $(-5,5)$; (iii) **C** $(-5,-3)$; (iv) **D** $(3,-5)$.

(i) To plot **A** move 5 units to the *right*, then *up* 4; the resulting point is in the first quadrant.

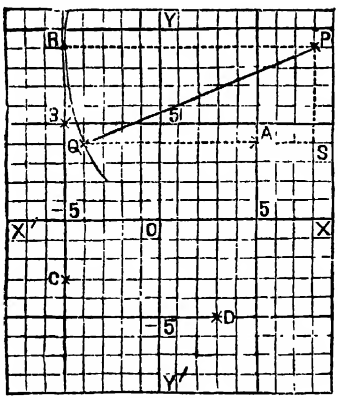


Fig. 1.

(ii) To plot **B** move 5 units to the *left*, then *up* 5; the resulting point is in the second quadrant.

(iii) To plot **C** move 5 units to the *left*, then *down* 3; the resulting point is in the third quadrant.

(iv) To plot **D** move 3 units to the *right*, then *down* 5; the resulting point is in the fourth quadrant.

Ex. 2. Plot the points **P** (8, 9), **Q** (−4, 4); and find the distance between them. [Fig. 1.

Beginning from **O** move 8 units to the *right*, then *up* 9; this is the point **P**.

Move 4 units to the *left*, then *up* 4; this is the point **Q**.

With centre **P** and radius **PQ** describe a circle cutting the horizontal line through **P** at the point **R**.

The length reqd. = **PQ** = **PR** = 13 units, from the diagram.

Or we may proceed thus :—

Draw through **Q** a line parallel to **XX′** to meet the ordinate of **P** at **S**. Then **PSQ** is a right-angled △ in which **QS** = 12, and **PS** = 5.

Now $PQ^2 = PS^2 + QS^2 = 5^2 + 12^2$ [Fig. 1.

$= 25 + 144 = 169$; $\therefore$ $PQ = 13$.

Ex. 3. Plot the points **A** (8, 12), **B** (−7, 12), **C** (−7, −6), **D** (8, −6); find lengths of the sides and the area of the quadrilateral **ABCD**.

After plotting the points as in the diagram below, we clearly see that **ABCD** is a rectangle. **BA**, **CD** are each 15 units and **DA**, **CB** are each 18 units.

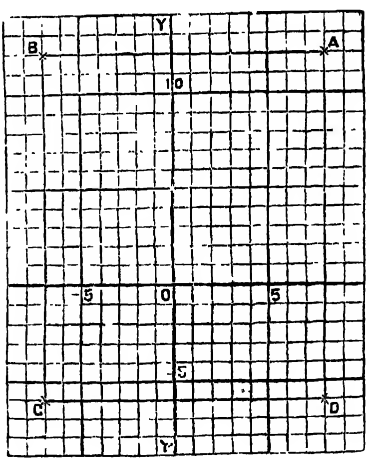


Fig. 2.

The rectangle is divided by the horizontal lines into 18 strips, and each strip contains 15 small squares; the area of **ABCD** is therefore 18×15 or 270 times the area of a small square.

In the diagram, since one division in the paper is one-tenth of an inch, therefore, the number 15, which gives the length of **BA** or **CD**, represents 15 tenths of an inch; **BA**, **CD** are therefore $1{\cdot}5''$. Similarly, **DA**, **CB** are $1{\cdot}8''$.

The area of a small square is one-hundredth of a square inch; therefore the area of **ABCD** is 270 hundredths of a square inch, that is, $2{\cdot}7$ square inches.

Ex. 4. Plot the points **A** $(4, 8)$, **B** $(9, -5)$, **C** $(-7, -5)$; find the area of the triangle formed by joining these points.

Plotting the points as shewn in the diagram below, we find that $\mathbf{BC} = 16$ units, and **AD**, the perpendicular from **A** on $\mathbf{BC} = 13$ units. Hence,

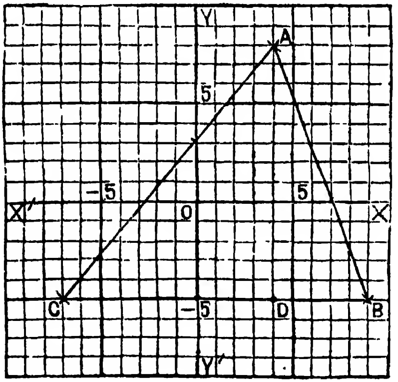


Fig. 3.

$$\text{Area of the } \triangle\ \mathbf{ABC} = \tfrac{1}{2}\ \mathbf{BC} \times \mathbf{AD} = \tfrac{1}{2} \times 16 \times 13 \text{ square units}$$
$$= 104 \text{ square units} = 1{\cdot}04 \text{ square inches.}$$

Ex. 5. Plot the points **A** $(1{\cdot}7, 0{\cdot}6)$, **B** $(-0{\cdot}9, 1{\cdot}6)$, **C** $(-1{\cdot}5, -0{\cdot}4)$; **D** $(0{\cdot}8, -0{\cdot}9)$ and find the area of the quadrilateral **ABCD**. (Scale $1'' = 1$).

Plotting the points as directed and drawing lines parallel to the axes (as shewn in the diagram below by dotted lines), the quadrilateral **ABCD** is divided into four right-angled triangles and a rectangle.

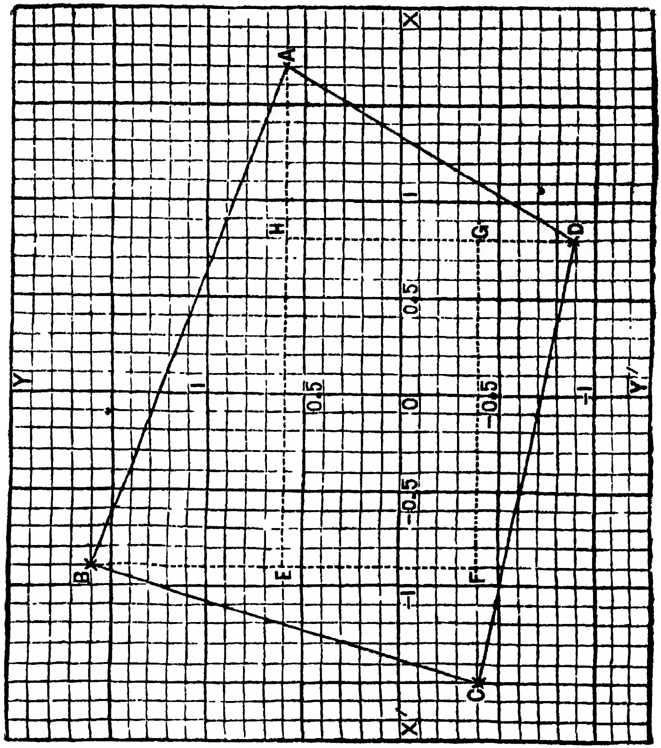


Fig. 4.

$\triangle$ ABE $= \frac{1}{2}$ AE $\times$ BE $= \frac{1}{2} \times 2{\cdot}6 \times 1 = 1{\cdot}3$ square inches.

$\triangle$ BFC $= \frac{1}{2}$ CF $\times$ BF $= \frac{1}{2} \times {\cdot}6 \times 2 = {\cdot}6$,, ,,

$\triangle$ CDG $= \frac{1}{2}$ CG $\times$ DG $= \frac{1}{2} \times 2{\cdot}3 \times {\cdot}5 = {\cdot}575$,, ,,

$\triangle$ DAH $= \frac{1}{2}$ AH $\times$ DH $= \frac{1}{2} \times {\cdot}9 \times 1{\cdot}5 = {\cdot}675$,, ,,

rect. EFGH $=$ EF $\times$ EH $= 1 \times 1{\cdot}7 = 1{\cdot}7$,, ,,

$\therefore$ the area of ABCD $= 4{\cdot}85$ square inches.

Exercise CIV.

1. Plot the following points on squared paper :—

$(7, 8), (0, 9), (-8, 9), (3, -8), (-5, -3), (-5, 5)$.

2 Plot the following pairs of points and draw the line which joins them :—

(1) $(7, 6), (3, -8)$. (2) $(-3, 5), (-5, 3)$.

(3) $(-6, 7), (3, -8)$. (4) $(6, 8), (-2, -4)$.

(5) $(-2, 0), (0, -8)$. (6) $(0, 0), (-8, -10)$.

3. Plot the points $(5, 2), (5, 1), (5, -2), (5, -4), (5, -3)$ and shew that they all lie on a straight line parallel to the axis of **Y**.

4. Plot the points $(8, 12), (-7, 12), (-7, -6), (8, -6)$ and find the sides and area of the figure formed by joining the points in succession.

5. Plot the points $(3, 4), (3, -4), (-3, 4), (-3, -4)$. Determine the number of square units in the area of the figure formed by joining them.

6. Plot the following pairs of points, and determine the co-ordinates of the mid. points of the lines joining each pair :—

(1) $(3, 4), (3, -4)$. (2) $(4, 3), (12, 7)$.

(3) $(-8, 0), (0, -10)$. (4) $(-3, 5), (-5, 3)$.

7. Plot the following points, and calculate their distances from the origin :—

(1) $(6, 8)$. (2) $(-15, -8)$. (3) $(-7, 24)$. (4) $(3, -8)$.

8. Plot the following pairs of points, and in each case find the distance between them :—

(1) $(9, 8), (-10, 19)$. (2) $(15, 0), (0, 8)$.

(3) $(15, -12), (-15, 4)$. (4) $(10, 4), (-5, 12)$.

(5) $(0, 0), (15, 20)$. (6) $(20, 8), (-15, 0)$.

Verify your results by measurement.

9. Find the perimeter of the triangle formed by joining the points $(7, 9), (-11, 20), (-17, -5)$.

10. Draw the figure whose angular points are given by

$(13, 0), (-13, -15), (15, -15), (15, 0)$.

Find the lengths of its sides and the area of the figure.

11. Plot the three points in each of the following examples and find in each case the area of the triangle of which the three points are the vertices :—

(1) $(-15, -15)$, $(15, -15)$, $(0, 10)$. (2) $(12, 14)$, $(-14, 4)$, $(12, -8)$.
(3) $(10, 5)$, $(-6, 5)$, $(6, 17)$. (4) $(13, 0)$, $(0, 8)$, $(13, 8)$.

12. Plot the following points :—(scale $1'' = 1$).

$(1{\cdot}5, 2{\cdot}5)$; $(-3{\cdot}2, -1{\cdot}3)$; $(-2{\cdot}3, 1{\cdot}4)$; $(2{\cdot}1, -1{\cdot}6)$.

13. Plot the following four points in each case and find the sides and area of each quadrilateral.

(1) $(2{\cdot}7, 3)$, $(0{\cdot}4, 3)$, $(0{\cdot}4, -1{\cdot}2)$, $(2{\cdot}7, -1{\cdot}2)$.
(2) $(1{\cdot}8, 1{\cdot}3)$, $(-2{\cdot}4, 1{\cdot}3)$, $(-2{\cdot}4, -0{\cdot}7)$, $(1{\cdot}8, -0{\cdot}7)$.

14. Plot the two following series of points :—

(i) $(5, 0)$, $(5, 2)$, $(5, 5)$, $(5, -1)$, $(5, -4)$.
(ii) $(-4, 8)$, $(-1, 8)$, $(0, 8)$, $(3, 8)$, $(6, 8)$.

Shew that they lie on two lines respectively parallel to the axis of y, and the axis of x. Find the co-ordinates of the point in which they intersect.

15. Plot the following five points and shew experimentally that they lie on a straight line.

$(0, 10)$, $(1, 12)$, $(3, 16)$, $(-2, 6)$, $(-5, 0)$.

16. Plot the points $(15, 0)$, $(19, 6)$, $(10, 14)$, $(-14, 8)$ and find the area of the quadrilateral formed by joining them.

17. Find the area (in squares of your paper) of the figures formed by joining the following points :—

(1) $(0, 0)$, $(17, 0)$, $(0, 12)$. (2) $(13, 0)$, $(0, 8)$, $(13, 8)$.

18. Plot the points $(3, 4)$, $(4, 8)$. Join them, and write down the abscissæ of the points on this line whose ordinates are respectively 0 and 12. Write down also the ordinates of the points whose abscissæ are respectively $1{\cdot}5$ and $3{\cdot}5$.

II. GRAPHS OF STRAIGHT LINES.

277. Any expression involving x is called a *function* of x and is usually denoted by $f(x)$. If y represent its value, then, from the equation $y = f(x)$, by giving to x a series of numerical values (generally increasing by small differences) we may obtain a corres-

ponding series of values for y. Now, if these values of x and y be marked off as abscissæ and ordinates respectively, we can plot a series of points in succession. By joining *all* these points in succession we shall obtain a line, either straight or curved, which is called the **graph** of the *function* $f(x)$, or the **graph** (properly **locus**) of the *equation* $y=f(x)$. Thus, the graph of the *function* $3x+4$ is the same as the graph or locus of the *equation* $y=3x+4$.

278. It is worth while to notice here *that all graphs of linear functions, i.e., graphs obtained from equations of the first degree, are straight lines.* The following points, as regards graphs of straight lines, are very important and should be committed to memory.

(i) *The co-ordinates of the origin are* (**o**, **o**).

(ii) *If a point lies on the axis of* x, *its ordinate is* **o**.

(iii) *If a point lies on the axis of* y, *its abscissa is* **o**.

Thus, the graph of $\boldsymbol{x=o}$ is the axis of $\boldsymbol{y}$; and the graph of $\boldsymbol{y=o}$ is the axis of $\boldsymbol{x}$.

(iv) The graph of $\boldsymbol{x=a}$, where $\boldsymbol{a}$ is constant, is a straight line parallel to the axis of $\boldsymbol{y}$ and at a distance a from the axis of y.

(v) The graph of $\boldsymbol{y=b}$, where $\boldsymbol{b}$ is constant, is a straight line parallel to the axis of x and at a distance b from the axis of x.

The student should illustrate (iv) and (v) by drawing graphs of $x=5$, $x=-8$ and so on ; and also by drawing graphs of $y=3$, $y=-7$ and so on.

Ex. 1. Draw the graph of $y=x$.

When $x=0$, $y=0$; *thus the origin is one point on the graph.*

Also, when $x=1, 2, 3, \ldots\ldots\ldots -1, -2, -3, \ldots\ldots\ldots$

$y=1, 2, 3, \ldots\ldots\ldots -1, -2, -3, \ldots\ldots\ldots$

Thus the graph passes through O, and represents a series of points each of which has its ordinate equal to its abscissa, and is clearly represented by the straight line POP′ in Fig. 6.

Ex. 2. Draw the graph of $y=3x$.

Tabulate the values of x and y as follows :—

$x=$	3	2	1	0	-1	-2
$y=$	9	6	3	0	-3	-6

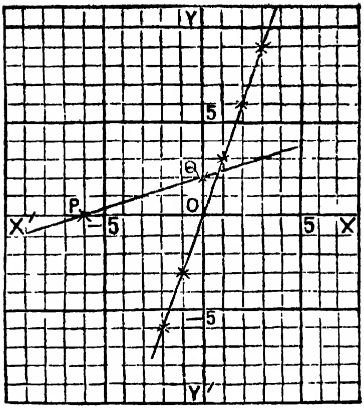


Fig. 5.

On joining the points thus found, the required graph will be a straight line of an unlimited length through **O** the origin, as shown in Fig. 5.

Ex. 3. Plot the graph of $y=x-4$.

Tabulate the values of x and y as follows :—

$x=$	5	**4**	3	2	1	**0**	−1	−2	..
$y=$	1	**0**	−1	−2	−3	**−4**	−5	−6	..

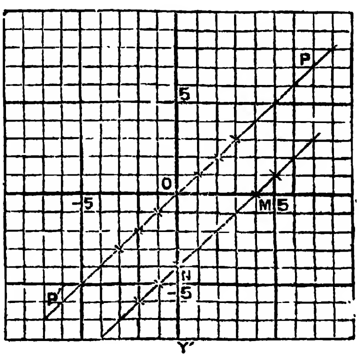


Fig. 6.

Joining these points, we obtain the line **MN**, parallel to that in **Ex. 1.** as shown in Fig. 6. The distances **ON**, **OM** (usually called the ***intercepts on the axes***) are obtained by separately putting $x=0$, $y=0$ in the equation of the graph. The value of y obtained by putting $x=0$ gives the intercept cut off from the straight line **OY**, while the value of x obtained by putting $y=0$ gives the intercept cut off from the straight line **OX**.

Note.—The student should notice that he could have saved the trouble of plotting the positions of several points if he could find the positions of only two particular points (which it is generally easier to find) namely the points where the graph cuts the axes. Because as it is known that the graph of a *linear equation* is a straight line, only two points in it will suffice to determine the whole straight line, since all that he will then have to do is to join these two points and to produce the join indefinitely both ways. (Art. 281).

Ex. 4. Draw the graph of the expression $2x+3$.

Let $y=2x+3$.

When

$x=$	-6	-3	-1	0	1	3	4
$y=$	-9	-3	1	3	5	9	11

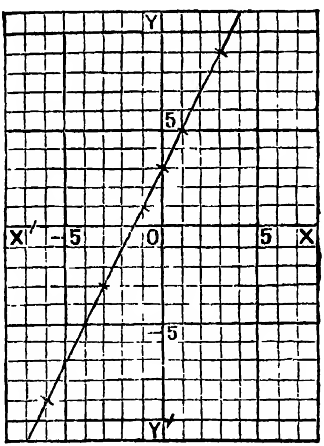


Fig. 7.

Now plotting the points, we notice that they lie in a straight line. This straight line, produced indefinitely both ways, is the required graph, as shown in Fig. 7.

Ex. 5. Draw the graphs represented by the equations :—

(i) $3y=4x$. (ii) $3y=4x+6$. (iii) $4y+3x=8$.

Putting the equations in equivalent forms, we have

(i) $y=\dfrac{4x}{3}$. (ii) $y=\dfrac{4x}{3}+2$. (iii) $y=2-\dfrac{3x}{4}$.

In (i) and (ii) find values of y corresponding to

$$x=-3, -2, -1, 0, 1, 2, 3,$$

and in (iii) find values of y corresponding to

$$x=-2, -1, 0, 1, 2, 3.$$

Thus, we have the following values of y :—

In (i) $y=-4, -2\frac{2}{3}, -1\frac{1}{3}, \ 0, \ 1\frac{1}{3}, \ 2\frac{2}{3}, \ 4.$

In (ii) $y=-2, -\frac{2}{3}, \ \frac{2}{3}, \ 2, \ 3\frac{1}{3}, \ 4\frac{2}{3}, \ 6.$

In (iii) $y= 3\frac{1}{2}, \ 2\frac{3}{4}, \ 2, \ 1\frac{1}{4}, \ \frac{1}{2}, \ -\frac{1}{4}.$

In plotting the corresponding points it will be found convenient to take *three* divisions of the paper as our unit in (i) and (ii) and *four* divisions as our unit in (iii).

The graphs are given in Fig. 8 below.

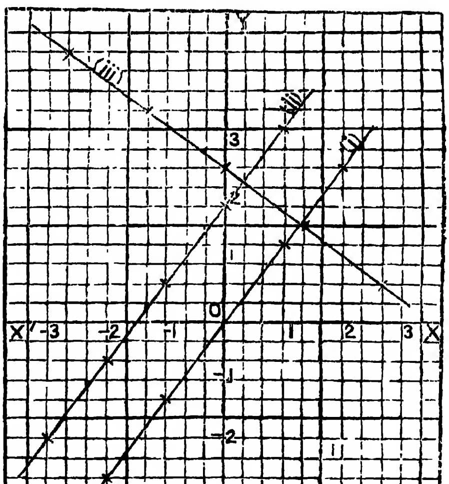


Fig. 8.

Exercise CV.

1. Trace the graphs corresponding to the following equations :—

(1) $y=2x$. (2) $y=2x+4$. (3) $y=2x-4$. (4) $y=-2x$.
(5) $y=x+6$. (6) $y-x=5$. (7) $y+x=0$. (8) $y=-2x+3$.
(9) $5y=9x$. (10) $y=2x+1$. (11) $7x-3y=0$. (12) $y=-2x-3$.
(13) $3y=6x+5$. (14) $x-3y=6$. (15) $3x+4y=0$.
(16) $3x+4y=12$. (17) $3x-2y=4$. (18) $4x-2y+5$
(19) $4x+y=9$ (20) $2y=6+x$. (21) $6y=3x-5$.

2. Draw the graphs of $3x-4y=5$ and $4x+3y=7$ and shew that they intersect at right angles.

3. Find the area included by the graphs of

$$y=x+4, y=x-4, y=-x+4, y=-x-4,$$

taking one-tenth of an inch as the unit of length.

4. What is the locus of a point in the following cases :—

(i) when its x is always -4? (ii) when its y is always -4?

5. Draw the graphs of the following equations :—

$$x+y=5,\ 2x-y=10,\ 2x+3y=-30,\ 3y-x=15.$$

If the paper is ruled to tenths of an inch, find the area of the space enclosed by these lines.

---

279. Equation of a Straight Line. Every equation of the first degree involving x and y only *represents a straight line.* Its most general form is $ax+by+c=0$, and is said to be a **linear equation**.

280. As the equation $ax+by+c=0$ can be reduced to either of the forms $y=ax$ or $y=ax+b$, it follows that

(i) for all numerical values of a the equation $y=ax$ represents a straight line passing through the origin ;

(ii) for all numerical values of a and b the equatlon $y=ax+b$ represents a straight line parallel to that given by $y=ax$. As the latter passes through the origin, the former lies b units above it (the distance between the lines being measured along the axis of y) when b is positive, but below it when b is negative.

In either case, a is called the **gradient** or **slope** of the line.

281. As a *linear equation* always represents *a straight line*, and as only one straight line can be drawn through two given points, we need only determine **any** two convenient points and the graph is the straight line joining them. Moreover, the student should carefully notice the fact that *a point does or does not lie on a graph*

according as its co-ordinates do or do not satisfy the equation of the graph.

Ex. 1. Shew that the points $(-2, 10)$, $(-1, 5)$, $(2, -10)$ lie on a straight line, and find its equation.

Let $y=ax+b$ be the required equation. As it passes through the first two given points, their co-ordinates satisfy the above equation.

Substituting $x=-2, y=10$, we have

$$10=-2a+b \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Again, substituting $x=-1, y=5$, we have

$$5=-a+b \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Now, solving equations (i) and (ii), we get

$$a=-5, \; b=0.$$

Hence $\qquad y=-5x$ or $y+5x=0$,

is the equation of the line passing through the first two points.

Since $x=2, y=-10$ satisfies this equation, the line also passes through $(2, -10)$.

282. Since a straight line can be drawn when *any* two points on it are given, sometimes we can conveniently draw a *linear* graph from the equation of a line by marking its intercepts on the axes, which may readily be found by putting $x=0$, $y=0$, successively in the equation.

Ex. 2. Draw the graph of $3y-x=6$.

For the intercepts on the axes, we have

when $y=0$, $x=-6$ (intercept on the x-axis),

and when $x=0$, $y=2$ (intercept on the y-axis).

Hence, the graph can now be drawn by joining the points $P(-6, 0)$, $Q(0, 2)$, as shown in Fig. 5.

283. Measurement on Different Scales. We have hitherto measured abscissæ and ordinates on the same scale, but points have often to be plotted whose co-ordinates differ considerably in magnitude. In such cases the plotting of points on the same scale requires either a very small unit length or a very large diagram. To obviate these, it will be found convenient to measure the variables namely x and y on different scales, but before making our choice we should find out as far as possible the greatest numbers that have to be represented.

Ex. 3. Draw the graph of $y=13x+6$.

When x has the values	$-1,\ 0,\ 1,\ 2,\ 3,$
the corresponding values of y are	$-7,\ 6,\ 19,\ 32,\ 45.$

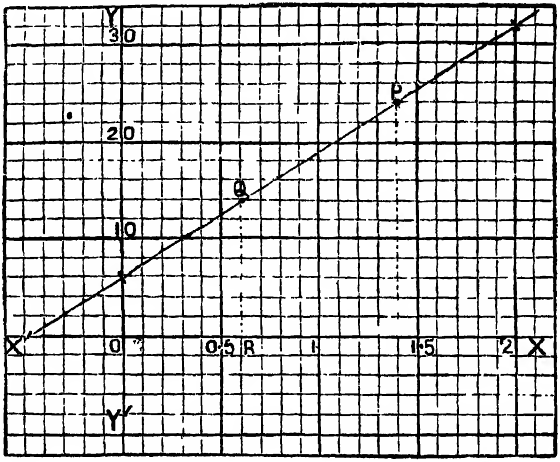


Fig. 9.

Thus, we see that some of the ordinates are much larger than the abscissæ, and rapidly increase as x increases. Here, equal horizontal and vertical units would give an inconvenient representation. To obviate this difficulty, take 1 inch along **OX** as the x-unit but let 1 inch along **OY** count as 20 y-units. The required graph is shown in Fig. 9 above, where the line has been drawn by joining the points (0, 6), (2, 32).

284. Interpolation. If one co-ordinate of an intermediate point on a graph accurately drawn from its plotted points be given, we can determine (without calculation) its other co-ordinate by measurement; but, in some cases, the results so obtained will only be approximate.

Ex. 1. From the graph of the function $13x+6$, find its value when $x=1{\cdot}4$; also find for what value of x the function becomes equal to 14.

Put $y=13x+6$, then the required graph is that given in Fig. 9.

Now we see that $x=1{\cdot}4$ at the point **P** and here $y=24$, nearly.

Again, $y=14$ at the point **Q**; and $x=$ **OR** $=0{\cdot}61$ approximately.

Exercise CVI.

1. Find the equations of the straight lines through the following pairs of points :—

(1) (5, 6), (−5, −3). (2) (3, 4) (−2, 6). (3) (6, 7), (−3, 7).
(4) (8, −1·5), (10, −3). (5) (−4, 0), (−2, 3). (6) (·5, −4), (3·2, 1·4)
(7) (−2, 11), (6, −5). (8) (6, −4), (−7, −3).

2. Shew that the three points (3, −1), (−2, 4), (5, −3) are in a straight line, and find the equation of the line.

3. Find the equation of the graph which passes through the points (0, 4), (−1, 1), (·3, 4·9), (2, 10), (·8, 6·4).

4. Find, without drawing the line, which, if any, of the points (3, 2), (4, 3), (−2, −2), (8, 6), (5, −4), lie on the line given by the equation $4x-5y=2$.

5. Draw the graph of the function $\frac{26-2x}{5}$. From the graph find the value of the function when $x=2·4$; also find for what value of x the function becomes equal to 8.

6. Draw the graph of $\frac{3x-7}{6}$ From the graph find the value of the function when $x=3·5$; also find for what value of x the function becomes equal to 1·1.

7. Find, without drawing the line, which of the given points (0, 2), (−4, −4), (4, 3), (2, 5), (4, 8) lie on the straight line represented by the equation $3x-2y+4=0$.

8. Find the equation of the graph which passes through the points (0, −5), (·5, −4), (1, −3), (3, 1), (3·2, 1·4), (3·6, 2·2).

9. What are the equations of the lines forming the sides of the triangle whose angles are at the points (1, 3), (2, −1), (−3, 4)?

10. By careful plotting and measurement find the length of the perpendicular drawn from

(1) the point (4, 5) upon the straight line $3x+4y=10$.

(2) the origin upon the straight line $\frac{1}{3}x-\frac{1}{4}y=1$.

III. APPLICATION TO SIMULTANEOUS EQUATIONS.

285. If two simultaneous equations between x and y be given, draw the graph of each and the co-ordinates of the point at which these graphs meet, will give a pair of values which will satisfy both equations.

Ex. 1. Solve graphically the equations :—

$$\text{(i) } x+2y=12, \qquad \text{(ii) } x-3y=2$$

In (i) when $x=0, y=6$; when $x=4, y=4$.

Thus the graph is the line joining P (0, 6) and P′ (4, 4).

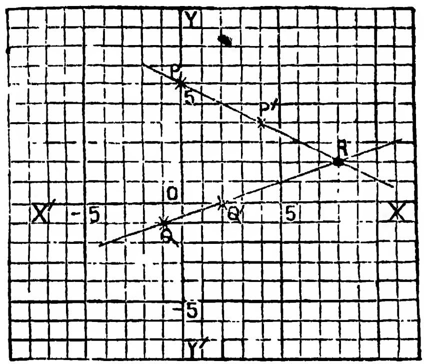


Fig. 10.

In (ii) when $x=-1, y=-1$; when $x=2, y=0$.

Thus the graph is the line joining Q $(-1,-1)$ and Q′ (2, 0).

Now, we see from the diagram that these lines intersect at the point R whose co-ordinates are 8, 2. Thus the solution of the given equations is $x=8, y=2$.

Verification. In the first equation, when

$$x=8,\ 8+2y=12,\ \therefore\ 2y=4,\ \therefore\ y=2.$$

$\therefore\ x=8, y=2$ satisfy the equation.

In the second equation, when $x=8$,

$$8-3y=2,\ \therefore\ -3y=-6,\ \therefore\ y=2.$$

$\therefore\ x=8, y=2$ satisfy this equation also.

Ex. 2. Draw the graphs of

$$\text{(i) } 4x=3y, \qquad \text{(ii) } y=5x-11, \qquad \text{(iii) } 5y=x+17 ;$$

and shew that they represent three straight lines which pass through one point. Find its co-ordinates.

In (i) when $x=0, y=0$; when $x=6, y=8$, and the graph is the line joining O (0, 0) and P (6, 8).

In (ii) when $x=1$, $y=-6$; when $x=2$, $y=-1$, and the graph is the line joining **Q** $(1,-6)$ and **Q′** $(2,-1)$.

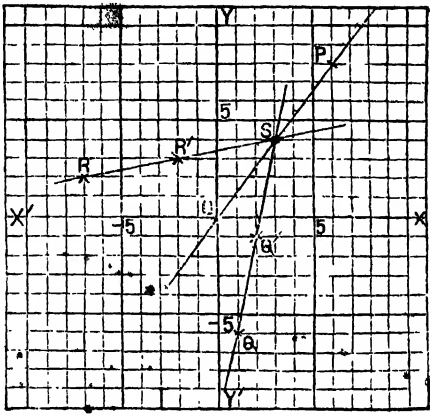


Fig. 11.

In (iii) when $x=-2$, $y=3$; when $x=-7$, $y=2$ and the graph is the line joining **R** $(-7, 2)$ and **R′** $(-2, 3)$.

Now, from the diagram, we see that these three straight lines all pass through the point **S** whose co-ordinates are 3, 4.

Exercise CVII.

1. Solve the following equations graphically, and verify your result by Algebra :—

(1) $3x-2y=4$, $5x+4y=14$.	(2) $4y=3x$, $4x-3y=14$.	(3) $x-2y+11=0$, $2x-3y+18=0$.
(4) $3x-2y=12$, $5x-7y=20$.	(5) $4x+y=10$, $3x-4y=17$.	(6) $2x+y=-1$, $8x+6y=3$.
(7) $5x+6y=60$, $2x-y=7$.	(8) $3y=4x$, $y+x=21$.	(9) $3x-2y=2$, $20x+24=25y$.
(10) $2x+3y=45$; $5x+4y=74$.	(11) $3x-4y=12$, $5x+2y=46$.	(12) $4x+3y=43$, $3x-2y=11$,

(13) $\frac{x+y}{3}=2+2y,\ \frac{2x-4y}{5}=\frac{23}{5}-y.$

(14) $x-\frac{y-2}{7}=5,\ \frac{x+10}{3}=4y-3.$

(15) $3x-\frac{y-3}{5}=6$

$4y+\frac{x-2}{3}=12.$

2. Shew that the straight lines given by the equations

$$15x+2y=27,\ 3x+7y=45,\ x+3y=19,$$

meet in a point Find its co-ordinates.

3. Find the co-ordinates of the vertices of the triangle whose sides are given by the equations :–

$$x-2y+4=0,\ x+y+1=0,\ 5x-y=7.$$

4. Show by solution of equations that the straight lines whose equations are

$$7x-3y=31,\ 9x-5y=41,\ 3x+y=11$$

all pass through one point. Verify by drawing the lines.

5. Draw the triangle whose sides are represented by the equations :—

$$3y-x=9,\ x+7y=11,\ 3x+y=13;$$

and find the co-ordinates of the vertices.

6. What must be the value of a in order that the three lines represented by the equations

$$3x+y-2=0,\ ax+2y-3=0,\ 2x-y-3=0,$$

may meet in a point?

IV. APPLICATIONS OF GRAPHS.

286. We shall now give some illustrations of the way in which graphs may be used as a "ready reckoner."

Ex. 1. If £1 is worth 25 francs, construct a graph from which you can read off the value of any number of shillings up to £3, in francs. Write down from the diagram the value of 35 shillings in francs, and 35 francs in shillings.

Measure shillings along **OX** to a scale of 1″ to 20 shillings, and measure francs along **OY** to a scale of 1″ to 50 francs.

If x shillings $=y$ francs, then $\frac{x}{20}=\frac{y}{25}$, or $y=\frac{5}{4}x$. This represents a straight line passing through the origin.

Take an abscissa **ON** = 60 units and an ordinate **NP** = 75 units. Join **OP**. Then **OP** is the required graph. (Note that different units are used to measure lengths along **OX** and **OY**).

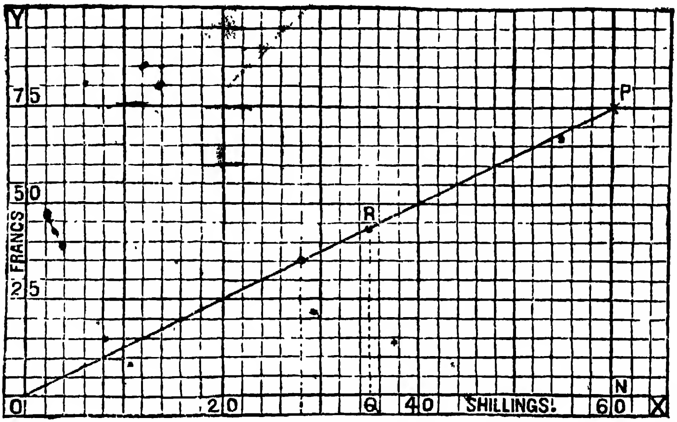


Fig. 12.

Since the abscissa at the point **Q** represents 35 shillings, therefore its ordinate **QR** represents 35 shillings in francs.

Hence, from the diagram 35 shillings = 44 francs.

Also from the diagram, 35 francs = 28 shillings.

Ex. 2. In a Fahrenheit thermometer the freezing point stands at 32° and the boiling point at 212°; in a Centigrade, the freezing point at 0°, and the boiling at 100°. Construct a graph to convert *F.* degrees into *C.* degrees, and *vice versa.* Read off 100°*F.* in *C.* degrees, and 40°*C.* in *F.* degrees.

Let x degrees in the Fahrenheit scale be the same temperature as y degrees in the Centigrade scale.

Then $\dfrac{y}{100} = \dfrac{x-32}{212-32} = \dfrac{x-32}{180}$; whence $9y = 5x - 160$.

When $x = 32$, $y = 0$; when $x = 50$, $y = 10$;

when $x = 68$, $y = 20$ and so on.

Since no point *to the left of or below the point* (68, 20) is required, it is convenient to measure the co-ordinates along lines drawn through this point parallel to the co-ordinate axes. Hence the

following graph (as shown in Fig. 13), passing through the points (68, 20), (122, 50).

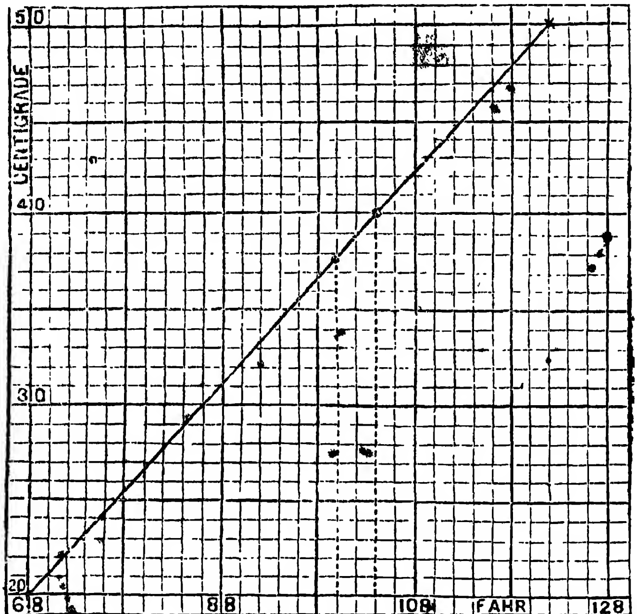


Fig. 13.

By measurement it be found that $100°F = 37°\cdot8C$, and $40°C = 104°F$.

Note.—The above device is often useful; it might be referred to as a *change of axes to parallel axes through the point* (68, 20).

Ex. 3. The expenses of a family when rice is at 20 seers for a rupee are *Rs.*50 a month; when rice is at 25 seers for a rupee the expenses are *Rs.*48 a month (other expenses remaining the same); what will they be when rice is at 30 seers for a rupee? (C.F.A. 1869) Also find how much rice can be had for a rupee when the expenses are *Rs.*60.

Let the expenses be *Rs.* y per month when rice sells at *Re.* x per seer; then the variable part may be denoted by *Rs.* ax, and the constant part by *Rs.* b. Hence x and y satisfy the linear equation $y = ax + b$, where a and b are constants. Hence the graph is a straight line.

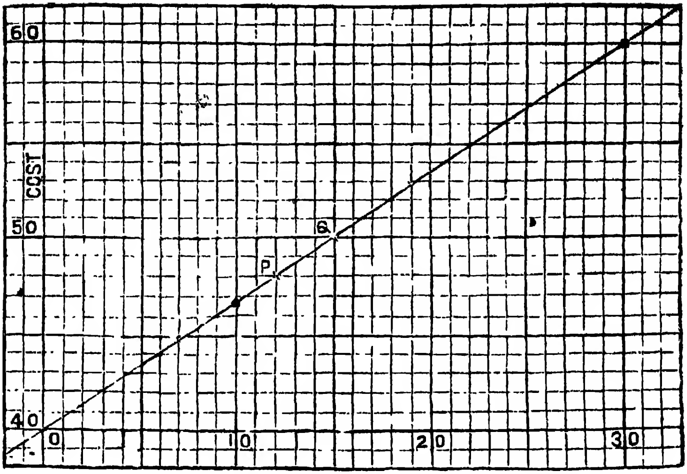


Fig. 14.

To determine a and b we have two pairs of corresponding values of x and y, giving

$$50=\tfrac{1}{20}a+b \text{ and } 48=\tfrac{1}{25}a+b,$$

whence $a=200$, $b=40$, and therefore $y=200x+40$.

From this equation, when $x=\frac{1}{20}$, $y=50$; and when $x=\frac{1}{25}$, $y=48$. Also, when $x=0$, $y=40$. Hence, it will be convenient if we begin to measure the co-ordinates at the point (0, 40).

Take 30 sides of a square along **OX** to represent 0·1 units and 10 sides of a square along **OY** to represent 10 units. Thus we find two points **P** and **Q** when $x=\frac{1}{25}$ and $\frac{1}{20}$ respectively. The line joining **PQ** and passing through the point (0, 40) is the required graph, (as shewn in Fig. 14).

By measurement, we find that when $x=\frac{1}{30}$, $y=46\frac{2}{3}$; and that when $y=60$, $x=\frac{1}{10}$. Thus the required answers are *Rs.*46. 10*a.* 8*p.* and 10 seers per rupee.

Ex. 4. Given that 1 centimetre = ·39 inches, draw a graph to convert inches into centimetres. Read off the value of 3·6 in. in centimetres and the value of 8·6 cms. in inches, as accurately as you can.

If x inches $= y$ centimetres, then $\frac{x}{39} = \frac{y}{100}$, or $y = \frac{100}{39}x$.

This equation represents a straight line through the origin.

When $x = 39$, $y = 100$. Hence the graph passes through the point (39, 100).

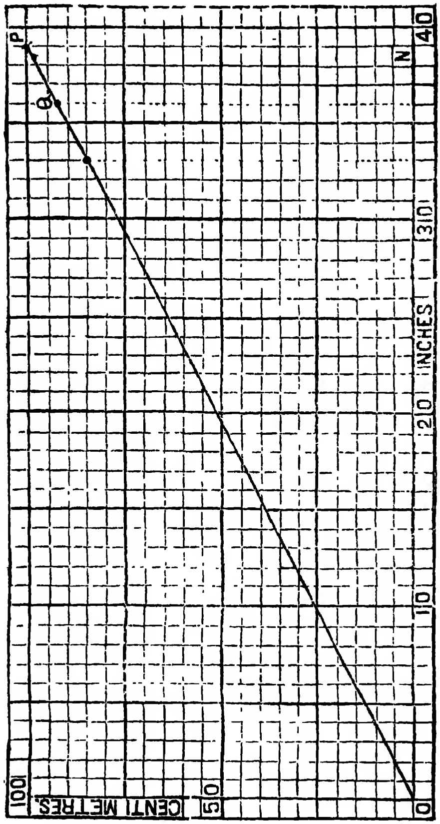


Fig. 15.

Take an abscissa ON $= 39$ units (39 sides of a sq.),

and an ordinate NP $= 100$ units (20 sides of a sq.)

Join **OP** ; then **OP** is the required graph, (as shown in Fig. 15.)

Take each horizontal side of a square to represent 0·1 inch and two vertical sides of a square to represent 0·1 cms.

The abscissa of the point **Q** represents 3·6 inches, therefore its ordinate represents 3·6 inches in centimetres.

Hence, from the diagram, 3·6 inches = 9·23 cms.

Again, from the diagram 8·6 cms. = 3·35 in.

Ex. 5. 60 oranges sell for six shillings and eight pence. Make a graph to shew the cost of any number up to 60, and from it write down the cost of 27 oranges, and the number of whole oranges you would get for 2*s* 3*d*.

Let x oranges cost y pence, then $\frac{x}{60} = \frac{y}{80}$, or $y = \frac{4}{3}x$.

When $x = 0$, $y = 0$; when $x = 60$, $y = 80$. This shews that the graph passes through the origin and the point (60, 80).

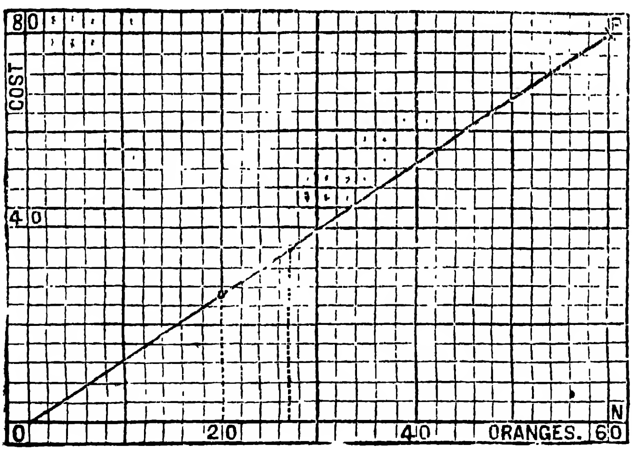


Fig. 16.

Along the abscissa take **ON** = 60 units (30 sides of a sq.),

and **NP** parallel to the ordinate = 80 units (20 sides of a sq.)

Join **OP.** Then **OP** is the required graph, (as shewn in Fig. 16).

Thus, when $x=27, y=36$; that is, 27 oranges cost 3*s*. Again, when $y=27$, $x=20$; thus for 2*s*. 3*d*. one can buy 20 oranges.

V. STATISTICS AND EASY PROBLEMS.

287. We have hitherto considered graphs to be straight lines drawn through a number of plotted points obtained by giving suitable values to x and y which satisfy *any linear equation*. The method is general and may also be applied when the relation between the variables is connected by an equation *which is not linear*. In such a case, the graph drawn through the plotted points will take the form of *some* curve. But in cases where no algebraical relation subsists between the quantities considered, and only a *limited number* of corresponding values is given and therefore only a limited number of points can be plotted, we may indicate the form of the graph which is *most probable*, the curve passing through some of the plotted points and lying evenly as possible among the others on either side of the curve.

In case of statistical results, where no great accuracy of detail is required, it is best to join successive points by *straight lines*. When the graph consists of a succession of straight lines each of which makes an angle with the two lines adjacent to it, the graph will then be represented by an irregular *broken line* to distinguish it from a *continuous curve* like a circle or a parabola. Problems on prices may also be represented graphically by broken lines.

Ex. 1. The following table gives statistics of the population of England and Wales, where P is the number of millions at the beginning of each of the years specified.

Year	1801	1811	1821	1831	1841	1851	1861	1871	1881	1891
	8·9	10·2	12·0	13·9	15·9	17·9	20·0	22·7	26·0	29·0

Draw a graph to exhibit the above. Estimate the population in 1837, and the year in which the population was 24 millions.

Plot the values of P vertically to a scale of 1″ to 10 millions, and those of time horizontally to a scale of 1″ to 20 years; also it will be convenient to begin measuring abscissæ at 1801 and ordinates at 8.

The graph is given in Fig. 17 on the next page.

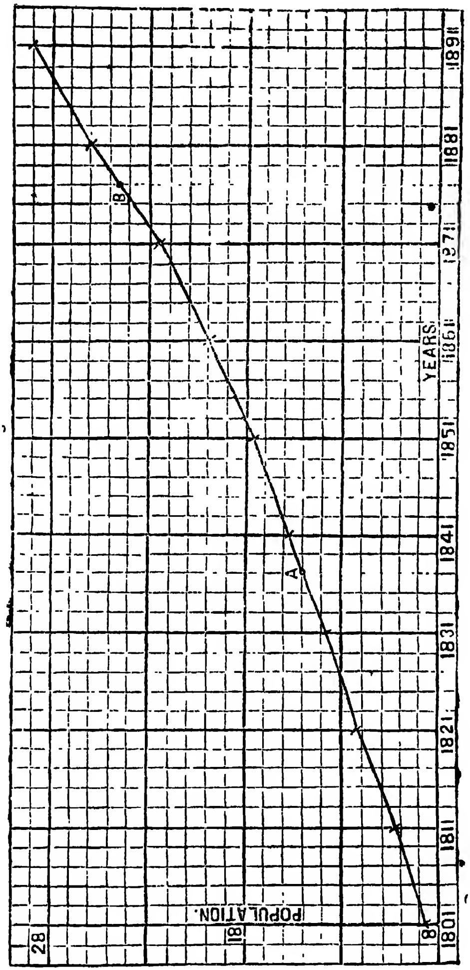


Fig. 17.

The population in 1837 at the point **A** will be found to be 15·1 millions and the year in which the population at **B** was 24 millions is 1875.

Ex. 2. The average annual premiums (£P) for whole life assurance of £100 for the age at entry (A years) is given as follows :—

A	20	25	30	35	40	45	50	55	60
		2·2	2·5	2·8	3·2	3·8	4 6	5 5	6·9

Estimate the premium for £1000 insurance at ages 28 and 43 to the nearest £.

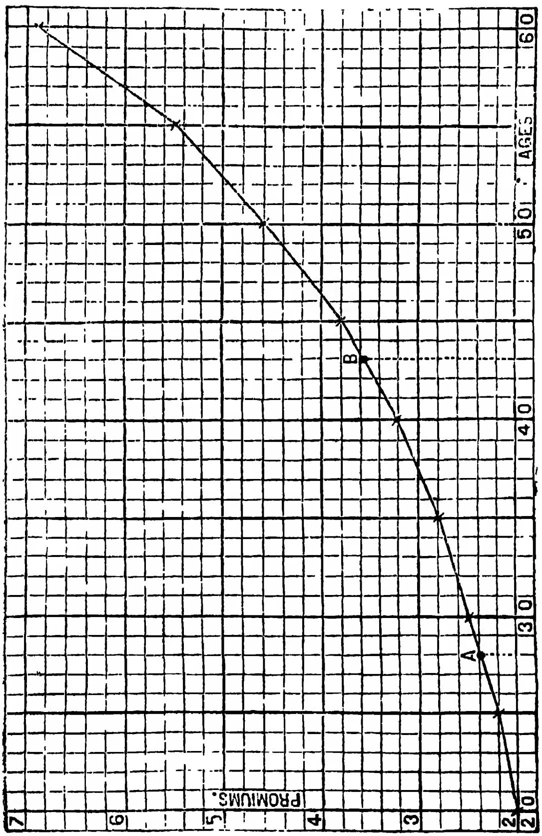


Fig. 18.

Reckoning ages along the abscissæ to the scale of 1″ to 10 years, and premiums along the ordinates to the scale of 1″ to £2, we plot the given points and thus obtain the required graph (as shewn in Fig. 18.)

The premiums of £100 at ages 28 and 43 at the points **A** and **B** respectively are £2·4 and £3·5. Thus the required premiums are £24 and £35.

Ex. 3. The price, £P, of certain engines of brake-horse power H is given as follows : –

H	3	$6\frac{1}{2}$	10	$14\frac{1}{2}$
P	105	160	208	255

What is the probable price of engines of 4 and of 12 horse-power?

Measure horse-power along **OX** to a scale of 1″ to 5, and the price along **OY** to a scale of 1″ to £100.

Plot the given points and join them successively by straight lines. The required graph is shewn in Fig. 19

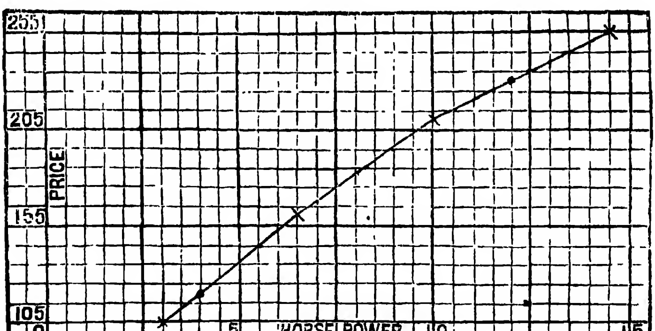


Fig. 19.

By measurement, we see that when $x=4$, $y=121$; and that when $x=12$, $y=229$. Thus the prices are £121 and £229 respectively.

Ex. 4. The temperature taken every two hours beginning at Noon is 61·0°, 66·7°, 67·5°, 58·5°, 54·6°, 51·4°.

Draw a curve to shew the variation of temperature and estimate the temperature at 3 P. M.

Measure times along abscissa to the scale of 1″ to 4 hours, and temperatures by ordinates to the scale of 1″ to 10 degrees.

Plot the given points and joining them, we obtain the graph represented by broken lines as shewn in Fig. 20.

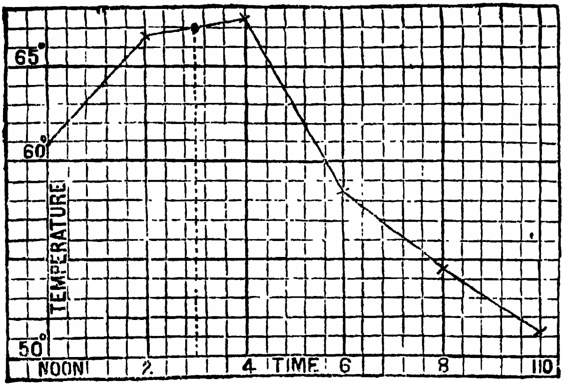


Fig. 20.

By measurement, we find that the temperature at 3 P. M. is 67·1°.

Ex. 5. Corresponding values of x and y are given in the following table :—

x	3	6·5	12	14	21	28·6	31·5
y	4	4·8	6·7	7	8·5	11	11·5

Draw the most probable graph, and find its equation. Find the value of x when $y = 10$, and the value of y, when $x = 36$.

Take 1 inch to represent 10 units along **OX** and also 10 units along **OY.**

Plotting carefully the given points, we see that a straight line can be drawn passing through only two of them and lying evenly among the others. The required graph is given in Fig. 21.

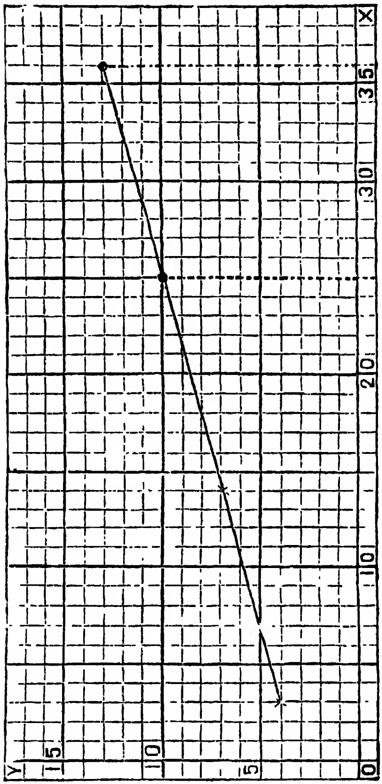


Fig 21.

Assume $y=ax+b$ for its equation. Find the values of a and b by substituting the co-ordinates of the two points through which the line passes.

Thus, putting $x=3$, $y=4$, we have $4=3a+b$;

Again, putting $x=14$, $y=7$, we have $7=14a+b$.

Solving these equations, we get $a=\frac{3}{11}$, $b=\frac{35}{11}$.

Hence the equation of the graph is $y=\frac{3}{11}x+\frac{35}{11}$ or $11y=3x+35$.

The co-ordinates of any number of points on the line can be obtained by trial.

Thus, when $y=10$, $x=25$; and that $y=13$, when $x=36$.

Ex. 6. A train travels at a uniform rate for an hour and a half, and covers 40 miles in that time. Draw the graph of its motion and write down the time it takes to travel 17 miles and how far it has travelled in 12 minutes. Give the results to the nearest mile and minute.

Measure distance along **OX** to the scale of 1″ to 20 miles, and times along **OY** to the scale of 1″ to 1 hour, so that each side of a square represents 6 min.

Along the abscissa measure **OA** $=40$ miles and draw **AB** at right angles to the abscissa $=1 \cdot 5$ hours. Since the train travels 40 miles in $1\frac{1}{2}$ hours, therefore **B** is its terminus after $1\frac{1}{2}$ hours.

Join **OB**. Then **OB** is the graph of the train's motion.

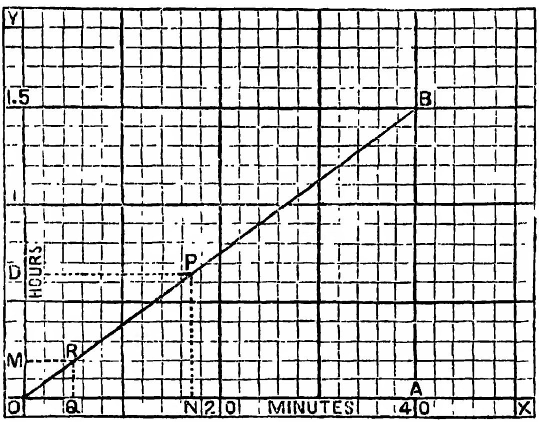


Fig. 22.

(i) To find the time it takes to travel 17 miles.

Take **ON** $=17$ miles and draw the corresponding ordinate **NP**.

Then drawing **PD** parallel to **OX**, we find the required time to be 38 minutes nearly, for **OD** $=\frac{19}{30}$ units.

(ii) To find the distance travelled in 12 minutes.

Take **OM** along **OY** = 12 min., and draw **MR** parallel to **OX** meeting the graph at **R**.

Then drawing the ordinate **RQ** at **R**, we find the required distance to be 5 miles nearly, for **OQ** = $2\frac{1}{2}$ units.

Ex. 7. **A** starts walking at the rate of 4 miles an hour, and 15 minutes later **B** starts at the rate of 8 miles an hour. Find, graphically, when and where **B** overtakes **A**.

Measure distances along **OX** to the scale of 1″ to 4 miles, and times along **OY** to the scale of 1″ to one hour.

Take a point **D** whose abscissa is 4 miles and ordinate 1 hour.

Join **OD**. Then **OD** is the graph of **A**'s motion.

Take a point **E** at 15 min. point in **OY**. Then this is **B**'s starting time. Now take a point **F**, whose abscissa is 8 miles and ordinate (reckoned from the level of **E**) 15 min. more than the time represented by the ordinate of **D**. Join **EF**. Then **EF** is the graph of **B**'s motion.

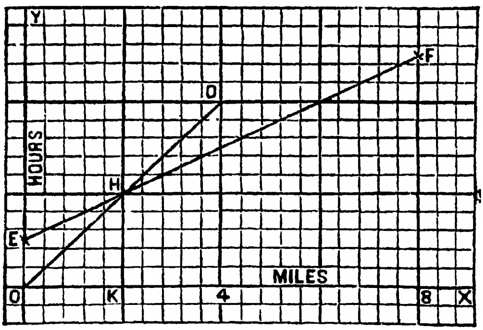


Fig. 23.

The point **H** where the graphs **OD** and **EF** meet, gives the place and time of meeting. Thus, we see that **B** overtakes **A** in half-an hour from **A**'s start, **A** having travelled 2 miles, for **HK** = $\frac{1}{2}$ and **OK** = 2.

Ex. 8. A man starts at noon at the rate of 4 miles an hour to walk from **A** to **B**, a distance of 29 miles; a second man bicycles from **B** to **A**, starting at 2 P. M., and riding at 10 miles an hour. Draw a graph to show where and when they meet and determine also from it the times when they are 10 miles apart.

On squared paper, take two points **A** and **B** on a vertical line

29 units apart. Take horizontally **AP** = 25 units (5 units to an hour) and **PQ** vertically = 20 units.

Join **AQ**. Then since the first man walks 20 miles in 5 hours (25 units), **AQ** is the graph of his motion, *i. e.*, the ordinate of any point on **AQ** denotes the distance he has walked in the time denoted by the abscissa of the point.

As regards the second man, take the point **D** in the horizontal line through **B**, 10 units (2 hours) from **B**, for he starts 2 hours after the first man.

Take horizontally **DF** = $12\frac{1}{2}$ units ($2\frac{1}{2}$ hours) and vertically **FE** = 25 units (for the second man travels 25 miles in $2\frac{1}{2}$ hours).

Join **DE**. Then **DE** is the graph of the second man's motion, reading his times along **BD** and distances travelled vertically downwards.

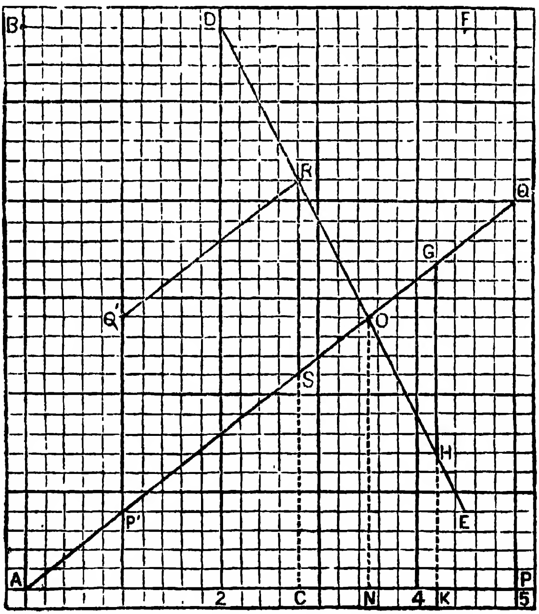


Fig. 24.

Hence, if **AQ** and **DE** meet at **O**, **AN** denotes the time when they meet, and **ON** the distance travelled by the first man.

Thus, from the Fig. we see that they meet at 3-30 P. M. and that the first man has walked 14 miles, for **AN** $= 3\frac{1}{2}$ and **ON** $= 14$.

(i) To find the time when they are first 10 miles apart.

Take a point **P′** on **AQ** where it passes through a corner of a square, and draw **P′Q′** vertically upwards $= 10$ units. Draw **Q′R** parallel to **AQ** to meet **DE** at **R**, and from **R** draw **RS** parallel to **P′Q′** to meet **AQ** at **S**. Then **RS** $=$ **P′Q′** $= 10$ units.

From the Fig. we see that the required time is 2-48 P. M., for **AC** (the abscissa of **S**) $= 2\frac{4}{5}$ units.

(ii) To find the time when they are 10 miles apart the second time.

On **OQ** take **OG** $=$ **OS** and from **G** draw **GH** parallel to **RS** to meet **DE** at **H**. Then **GH** $=$ **RS** $= 10$ miles.

From the Fig. we see that the required time is 4·12 P. M., for **AK** (the abscissa of **H**) $= 4\frac{1}{5}$ units.

Ex. 9. **A** walks at 4 miles an hour, but takes a rest of half an hour at the end of every 4 miles. **B** starting at the same time and walking at a uniform rate, without any rests, catches **A** up just as he is starting after his third rest. Find, graphically, **B**'s rate of travelling.

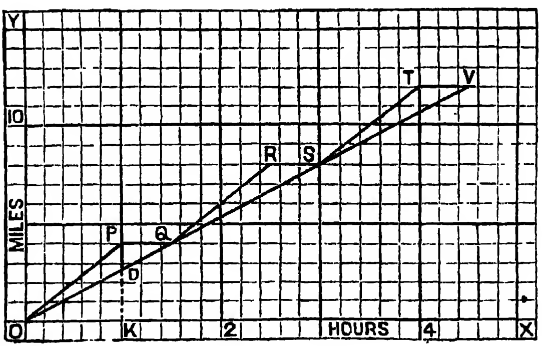


Fig. 25.

Reckon times along the abscissa to the scale of 1″ to 2 hours, and distances along the ordinates to the scale 1″ to 10 miles. Referring to Fig. 25, we see that **OP** is **A**'s graph for the first hour, and **PQ** is his graph for the next half hour, as he stops for that time. In the same way **QR**, **RS**, **ST** and **TV** are his successive graphs.

Again, since **B** starting at **O**, catches **A** at **V**, therefore **OV** is his graph, and the ordinate of $V=12$ and abscissa $=4\frac{1}{2}$.

To find **B**'s travelling rate per hour.

Take $OK=1$ hour and draw **KD** at right angles to **OK** to meet **OV** at **D**.

Hence, **B**'s rate of travelling per hour is denoted by the ordinate **DK**, which $=2\cdot7$ miles nearly.

† **Ex. 10.** At what times between 4 and 5 o'clock are the two hands of a watch (i) together, (ii) 15 minute-spaces apart?

Take abscissæ to represent the time in minutes after 4 o'clock and ordinates to represent the number of minute-spaces past 12 o'clock. Along abscissæ, take 1″ to represent 20 minutes and along ordinates, take 1″ to represent 20 minute-spaces.

The graph of the motion of the long hand is a straight line, for it moves at the constant rate of 1 minute-space per minute.

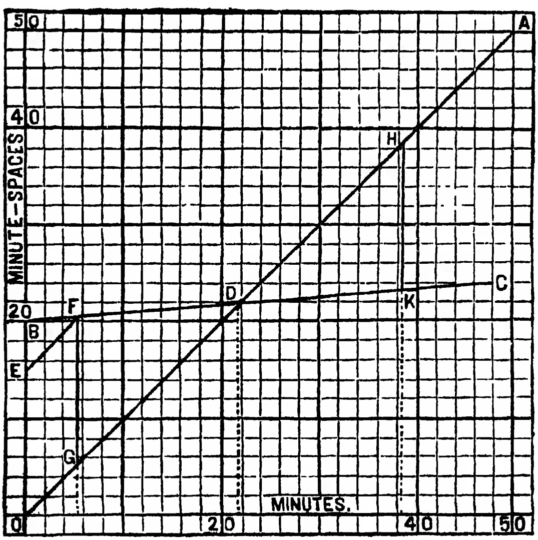


Fig. 26.

This line goes through the origin. Draw **OA** passing through **O** and terminated at (50, 50).

At 4 o'clock, the short hand is 20 minute-spaces in advance of 12 o'clock; and as the abscissa of its position is zero, it is the point **B**(0, 20). Again, since the short-hand moves at the constant rate of 1 minute-space in 12 minutes, another convenient point may be denoted by **C** (48, 24).

(i) Draw **BC** cutting **OA** in **D**. Then **D** is the position in which the two hands are together; and as the abscissa of **D** is 21·8, the hands are together at 21·8 min. past 4 o'clock (nearly).

(ii) To find the time when the hands are 15 minute-spaces apart.

Along **OY** take **OE** = 15 units, and draw **EF** parallel to **OA** meeting **BC** in **F**. Draw **FG** parallel to **OE** meeting **OA** in **G**. Then **FG** = **OE** = 15. The abscissa of **G** represents the time required, which = 5·5 min. past 4 (nearly).

Again, in **DA** take **DH** = **DG** and draw **HK** parallel to **OY** meeting **BC** in **K**. Then **HK** = **FG** = 15 units. The abscissa of the point **K** represents the time when the hands are again 15 minutes apart. Hence the required time = 38·2 min. after 4 (nearly).

Exercise CVIII.

1. Given that 1 kilogramme = 2·2 lbs., draw a graph which will enable you to read off any number of lbs. in kilogrammes (up to 50 lbs.), and read off the values of 25 and 38 kilogrammes in lbs., and of 32·5 and 38 lbs. in kilogrammes.

2. If 3·26 in. are equivalent to 8·28 cm., show how to find graphically the number of inches corresponding to a given number of centimetres. Obtain the number of inches in a metre, and the number of centimetres in a yard. Find the equation of the graph.

3. If C is the circumference of a circle and D its diameter, $C = \frac{22}{7}D$. Draw a graph and from it read off the circumferences of circles whose diameters are 4 in., 11 in, 20 in., and the radii of circles whose circumferences are 47 in. and 31·4 in.

4. The highest marks obtained in an examination are 132 and the marks are to be reduced so that the highest marks may be 100. Show how to do this graphically and state what marks will be assigned to papers which obtained (i) 100, (ii) 70 marks, giving the marks to the nearest integer.

5. The readings on a Centigrade thermometer in degrees and

the corresponding readings on a Fahrenheit thermometer in degrees are given in the following table :—

C	5	10	15	20	30	50	80
F	41	50	59	68	86	122	176

Illustrate graphically the connection between the two scales. Express 140° *F.* in Centigrade.

6. Construct a graph to exhibit the following :-

Premiums of Life-insurance at various ages (for £100).

Age in years	20	25	30	35	40	45	
Premium	£2. 8s.	£2. 16s.	£3. 6s.	£4. 2s.	£5. 4s.	£7. 3s.	£11

Estimate the premium to be paid at 27 and 37 years.

7. The temperature taken every two hours one day showed :

Midnight,	41°·0	2 P. M.,	51°·2
2 A. M.,	40°·8	4 P. M.,	53°
4 A. M.,	40°·7	6 P. M.,	46°·5
6 A. M.,	39°·5	8 P. M.,	46°·3
8 A. M.,	40°·8	10 P. M.,	46°·7
10 A. M.,	44°·5	Midnight,	47°·4
Noon,	48°		

Draw a graph to show the variation of temperature throughout the day, and estimate the temperature at 3 P. M.

8. The price (in pence) of an ounce (Troy) of silver on Jan. 1st in each of the following years was as follows :—

1891	1892	1893	1894	1895	1896	1897	1898	1899	1900
45	40	36	29	30	31	28	27	27	28

Draw a graph showing these changes in value.

9. Given that 1 inch=2·54 centimetres, construct a graph to convert centimetres into inches. Read off the value of 5·6 cms. in

inches, and the value of 4·9 inches in centimetres, as accurately as you can.

10. On an examination paper of maximum 69 the marks gained by 10 candidates were :—

Candidates	1	2	3	4	5	6	7	8	9	10
Marks	60	54	46	35	32	29	27	26	25	12

Draw a graph to raise the maximum to 100, and read off (to the nearest integer) the raised marks of the candidates.

11. The number of thousands (N) of people who emigrated Ireland between 1876 and 1885 is given in the table :—

Year	1876	1877	1878	1879	1880	1881	1882	1883	1884	1885
N	37·5	38·5	41·1	47·0	95·5	78·4	89·1	108·7	75·8	62·0

Illustrate the above graphically.

12. A man spends *Rs.*750 in 64 days. Draw a graph to give his expenditure in any number of days. Write down his expenditure in 17, 35 and 49 days, to the nearest rupees.

13. The mean temperature on the first day of each month, on an average of 50 years, had the following values :—

Jan. 1, 37° ;	May 1, 50° ;	Sept. 1, 59° ;
Feb. 1, 38° ;	June 1, 57° ;	Oct. 1, 54° ;
Mar. 1, 40° ;	July 1, 62° ;	Nov. 1, 46° ;
April 1, 45° ;	Aug. 1, 62° ;	Dec. 1, 41°.

Draw a graph to represent these variations.

14. The first 100 copies of a pamphlet cost 27*s.* to print, but every 100 in excess of the first costs only 3*s.* ; make a graph to show the cost of any number up to 800, and read off the cost of 370 copies. Write down the number of copies you would get for £2. 2*s.* 6*d.*

15. The top boy in a form gets 88 marks, and the last boy 33. These have to be scaled so that the top boy gets 100 and the last boy 0. Draw a graph which will effect this, and read off (to the nearest integer) the scaled marks of the boys who get 65, 54, 49. Find the equation between x the actual marks gained, and y the corresponding scaled marks.

16. I want a ready means of finding approximately 0·866 of any number up to 10. Justify the following construction. Join the origin to a point **P** whose co-ordinates are 10 and 8·66 (1 inch being taken as unit): then the ordinate of any point on **OP** is 0·866 of the corresponding abscissa. Read off from the diagram,

$$0·866 \text{ of } 3,\ 0·866 \text{ of } 6·5,\ 0·866 \text{ of } 4·8 \text{ and } \frac{1}{0·866} \text{ of } 5.$$

17. If the cost of maintaining a family be *Rs.*50 a month, when rice is 12 seers a rupee, and *Rs.*48 when rice is 14 seers a rupee (the other expenses remaining the same); what will be the cost when rice is 16 seers a rupee?

18. For a dinner at which there are 60 guests a restaurant keeper charges 10*s.* 6*d.* per head, but if there are 100 guests the charge is 8*s.* 6*d.* per head. What will be the probable charge per head for 75 guests?

19. In a Reaumur thermometer the freezing point stands at 0° and the boiling point at 80°; in a Fahrenheit, the freezing point at 32°, and the boiling point at 212°. Construct a graph to convert *R.* degrees into *F.* degrees and *vice versa*. Read off 60° *R.* in *F.* degrees, and 43°*F* in Reaumur degrees.

20. For a certain book it costs a publisher £100 to prepare the type and 2*s.* to print each copy. Find an expression for the total cost in pounds of x copies. Also make a diagram on the scale of 1 inch to 1000 copies, and 1 inch to £100 to show the total cost of any number of copies up to 5000. Read off the cost of 2500 copies, and the number of copies costing £525.

21. Two men start to meet each other at 9 P. M., from places 31 miles apart; if one of them walks $4\frac{1}{4}$ miles an hour and the other $3\frac{1}{2}$ miles an hour, when will they meet, and how far will each have travelled?

22. **A** and **B** walk respectively $5\frac{1}{4}$ and $3\frac{1}{2}$ miles an hour. They are 25 miles apart and walk to meet one another but **B** starts 2 hours before **A**. How far will **A** have to walk?

23. In a 100 yds. race, **A** can beat **B** by 20 yds., and **B** can beat **C** by 10 yds. How many yards start can **A** give **C** that there may be a dead heat?

24. A man bicycles from **A** to **B** at 10 miles an hour, and returns from **B** to **A** at 15 miles an hour. If he takes 5 hours to go there and back, find the distance from **A** to **B**. Find also his average speed her hour.

25. A train leaves **A** for **B** at 9-15 A.M. and travels at the rate of 30 miles per hour. At 9-35 A. M., a second train starts, and travels at the rate of 35 miles an hour. If both trains arrive at **B**

at the same time, find the distance from **A** to **B**, and the time each train takes.

26. **A** starts at 8 A. M. to walk from **P** to **Q**, a distance of 30 miles. At noon he meets **B**, who started from **Q** to **P** at 7-30 A.M. If **A** reaches **Q** at 6 P. M., when will **B** reach **P**?

27. **A** starts walking at the rate of 100 yds. in 30 secs. and **B** starts from the same spot 6 secs. later at the rate of 100 yds. in 12 secs. Draw a graph to find when and where **B** catches **A** up.

28. At what times between 3 and 4 o'clock are the two hands of a watch (i) together, (ii) 10 minute-spaces apart?

29. A monkey, climbing up a greased pole, ascends 2 ft. and slips down 1 foot in alternate seconds, until he reaches the top of the pole. If the pole be 6 feet high, how long will it take him to reach the top?

30. **A** does a journey of 42 miles in $5\frac{1}{4}$ hours, and **B** starting an hour later does the reverse journey in 4 hours. Find, graphically, as accurately as you can, how far their meeting place is from **A**'s starting point. In how many minutes after **B**'s start were they first 20 miles apart?

31. **A** starts from Calcutta for Mankar, a distance of 91 miles, at 6 A. M., walking $3\frac{1}{2}$ miles an hour; **B** starts from Mankar 12 hours later and reaches Calcutta at the same time as **A**. What was **B**'s speed per hour?

32. **A** travelling at 4 miles an hour, walks 4 miles, then rests for half-an hour, then walks 8 miles further, and then walks straight back at the same rate. He meets **B**, who walks uniformly and without resting, a mile and a half from home. Find **B**'s rate of travelling, if he started at the same time as **A**.

33. At what times between 5 and 6 o'clock are the two hands of a clock (i) together, (ii) at right angles, (iii) directly opposite to each other?

34. In what proportion must tea at *Re.*1. 4*a.* per seer be mixed with tea at *Rs.*2 per seer, so that the mixture may be sold at *Re.*1. 12*a.* per seer?

35. **A** starts from Calcutta to walk to Burdwan, a distance of 68 miles, at 3 miles an hour; two hours later **B** starts from Burdwan for Calcutta at 5 miles an hour. When will **A** and **B** meet? When will they be 20 miles apart?

36. **A** starts from a place **X**, for a place **Y**, a distance of 80 miles at 6 A. M., walking $3\frac{1}{3}$ miles an hour; **B** starts 4 hours later and reaches **Y** at the same time as **A**. What was **B**'s speed per hour?

37. **A** travels at 5 miles an hour, but takes a rest of half-an hour at the end of each hour. **B** starting 2 hours after **A** and

travelling uniformly, without resting, overtakes **A** $17\frac{1}{2}$ miles from home. Find, graphically, **B**'s rate of travelling per hour.

38. In a 100 yds. race **A** beats **B** by 9 yards. and in 100 yds. **C** beats **B** by 8 yards. If **A**'s time for the hundred yards is $10\frac{2}{5}$ secs., what are **B**'s and **C**'s times?

39. How much tea at *Rs.*3 per lb. must I mix with 12 lbs. at *Re.*1. 13*a.* 4*p.* per lb. to make a mixture worth *Rs.*2. 2*a.* 8*p.* per lb?

40. The salary of a clerk is increased each year by a fixed sum. After 6 years' service his salary is raised to *Rs.*128, and after 15 years to *Rs.*200. Draw a graph from which his salary may be read off for any year, and determine from it (i) his initial salary, (ii) the salary he should receive for his 21st year.

CHAPTER XI.

INDICES AND SURDS.

I. THEORY OF INDICES.

288. It was noticed in Art. 73, that powers of the same quantity were multiplied by adding their indices; in Art. 105 that one power of a quantity is divided by another power of the same quantity by subtracting the index of the latter from that of the former; and in Art. 167 that any power of a power of a quantity is obtained by multiplying together the indices of the two powers. We shall now prove the above rules to be *generally* true, which were there only shewn to be true in particular instances.

289. The following are the three fundamental laws for positive indices.

When m *and* n *are positive integers, then*

$$a^m \times a^n = a^{m+n} \quad \ldots\ldots\ldots\ldots \text{I.}$$

$$a^m \div a^n = a^{m-n} \quad \ldots\ldots\ldots\ldots \text{II.}$$

$$(a^m)^n = a^{mn} \quad \ldots\ldots\ldots\ldots \text{III.}$$

The first is called the **Index Law**, as being the basis of the other two laws, for they may be deduced from the first.

I. If m and n be any positive integers, to prove that

$$a^m \times a^n = a^{m+n}.$$

$$\left.\begin{aligned} \text{Since } a^m &= a \times a \times a \times \&\text{c.} \ldots\ldots \text{to } m \text{ factors,} \\ \text{and } a^n &= a \times a \times a \times \&\text{c.} \ldots\ldots \text{to } n \text{ factors;} \end{aligned}\right\} \text{Art. 20.}$$

$\therefore\ a^m \times a^n = a \times a \times a \ldots\ldots$ to m factors $\times a \times a \times a \ldots\ldots$ to n factors,

$= a \times a \times a \ldots\ldots$ to $(m+n)$ factors,

$= a^{m+n}$, by Art. 20.

Similarly, if p is also a positive integer, we have

$$a^m \times a^n \times a^p = a^{m+n} \times a^p = a^{m+n+p}, \text{ and so on.}$$

Hence, generally,

$$a^m \times a^n \times a^p \times \ldots\ldots = a^{m+n+p+\cdots\cdots} \ldots\ldots \textbf{IV.}$$

where $m, n, p, \ldots\ldots\ldots\ldots$ are all positive integers.

II. If m and n be any positive integers and $m > n$, then

$$a^m \div a^n = a^{m-n}.$$

For $a^m \div a^n = \dfrac{a^m}{a^n} = \dfrac{a \times a \times a \times \ldots\ldots \text{to } m \text{ factors}}{a \times a \times a \times \ldots\ldots \text{to } n \text{ factors}}$

$= \dfrac{a \times a \times a \ldots\ldots \text{to } (m-n) \text{ factors} \times a \times a \times a \ldots\ldots \text{to } n \text{ factors}}{a \times a \times a \ldots\ldots \text{to } n \text{ factors}}$

$= a \times a \times a \times \ldots\ldots\ldots$ to $(m-n)$ factors

$= a^{m-n}$, by Art. 20.

III. If m and n be any positive integers, to prove that

$$(a^m)^n = a^{mn}.$$

For, $(a^m)^n = a^m \times a^m \times a^m \times \ldots\ldots$ to n factors

$= (a \times a \times a \ldots\ldots$ to m factors)

$\times (a \times a \times a \ldots\ldots$ to m factors) $\times \ldots\ldots$ to n brackets

$= a \times a \times a \times \ldots\ldots$ to mn factors

$= a^{mn}$, by Art. 20.

290. To prove that **II.** and **III.** are deducible from **I.**

(1) Since $a^p \times a^n = a^{p+n}$, when p and n are any positive integers,

$\therefore\ a^{p+n} \div a^n = a^p$, by *Def.* of Division.

Let $p + n = m$, so that $p = m - n$.

$\therefore$ from the above, we obtain

$$a^m \div a^n = a^{m-n}, \text{ which is } \textbf{II.}$$

(2) Again, since, from **IV.** we have

$$a^m \times a^p \times a^q \times \ldots\ldots\ldots = a^{m+p+q+\cdots\cdots}$$

Let $m = p = q = \ldots\ldots$ and let their *number* be n.

$\therefore\ a^m \times a^m \times a^m \times \ldots\ldots$ to n factors

$= a^{m+m+m+\cdots\cdots \text{ to } n \text{ terms}}$

$= a^{mn}$, which is **III.**

291. Hence $(a^m)^n = a^{mn} = (a^n)^m$.

For $(a^m)^n = a^m.a^m.a^m \ldots$ to n factors $= a^{m+m+m+\ldots \text{to } n \text{ terms}} = a^{mn}$,

and $(a^n)^m = a^n.a^n.a^n \ldots$ to m factors $= a^{n+n+n+\ldots \text{to } m \text{ terms}} = a^{nm}$;

$\therefore$ since $a^{mn} = a^{nm}$, we have $(a^m)^n = a^{mn} = (a^n)^m$:

that is, *the* n*th power of the* m*th power of* a = *the* m*th power of the* n*th power of* a, and either of them is found by multiplying the two indices.

292. Hence also $\sqrt[n]{a^m} = (\sqrt[n]{a})^m$.

For, let $\sqrt[n]{a^m} = x^m$, then $a^m = (x^m)^n = (x^n)^m$; by Art. 290.

hence $a = x^n$, and $\therefore \sqrt[n]{a} = x$, and $(\sqrt[n]{a})^m = x^m$.

But also, by our first supposition, $\sqrt[n]{a^m} = x^m$;

hence, we have $\sqrt[n]{a^m} = (\sqrt[n]{a})^m$;

that is, n*th root of the* m*th power of* a = *the* m*th power of the* n*th root of* a.

293. These results refer as yet only to positive integral indices, which in Art. 20 were first used to express briefly the repetition of the same factor in any product.

But now, suppose we write down a quantity, with a positive *fraction* for an index, such as $a^{\frac{p}{q}}$, and agree that such a symbol shall be treated by the same *Index Law* as if the index were an *integer* :— what would such a symbol, so treated, denote ?

Since it follows from the *Index Law*, in the case of *positive integers*, that $(a^m)^n = a^{mn}$, we should have here also $\left(a^{\frac{p}{q}}\right)^q = a^{\frac{pq}{q}} = a^p$; and hence it appears, that $a^{\frac{p}{q}}$ would denote such a quantity as, *when raised to the* q*th power*, becomes equal to a^p. But that quantity, whose qth power $= a^p$, is the q*th root of* a^p (Art. 31); and, therefore, $a^{\frac{p}{q}} = \sqrt[q]{a^p}$, or $= (\sqrt[q]{a})^p$ by Art. 292.

Hence, when a fractional index is employed with any quantity, the *numerator* denotes a **power**, and the *denominator* a **root** to be taken of it.

Thus, $a^{\frac{1}{2}}$ = 2nd *root* of 1st *power* of $a = \sqrt{a}$, $a^{\frac{1}{3}} = \sqrt[3]{a}$, $a^{\frac{1}{4}} = \sqrt[4]{a}$, &c.

$a^{\frac{2}{3}}$ = ***cube root*** of *square* of $a = \sqrt[3]{a^2}$,

or $=$ ***square*** of *cube root* of $a = (\sqrt[3]{a})^2$.

So $a^{\frac{3}{4}} = \sqrt[4]{a^3}$ or $(\sqrt[4]{a})^3$; $a^{\frac{1}{2}} = a^{\frac{2}{4}} = a^{\frac{3}{6}} =$ &c., or $\sqrt{a} = \sqrt[4]{a^2} = \sqrt[6]{a^3} =$ &c.

294. Again, if we write down a quantity with a *negative* index, as a^{-p} (where p may now be *integral* or *fractional*), and agree that this symbol shall be treated by the same *Index Law* as if the index were *positive*,—what would such a symbol, so treated, denote?

By this Law, we should have $a^{m+p} \times a^{-p} = a^{m+p-p} = a^m$;

but we have also $a^{m+p} \div a^p = \dfrac{a^{m+p}}{a^p} = \dfrac{a^m.a^p}{a^p} = a^m$:

so that, to *multiply* by a^{-p}, is the same as to *divide* by a^p:

and, therefore, $1 \times a^{-p} = 1 \div a^p$, or $a^{-p} = \dfrac{1}{a^p}$.

Hence, any quantity with a *negative* index denotes the **reciprocal** of the same with the same *positive* index.

Thus, $a^{-1} = \dfrac{1}{a}$, $a^{-3} = \dfrac{1}{a^3}$, $a^{-\frac{1}{2}} = \dfrac{1}{a^{\frac{1}{2}}} = \dfrac{1}{\sqrt{a}}$, or $= \sqrt{a^{-1}} = \sqrt{\dfrac{1}{a}}$;

$$a^{-\frac{2}{3}} = \frac{1}{a^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{a^2}}, \text{ or} = \sqrt[3]{a^{-2}} = \sqrt[3]{\frac{1}{a^2}}.$$

Hence, also *any power in the numerator of a quantity may be removed into the denominator, and vice versâ, by merely changing the sign of its index.*

Thus, $a^{-3}b^2c^{-1} = \dfrac{a^{-3}b^2}{c} = \dfrac{a^{-3}}{b^{-2}c} = \dfrac{b^2c}{a^3} = \&c.$

295. Lastly, if we write down a quantity with *zero* for an index, as a^0, and agree that this symbol shall be treated as if the index were an actual number,—what then would it denote?

Since, by the *Index Law*, $a^0 \times a^m = a^{0+m} = a^m$;

dividing both sides by a^m, $a^0 = 1$.

Hence, it follows that a^0 *is only equivalent to 1, whatever be the value of a.*

296. In actual practice, such a quantity as a^0 would only occur in certain cases, where we wish to keep in mind from what a certain number may have arisen.

Thus, $(a^3 + 2a^2 + 3a + 4a^0 + \&c.) \div a^2 = a + 2 + 3a^{-1} + 4a^{-2} + \&c.$,

where the 2 has lost all sign of its having been originally a coefficient of some power of a; if, however, we write the quotient $a + 2a^0 + 3a^{-1} + 4a^{-2} + \&c.$, we preserve an indication of this, and have, as it were, a connecting link between the positive and negative powers of a.

297. The quantity $a^{\frac{p}{q}}$ is still called *a to the power of* $\frac{p}{q}$, and similarly in the case of a^{-n}, a^0; but the word *power* has here lost its original meaning, and denotes merely *a quantity with an index*, whatever that index may be, subject, in all cases, to the *Index Law*.

298. To prove that $(a^m b^n)^p = a^{mp} b^{np}$.

Let m and n have *any value whatever*.

(1) Let p be a *positive integer*.

$$(a^m b^n)^p = (a^m b^n) \times (a^m b^n) \times \ldots\ldots \text{to } p \text{ brackets}$$
$$= (a^m \times a^m \times \ldots\ldots \text{to } p \text{ factors}) \times (b^n \times b^n \times \ldots\ldots \text{to } p \text{ factors})$$
$$= a^{mp} b^{np}, \text{ by Art. 289 III.}$$

(2) Let p be a *positive fraction*.

Let $p = \frac{r}{s}$, where r and s are positive integers, so that $r = ps$.

$$(a^m b^n)^p = (a^m b^n)^{\frac{r}{s}} = \sqrt[s]{(a^m b^n)^r} = \sqrt[s]{(a^{mr} b^{nr})}, \text{ by (1)}$$
$$= \sqrt[s]{(a^{mps} b^{nps})}, \text{ for } r = ps,$$
$$= \sqrt[s]{(a^{mp} b^{np})^s} \text{ by (1)}$$
$$= \sqrt[s]{(a^{mp} b^{np})^s} = a^{mp} b^{np}.$$

(3) Let p be *any negative quantity*.

Let $p = -r$, where r is a positive integer or fraction.

$$(a^m b^n)^p = (a^m b^n)^{-r} = \frac{1}{(a^m b^n)^r} = \frac{1}{a^{mr} b^{nr}}, \text{ by (1) and (2)}$$
$$= a^{-mr} b^{-nr} = a^{mp} b^{np}, \text{ writing } p \text{ for } -r.$$

Hence $(a^m b^n)^p = a^{mp} b^{np}$, for all values of m, n and p.

299. To prove that $\left(a^{m^{n-1}}\right)^m = a^{m^n}$.

$$\left(a^{m^{n-1}}\right)^m = (a^p)^m, \text{ if } p = m^{n-1}$$
$$= a^{mp}, \text{ by Art. 289 III.}$$

But, $mp = m \times m^{n-1} = m^{1+n-1}$, by *Index Law*
$$= m^n.$$

$$\therefore \left(a^{m^{n-1}}\right)^m = a^{m^n}.$$

Here, note carefully the distinction between $a^{m^{n-1}}$ and $\left(a^m\right)^{n-1}$.

The last $= a^{m(n-1)} = a^{mn-m}$.

Exercise CIX.

Express, with *fractional* indices,

1. $\sqrt{x^3}+\sqrt[3]{x^4}+(\sqrt{x})^5+(\sqrt[3]{x})^2$.
2. $\sqrt[3]{(a^3b^8)}+\sqrt[3]{(a^2b^4)}+\sqrt[3]{(ab^6)}+\sqrt[3]{(a^5b^3)}$.
3. $a\sqrt[3]{b^8}+(\sqrt{a})^5+\sqrt[3]{(a^6b)}+\sqrt[6]{(a^6b^4)}$.
4. $\sqrt[4]{(a^2b^8)}+a(\sqrt[3]{b})^6+\sqrt[6]{(a^2b^{10})}+\sqrt{(a^3b^4)}$.

Express, with *negative* indices, so as to remove all powers, (i) into the numerators, and (ii) into the denominators,

5. $\frac{1}{a}+\frac{2}{b^2}+\frac{3}{c^3}+\frac{4a}{b}+\frac{5b}{a}$. 6. $\frac{a^3}{b^3}+\frac{3a^2}{b}+\frac{5a}{b^2}+\frac{4b}{a^2}+\frac{2b^3}{a^3}$.

7. $\frac{a^3}{3b^2c^2}+\frac{4c^2}{a^2b}+\frac{2bc}{a}+\frac{1}{3abc}$. 8. $\frac{ab}{2\sqrt[3]{c}}+\frac{2b^2c^2}{3\sqrt{a^5}}+\frac{3}{4\sqrt[3]{(a^2bc^2)}}+\frac{5c}{a\sqrt[4]{a^3}}$.

Express, with the *sign of Evolution*,

9. $a^{\frac{1}{2}}+2a^{\frac{2}{3}}+3a^{\frac{3}{4}}+4a^{\frac{1}{5}}+a^{\frac{3}{4}}$. 10. $\frac{a^{\frac{1}{4}}}{b^{\frac{3}{4}}}+\frac{a^{\frac{2}{3}}b^{\frac{1}{3}}}{2c^{\frac{1}{2}}}+\frac{2a^{\frac{1}{4}}c^{\frac{3}{4}}}{3b^{\frac{3}{2}}}+\frac{b^{\frac{2}{5}}c^{\frac{3}{5}}}{4a^{\frac{1}{5}}}+\frac{b^{\frac{1}{6}}c^{\frac{5}{6}}}{5a^{\frac{3}{4}}}$.

Express, with *positive* indices, and with the *sign of Evolution*,

11. $a^{-1}bc+ab^{-2}c+a^{-1}b^{-1}c^{-1}+a^{-1}b^{-2}c^3$. 12. $a^{-\frac{2}{5}}+a^{\frac{1}{2}}b^{-\frac{4}{3}}+a^{-\frac{3}{2}}b^{\frac{2}{3}}+b^{-\frac{5}{3}}$.

13. $\frac{a^{-2}b^{-2}}{c^{-1}}+\frac{2a}{b^{-1}c^{-1}}+\frac{3b^{-1}c^{-2}}{a^{-3}}+\frac{1}{a^{-1}b^{-2}c^{-3}}$.

14. $\frac{a^{-2}}{b^{-\frac{1}{3}}}+\frac{b^{-\frac{2}{3}}}{a^{-\frac{3}{2}}}+\frac{b^{-\frac{3}{4}}}{a^{-\frac{2}{3}}}+\frac{a^{-\frac{1}{2}}}{b^{-2}}$.

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300. It follows, then, that, *whatever* be the indices,

$$a^m\times a^n=a^{m+n},\ a^m\div a^n=a^{m-n},\ (a^m)^n=a^{mn};$$

so that (i) to *multiply* any powers of the same quantity, we must *add* the indices, (ii) to *divide* any one power of a quantity by another power of the same quantity, we must *subtract* the index of the divisor from that of the dividend, and (iii) to obtain any *power of a power* of a quantity, we must multiply together the two indices.

Thus, $a^3\times a^{-2}=a^{3-2}=a$; $a^3\div a^{-\frac{1}{2}}=a^{3+\frac{1}{2}}=a^{\frac{7}{2}}$; $a^{-\frac{1}{2}}\div a^{-\frac{3}{5}}=a^{-\frac{1}{2}+\frac{3}{5}}$ $=a^{\frac{1}{10}}$; $(a^3)^{-2}=a^{-6}$; $(a^{-3})^{-\frac{1}{2}}=a^{\frac{3}{2}}$; $\{(a^{-\frac{1}{2}})^{\frac{2}{3}}\}^{-12}=a^4$.

Ex. 1. $a^{-3}b^{-3}\times a^{\frac{3}{2}}b^{-\frac{5}{2}}\times a^{\frac{7}{2}}b^{-\frac{15}{2}}=a^{-3+\frac{3}{2}+\frac{7}{2}}\times b^{-3-\frac{5}{2}-\frac{15}{2}}$

$$=a^{-3+5}\times b^{-3-10}=a^2b^{-13}=\frac{a^2}{b^{13}}.$$

Ex. 2. $a^3b^{-\frac{2}{3}}c^{\frac{3}{5}}\div a^{-5}b^{-\frac{3}{4}}c^{\frac{5}{6}}=a^{3+5}\times b^{-\frac{2}{3}+\frac{3}{4}}\times c^{\frac{3}{5}-\frac{5}{6}}=a^8b^{\frac{1}{12}}c^{-\frac{7}{30}}.$

Ex. 3. $\{\sqrt{ab^{-2}\sqrt{ab}}\}^4=\{a^{\frac{1}{2}}b^{-1}\times a^{\frac{1}{4}}b^{\frac{1}{4}}\}^4=(a^{\frac{3}{4}}b^{-\frac{3}{4}})^4=a^3b^{-3}.$

Exercise CX.

Find the value of

1. $16^{-\frac{1}{2}}$. **2.** $27^{-\frac{2}{3}}$. **3.** $16^{\frac{3}{4}}$. **4.** $32^{-\frac{3}{5}}$. **5.** $625^{\frac{3}{4}}$.

6. $(27b^{-6}c^3)^{\frac{1}{3}}$. **7.** $(\frac{1}{16}x^{12}y^{-8})^{\frac{1}{4}}$. **8.** $(1024^{-\frac{4}{5}})^{\frac{1}{2}}$. **9.** $343^{-\frac{2}{3}}$.

Simplify the following :—

10. $\{(a^{-3}b^2)^{\frac{1}{2}}\}^{-\frac{2}{3}}$. **11.** $\sqrt[3]{a^2\sqrt{a^{-1}}}$. **12.** $\sqrt{a^{-1}\sqrt{a^3\sqrt{a^{-4}}}}$.

13. $\{\sqrt{a^2b\sqrt[3]{a^{-4}b^{-2}}}\}^6$. **14.** $\{x^{-\frac{3}{2}}y.(xy^{-2})^{-\frac{1}{2}}.(x^{-1}y)^{-\frac{2}{3}}\}^3$.

15. $\{x^{\frac{1}{2}}y^{-\frac{1}{4}}\sqrt{(x^{\frac{1}{3}}y^{\frac{1}{2}}\sqrt{y^3})}\}^3$. **16.** $\sqrt[12]{\{x^3y^3\sqrt[3]{xyz^4\sqrt[4]{(x^{-1}y^{-2}z^{-3})}}\}^{12}}$.

17. $a^{m-n}\times a^{2n-2m}\div a^{n-m}$. **18.** $\sqrt[m]{(a^{2m-n}b^{5m+1}c^{3p})}\times\sqrt[m]{(a^nb^{m-1}c^{m-3p})}$.

19. $\dfrac{x^{m+2n}x^{3m-8n}}{x^{5m-6n}}$. (C. E. 1874). **20.** $\dfrac{x^{a+b}x^{a-b}x^{c-2a}}{x^{c-a}}$. (C. E. 1870).

21. $\{(x^{a+b-c}\times x^{a-b+c})^b\}^c$. (M. M. 1889).

22. $(a+b)^m\times(a-b)^m\times(a^2+b^2)^m$.

23. $\left(\dfrac{x^a}{x^b}\right)^{a+b}\div\left(\dfrac{x^a}{x^{a-b}}\right)^b$. (M. M. 1890).

24. $\left(\dfrac{x^p}{x^q}\right)^{p+q}+\left(\dfrac{x^{p+q}}{x^{p-q}}\right)^{\frac{p^2}{q}}$. (C. E. 1902).

25. $\dfrac{\left(p+\frac{1}{q}\right)^m\left(p-\frac{1}{q}\right)^n}{\left(q+\frac{1}{p}\right)^m\left(q-\frac{1}{p}\right)^n}$. (B. M. 1889).

26. $\dfrac{\left(p^2-\frac{1}{q^2}\right)^p\left(p-\frac{1}{q}\right)^{q-p}}{\left(q^2-\frac{1}{p^2}\right)^q\left(q+\frac{1}{p}\right)^{p-q}}$. (B. M. 1891).

27. $\left\{\sqrt[3]{\frac{x^2}{y^4}}\times\sqrt{\frac{y^3}{x^5}}\right\}^{12}\times x^{22}$. (M. M. 1894).

28. $\left(\frac{x^m}{x^n}\right)^{m+n}\times\left(\frac{x^n}{x^l}\right)^{n+l}\times\left(\frac{x^l}{x^m}\right)^{l+m}$. (C. E. 1900).

29. $\dfrac{(x^{l+2m})^2\times(x^{m+2n})^2\times(x^{n+2l})^2}{(x^l x^m x^n)^6}$.

30. $\dfrac{\{(a^m)^{\frac{1}{r}}(a^q)^{\frac{1}{n}}\}^{nr}}{\{\sqrt[q]{(b^n)}\sqrt[m]{(b^r)}\}^{mq}}\div\left\{\left(\frac{a}{b}\right)^{qc}\right\}^{r}$.

31. $\dfrac{2^n\times 4^{n+1}}{8^{n-2}}$.

32. $\dfrac{5^{-n}\times 25^{2n-2}}{5^{3n-2}\times 10^{-1}}$.

33. $\dfrac{2^{n+1}}{(2^n)^{n-1}}\div\dfrac{4^{n+1}}{(2^{n+1})^{n-1}}$.

34. If $m=a^x$, $n=a^y$ and $a^2=(m^y n^x)^z$, shew that $xyz=1$. (B. M. 1890).

35. If $x^y=y^x$, prove that $\left(\frac{x}{y}\right)^{\frac{x}{y}}=x^{\frac{x}{y}-1}$; and if $x=2y$, prove that $y=2$.

II. ALGEBRAICAL OPERATIONS INVOLVING FRACTIONAL AND NEGATIVE INDICES.

301. The ordinary methods of operation employed in Multiplication, Division, &c., of positive integral indices are applicable to expressions involving fractional and negative indices. We now give some illustrative Examples.

Ex. 1. Multiply $a^{\frac{5}{2}}+a^2b^{\frac{1}{3}}+a^{\frac{3}{2}}b^{\frac{2}{3}}+ab+a^{\frac{1}{2}}b^{\frac{4}{3}}+b^{\frac{5}{3}}$ by $a^{\frac{1}{2}}-b^{\frac{1}{3}}$.

$$\begin{array}{l}
a^{\frac{5}{2}}+a^2b^{\frac{1}{3}}+a^{\frac{3}{2}}b^{\frac{2}{3}}+ab+a^{\frac{1}{2}}b^{\frac{4}{3}}+b^{\frac{5}{3}}\\
a^{\frac{1}{2}}-b^{\frac{1}{3}}\\
\hline
a^3+a^{\frac{5}{2}}b^{\frac{1}{3}}+a^2b^{\frac{2}{3}}+a^{\frac{3}{2}}b+ab^{\frac{4}{3}}+a^{\frac{1}{2}}b^{\frac{5}{3}}\\
\quad -a^{\frac{5}{2}}b^{\frac{1}{3}}-a^2b^{\frac{2}{3}}-a^{\frac{3}{2}}b-ab^{\frac{4}{3}}-a^{\frac{1}{2}}b^{\frac{5}{3}}-b^2\\
\hline
a^3 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -b^2.
\end{array}$$

Here in the first line

$$a^{\frac{5}{2}}\times a^{\frac{1}{2}}=a^{\frac{5}{2}+\frac{1}{2}}=a^{3},\quad a^{2}b^{\frac{1}{3}}\times a^{\frac{1}{2}}=a^{2+\frac{1}{2}}b^{\frac{1}{3}}=a^{\frac{5}{2}}b^{\frac{1}{3}},\text{ and so on.}$$

Ex. 2. Divide $x^{\frac{5}{2}}-a^{\frac{1}{2}}x^{2}-4ax^{\frac{3}{2}}+6a^{\frac{3}{2}}x-2a^{2}x^{\frac{1}{2}}$ by $x^{\frac{3}{2}}-4ax^{\frac{1}{2}}+2a^{\frac{3}{2}}$.

$$x^{\frac{3}{2}}-4ax^{\frac{1}{2}}+2a^{\frac{3}{2}}\,\big)\,x^{\frac{5}{2}}-a^{\frac{1}{2}}x^{2}-4ax^{\frac{3}{2}}+6a^{\frac{3}{2}}x-2a^{2}x^{\frac{1}{2}}\,\big(\,x-a^{\frac{1}{2}}x^{\frac{1}{2}}.$$

$$\begin{array}{l} x^{\frac{5}{2}}\qquad\qquad -4ax^{\frac{3}{2}}+2a^{\frac{3}{2}}x \\ \hline -a^{\frac{1}{2}}x^{2}\qquad +4a^{\frac{3}{2}}x-2a^{2}x^{\frac{1}{2}} \\ -a^{\frac{1}{2}}x^{2}\qquad +4a^{\frac{3}{2}}x-2a^{2}x^{\frac{1}{2}} \\ \hline \end{array}$$

Ex. 3. Find the square root of $4x^{\frac{3}{2}}-12x^{\frac{3}{4}}+25-24x^{-\frac{3}{4}}+16x^{-\frac{3}{2}}$.

$$4x^{\frac{3}{2}}-12x^{\frac{3}{4}}+25-24x^{-\frac{3}{4}}+16x^{-\frac{3}{2}}\,\big(\,2x^{\frac{3}{4}}-3+4x^{-\frac{3}{4}}.$$

$$\begin{array}{ll} & 4x^{\frac{3}{2}} \\ \hline 4x^{\frac{3}{4}}-3\,\big) & -12x^{\frac{3}{4}}+25 \\ & -12x^{\frac{3}{4}}+9 \\ \hline 4x^{\frac{3}{4}}-6+4x^{-\frac{3}{4}}\,\big) & 16-24x^{-\frac{3}{4}}+16x^{-\frac{3}{2}} \\ & 16-24x^{-\frac{3}{4}}+16x^{-\frac{3}{2}} \\ \hline \end{array}$$

Ex. 4. Simplify $\dfrac{a^{\frac{3n}{2}}-a^{-\frac{3n}{2}}}{a^{\frac{n}{2}}-a^{-\frac{n}{2}}}$ (C. F. A. 1861)

$$\text{and}\quad \frac{\sqrt[3]{a^{4}}+\sqrt[3]{a^{2}b^{2}}-\sqrt[3]{a^{3}b}}{\sqrt[3]{a^{4}}+\sqrt[3]{ab^{3}}-\sqrt[3]{a^{3}b}-\sqrt[3]{b^{4}}}$$

(1) Let $a^{\frac{n}{2}}=x$ and $a^{-\frac{n}{2}}=y$.

Then the Exp. $=\dfrac{x^{3}-y^{3}}{x-y}=\dfrac{(x-y)(x^{2}+xy+y^{2})}{x-y}=x^{2}+xy+y^{2}$

$$=a^{n}+1+a^{-n},\text{ for } xy=a^{0}=1.$$

(2) Let $\sqrt[3]{a}=x$ and $\sqrt[3]{b}=y$.

$$\text{Then the Exp.}=\frac{x^4+x^2y^2-x^3y}{x^4+xy^3-x^3y-y^4}=\frac{x^2(x^2+y^2-xy)}{x(x^3+y^3)-y(x^3+y^3)}$$

$$=\frac{x^2(x^2+y^2-xy)}{(x-y)\,(x^3+y^3)}=\frac{x^2(x^2-xy+y^2)}{(x-y)(x+y)(x^2-xy+y^2)}$$

$$=\frac{x^2}{x^2-y^2}=\frac{\sqrt[3]{a^2}}{\sqrt[3]{a^2}-\sqrt[3]{b^2}}.$$

Ex. 5. Divide $x^{2^n}-y^{2^n}$ by $x^{2^{n-1}}+y^{2^{n-1}}$. (C. E. 1879).

Since $x^{2^n}-y^{2^n}=(x^{2^{n-1}})^2-(y^{2^{n-1}})^2$, Art. 289.

$=(x^{2^{n-1}}+y^{2^{n-1}})(x^{2^{n-1}}-y^{2^{n-1}})$. Art. 124.

$\therefore$ the quotient $=x^{2^{n-1}}-y^{2^{n-1}}$.

Exercise CXI.

Multiply

1. $x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y$ by $x^{\frac{1}{2}}-y^{\frac{1}{2}}$. (C. E. 1861).
2. $a^{\frac{3}{4}}+a^{\frac{1}{2}}b^{\frac{1}{2}}+a^{\frac{1}{4}}b+b^{\frac{3}{2}}$ by $a^{\frac{1}{4}}-b^{\frac{1}{2}}$.
3. $x^{\frac{1}{1}}y+y^{\frac{2}{3}}$ by $x^{\frac{1}{3}}-y^{\frac{1}{3}}$. (C. E. 1863).
4. $7x^{\frac{1}{2}}-3y^{\frac{1}{3}}+2x^{\frac{1}{3}}y^{\frac{2}{3}}$ by $6x^{\frac{1}{3}}-2y^{\frac{1}{3}}+7x^{\frac{2}{3}}y^{\frac{1}{3}}$. (C. E. 1858).
5. $ax^{\frac{1}{4}}+3a^{\frac{1}{2}}x^{\frac{1}{2}}+4x^{\frac{3}{4}}$ by $a-3a^{\frac{1}{2}}x^{\frac{1}{4}}+4x^{\frac{1}{2}}$. (C. E. 1890).
6. $x+2y^{\frac{1}{2}}+3z^{\frac{1}{3}}$ by $x-2y^{\frac{1}{2}}+3z^{\frac{1}{3}}$.
7. $a^{\frac{5}{2}}+2a^2b^{\frac{1}{3}}+4a^{\frac{3}{2}}b^{\frac{2}{3}}+8ab+16a^{\frac{1}{2}}b^{\frac{4}{3}}+32b^{\frac{5}{3}}$ by $a^{\frac{1}{2}}-2b^{\frac{1}{3}}$. (B. M. 1859).
8. $x+y+z-\sqrt{(xy)}-\sqrt{(yz)}-\sqrt{(zx)}$ by $\sqrt{x}+\sqrt{y}+\sqrt{z}$. (C. E. 1864).
9. $x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$ by $x^{\frac{1}{3}}-y^{\frac{1}{3}}$. (C. E. 1866).
10. $x^{-\frac{2}{3}}+x^{-\frac{1}{3}}+1$ by $x^{-\frac{1}{3}}-1$.
11. $a^{\frac{3}{2}}-a^{\frac{2}{3}}+a^{-\frac{1}{3}}-a^{-\frac{3}{2}}$ by $a^{\frac{3}{2}}+a^{\frac{2}{3}}-a^{-\frac{1}{3}}-a^{-\frac{3}{2}}$.
12. $x^{\frac{5}{8}}+x^{\frac{1}{2}}y^{-\frac{1}{8}}+x^{\frac{3}{8}}y^{-\frac{1}{4}}+x^{\frac{1}{4}}y^{-\frac{3}{8}}+x^{\frac{1}{8}}y^{-\frac{1}{2}}+y^{-\frac{5}{8}}$
    by $x^{\frac{3}{8}}-x^{\frac{1}{4}}y^{-\frac{1}{8}}+x^{\frac{1}{8}}y^{-\frac{1}{4}}-y^{-\frac{3}{8}}$.

Divide

13. $16x-y^2$ by $2x^{\frac14}-y^{\frac12}$. **14.** $x^{-1}-y^{-1}$ by $x^{-\frac13}-y^{-\frac13}$.

15. $a^{-3}-64b^2$ by $a^{-\frac12}+2b^{\frac13}$. **16.** $x-2x^{\frac12}+1$ by $x^{\frac13}-2x^{\frac16}+1$.

17. $x^{\frac43}+x^{\frac23}y^{\frac12}+y$ by $x^{\frac23}-x^{\frac13}y^{\frac14}+y^{\frac12}$. (C. E. 1860).

18. $(x^{\frac23}-a^{\frac23})(x^{\frac23}+a^{\frac23})$ by $x^{\frac13}+a^{\frac13}$. (C. E. 1859).

19. $x^{\frac43}+a^{\frac23}x^{\frac23}+a^{\frac43}$ by $x^{\frac23}+a^{\frac13}x^{\frac13}+a^{\frac23}$. (C. F. A. 1861).

20. $x+6a^{\frac15}x^{\frac45}+6a^{\frac35}x^{\frac25}+a+5a^{\frac25}x^{\frac35}+7a^{\frac45}x^{\frac15}$ by $x^{\frac15}+a^{\frac15}$. (C. E. 1891).

21. $8a^{\frac32}+b^{-\frac32}-c+6a^{\frac12}b^{-\frac12}c^{\frac13}$ by $2a^{\frac12}+b^{-\frac12}$.

22. $x^2y^{-2}+x^{-2}y^2+2$ by $x^{\frac23}y^{-\frac23}+x^{-\frac23}y^{\frac23}-1$. (M. F. A. 1894).

Find the square of

23. $a^{\frac15}-b^{\frac17}+c^{\frac13}$. (C. E. 1862). **24.** $a^{\frac23}-2a^{\frac13}+3-2a^{-\frac13}+a^{-\frac23}$.

Find the cube of

25. $a^{\frac13}b^{-1}+a^{-\frac13}b$. **26.** $\frac23x^{\frac23}y^{-\frac13}-\frac32x^{-\frac13}y^{\frac23}$ **27.** $a^{\frac23}-2a^{\frac13}b^{\frac16}+3b^{\frac13}$.

Find the fourth and fifth powers of

28. $x^{\frac14}-y^{\frac12}$ **29.** $a^{\frac12}b^{-\frac12}-a^{-\frac12}b^{\frac12}$. **30.** $a^{\frac12}-a^{-\frac12}$.

Find the square roots of

31. $\frac1{16}x^3-\frac16x^2-\frac14x^{\frac32}+\frac19x+\frac13x^{\frac12}+\frac14$. (B. M. 1886).

32. $x^{\frac83}-2a^{-\frac35}x^{\frac{11}{3}}+2a^{\frac45}x^{\frac45}+a^{-\frac65}x^{\frac{14}{3}}-2a^{\frac15}x^{\frac75}+a^{\frac85}$. (C. E. 1880).

33. $1+4\frac1{16}x-\frac{3+3x}{2}\sqrt{x}+x^2$. (P. E. 1888.)

34. $a^2b^{-2}+2ab^{-1}+3+2a^{-1}b+a^{-2}b^2$.

35. $a^{\frac43}-3a+\frac{33}{4}a^{\frac23}-21a^{\frac13}+45-63a^{-\frac13}+90a^{-\frac23}-108a^{-1}+81a^{-\frac43}$.

36. $a+2\sqrt{(2ab)}+2b+4\sqrt{(2ac)}+8\sqrt{(bc)}+8c$. (M. M. 1881).

37. $\frac{x^2}{y^2}+\frac{y^2}{x^2}-\left(\frac{x}{y}+\frac{y}{x}\right)\sqrt2+2\frac12$.

38. $\frac{(x+y)^2}{y}-x^{\frac14}y^{-\frac14}(x-\frac14x^{\frac14}y^{\frac34}+y)$. (M. F. A. 1889).

39. $3x^{\frac{1}{3}}(3x+4)+2x^{-1}(2x^{\frac{1}{3}}+1)-2\sqrt{2x^{-\frac{5}{6}}}(3x+2)$. (M. F. A. 1895).

40. $x^{\frac{4}{3}}y^{\frac{2}{3}}(y^{\frac{1}{6}}+x^{-\frac{1}{2}}y^{-\frac{1}{2}})^2+y^{-\frac{1}{3}}(x^{\frac{1}{6}}-2x^{-\frac{1}{2}}y^{-\frac{1}{2}})^2-x^{\frac{1}{6}}y^{-\frac{1}{6}}(x^{\frac{1}{6}}y^{-\frac{1}{6}}+4)$.
(M. F. A. 1896).

Find the cube root of

41. $a^{-\frac{3}{2}}x^{\frac{3}{2}}-3a^{-1}x+6a^{-\frac{1}{2}}x^{\frac{1}{2}}-7+6a^{\frac{1}{2}}x^{-\frac{1}{2}}-3ax^{-1}+a^{\frac{3}{2}}x^{-\frac{3}{2}}$.

Find the fourth roots of

42. $x^4y^{-\frac{4}{3}}-4x^{\frac{5}{2}}y^{-\frac{1}{3}}+6xy^{\frac{2}{3}}-4x^{-\frac{1}{2}}y^{\frac{5}{3}}+x^{-2}y^{\frac{8}{3}}$.

43. $16x^6-96x^{\frac{9}{2}}y^{\frac{3}{4}}+216x^3y^{\frac{3}{2}}-216x^{\frac{3}{2}}y^{\frac{9}{4}}+81y^3$.

Find the H. C. F. of

44. $e^{2x}x^3+e^{2x}-x^3-1$ and $e^{2x}x^2+2e^xx^2-2e^x+x^2-2e^x-1$.

45. $\frac{1}{3}x^2+\frac{1}{6}x\sqrt{(x+1)}-x-1$ and $x^2-\frac{1}{4}x-\frac{1}{4}$.

Find the L. C. M. of

46. $3ax^2-3a^2x$, x^2-a^2, x^2+ax, $\sqrt{(3ax)}$ and $\sqrt{x}-\sqrt{a}$. (C. E. 1873).

47. Multiply $a^{2n}-a^nx^n+x^{2n}$ by a^n+x^n. (C. E. 1879).

48. Divide $a^{5m}+b^{5n}$ by a^m+b^n. (C. E. 1901).

49. Simplify $\{\sqrt{(a^2+\sqrt[3]{a^4b^2})}+\sqrt{(b^2+\sqrt[3]{a^2b^4})}\}^{\frac{2}{3}}$.

50. Shew that $\dfrac{x^{2^n}-y^{2^n}}{x-y}=(x+y)(x^2+y^2)(x^4+y^4)\ldots\ldots\left(x^{2^{n-1}}+y^{2^{n-1}}\right)$.

III. ELEMENTARY SURDS.

302. It was stated in Art. 177, that, when any root of a quantity cannot be exactly obtained, it is expressed by the use of the sign of Evolution, and called an **Irrational** or **Surd quantity.**

Thus, $\sqrt{2}$, $\sqrt[3]{(3ab)}$ and $\sqrt[5]{(a^3+b^3)}$ are *Surds.*

303. The **order** of a surd is denoted by the root-symbol or surd-index.

Thus, $\sqrt[3]{a}$ and $\sqrt[n]{a}$ are surds of the *third* and *nth orders* respectively.

304. Surds of the second order are called **Quadratic surds** and of the third order are called **Cubic surds.**

Thus $\sqrt{2}$, $\sqrt{(a+b)}$ are *Quadratic surds*, and $\sqrt[3]{a}$, $\sqrt[3]{5}$ are called *Cubic surds*.

305. Since every fractional index indicates by its denominator a root to be extracted, all quantities having such indices are expressed as **surds**.

I. Reduction of Surds.

306. In the case of a **numerical** surd, expressed with a fractional index, should the numerator be any other than *unity*, we may take at once the required power, and so have unity only for the numerator, and a simple root to be extracted.

Thus, $2^{\frac{2}{3}}=(2^2)^{\frac{1}{3}}=4^{\frac{1}{3}}$ or $\sqrt[3]{4}$; $3^{-\frac{3}{5}}=(3^{-3})^{\frac{1}{5}}=(\frac{1}{27})^{\frac{1}{5}}$ or $\sqrt[5]{\frac{1}{27}}$.

307. Quantities are often expressed in the **form** of surds, which are not *really* so, *i.e.*, when we *can*, if we please, extract the roots indicated.

Thus, $\sqrt{a}$, $\sqrt[3]{7}$, $(a^2+ab+b^2)^{\frac{1}{2}}$ are *actually* surds, whose roots we cannot obtain; but $\sqrt{a^2}$, $\sqrt[3]{27}$, $(4a^2+4ab+b^2)^{\frac{1}{2}}$ are *apparently* so, and are respectively equivalent to a, 3, $2a+b$.

308. Conversely, any rational quantity may be expressed in the form of a surd, by raising it to the power indicated by the denominator of the surd-index.

Thus, $2=4^{\frac{1}{2}}=\sqrt[3]{8}=$&c.; $a=\sqrt[3]{a^3}$; $\frac{1}{2}a=(\frac{1}{4}a^2)^{\frac{1}{2}}$;

$$a+b=(a^2+2ab+b^2)^{\frac{1}{2}}.$$

309. In like manner, a **mixed** surd, *i.e.*, a product partly rational and partly surd, may be expressed as an entire surd, by raising the rational factor to the power indicated by the denominator of the surd-index, and placing beneath the sign of Evolution the product of this power and the surd-factor.

Thus, $2\sqrt{3}=\sqrt{4}\times\sqrt{3}=\sqrt{12}$; $3\cdot2^{\frac{2}{3}}=3\sqrt[3]{4}=\sqrt[3]{(27)}\times\sqrt[3]{4}=\sqrt[3]{(108)}$.

$$2a\sqrt{b}=\sqrt{(4a^2b)}\,;\quad 4a\sqrt[3]{\frac{c}{2a}}=\sqrt[3]{\left(\frac{64a^3c}{2a}\right)}=\sqrt[3]{(32a^2c)}.$$

310. Conversely, a surd may often be reduced to a *mixed* form by separating the quantity beneath the sign of Evolution into factors, of one of which the root required may be obtained, and set outside the sign.

Thus, $\sqrt{(20)}=\sqrt{(4\times5)}=2\sqrt{5}$; $\sqrt[3]{(24)}=\sqrt[3]{(8\times3)}=2\sqrt[3]{3}$;

$$\sqrt{(\tfrac{48}{25}a^3b)}=\tfrac{4}{5}a\sqrt{(3ab)}\,;\quad \sqrt[4]{(\tfrac{32}{81}a^5b^4c^3)}=\tfrac{2}{3}ab\sqrt[4]{(2ac^3)}.$$

311. A surd is reduced to its simplest form, when the quantity beneath the root, or surd-factor, is made as *small* as possible, but so as still to remain *integral*.

Hence, if the surd-factor be a *fraction*, its numerator and denominator should both be multiplied by such a number, as will allow us to take the latter from under the root.

Thus, $\sqrt{\frac{2}{3}} = \sqrt{\left(\frac{2.3}{3^2}\right)} = \frac{1}{3}\sqrt{6}$; $\frac{5}{2}\sqrt[3]{\left(\frac{24}{5}\right)} = 5\sqrt[3]{\frac{3}{5}} = 5\sqrt[3]{\left(\frac{3.5^2}{5^3}\right)}$
$= \sqrt[3]{(75)}$.

These latter forms allow of our calculating more easily the numerical values of the surd quantities. Thus, to find that of $\sqrt{\frac{2}{3}}$, we should have had to extract both $\sqrt{2}$ and $\sqrt{3}$, and then to divide the one by the other, a tedious process, since each would be expressed by decimals that do not terminate; whereas, in $\frac{1}{3}\sqrt{6}$, we have only to find $\sqrt{6}$, and divide this by the integer 3.

Ex. Given $\sqrt{3} = 1{\cdot}73205......$, find the value of $\frac{5}{\sqrt{3}}$.

$$\frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{\sqrt{3}\times\sqrt{3}} = \frac{5}{3}\sqrt{3} = \frac{5\times 1{\cdot}73205...}{3} = \frac{8{\cdot}66025...}{3} = 2{\cdot}88675...$$

312. Surds which are not of the same order can be transformed into equivalent surds which are of the same order.

Ex. 1. Express $\sqrt[4]{(11)}$ and $\sqrt[6]{(13)}$ as surds of the same order.

Here, the L. C. M. of the root figures 4 and 6 is 12.

Therefore, $\sqrt[4]{(11)} = 11^{\frac{1}{4}} = 11^{\frac{3}{12}} = \sqrt[12]{(11^3)} = \sqrt[12]{(1331)}$,

and $\sqrt[6]{(13)} = 13^{\frac{1}{6}} = 13^{\frac{2}{12}} = \sqrt[12]{(13^2)} = \sqrt[12]{(169)}$.

313. To **compare** surds with one another in magnitude, express them as *entire* surds, and then reduce their indices, if necessary, to a common denominator, simplifying as in Art. 306: their relative values will be now apparent.

Ex. 2. Which is the greater $3\sqrt{2}$ or $2\sqrt[3]{3}$?

Now $3\sqrt{2} = \sqrt{18} = 18^{\frac{1}{2}}$, and $2\sqrt[3]{3} = \sqrt[3]{24} = 24^{\frac{1}{3}}$.

Also $18^{\frac{1}{2}} = 18^{\frac{3}{6}} = \sqrt[6]{18^3} = \sqrt[6]{5832}$, and $24^{\frac{1}{3}} = 24^{\frac{2}{6}} = \sqrt[6]{24^2} = \sqrt[6]{576}$.

The former is therefore the greater, since 5832 is greater than 576.

314. Similar surds are those which have, or may be made to have, the *same* surd-factors.

Thus, $3\sqrt{a}$ and $\sqrt{a}$, $2a\sqrt[3]{x}$ and $3b\sqrt[3]{x}$, are pairs of *similar surds*; and $\sqrt{8}$, $\sqrt{(50)}$ and $\sqrt{(18)}$ are also *similar*, because they may be written $2\sqrt{2}$, $5\sqrt{2}$ and $3\sqrt{2}$.

Exercise CXII.

Express the following with indices, whose numerator is unity.

1. $4^{\frac{3}{4}}$. **2.** $9^{\frac{2}{3}}$. **3.** $3^{-\frac{2}{3}}$. **4.** $2^{-\frac{3}{4}}$. **5.** $(\frac{2}{3})^{-\frac{1}{4}}$. **6.** $(\frac{1}{2})^{-\frac{3}{5}}$.

Express as surds of the second and third orders.

7. 5. **8.** $2\frac{1}{2}$. **9.** $\frac{2}{3}a$. **10.** $\frac{3}{2}a^2$. **11.** $\frac{1}{2}(a+b)$.

Express with indices $\frac{1}{3}$ and $-\frac{1}{4}$.

12. 3^{-2}. **13.** $(3\frac{1}{3})^{-1}$. **14.** a^{2}. **15.** $ab^{-1}c^{-1}$.

Reduce to entire surds :—

16. $5\sqrt{5}$. **17.** $\frac{1}{2}\sqrt{\frac{3}{4}}$. **18.** $\frac{2}{3}\cdot 3^{\frac{3}{2}}$. **19.** $\frac{3}{5}\sqrt{1\frac{2}{3}}$. **20.** $\frac{1}{2}(\frac{3}{4})^{-\frac{1}{2}}$.

21. $25(1\frac{1}{4})^{-\frac{3}{2}}$. **22.** $3\sqrt[3]{2}$. **23.** $8.2^{-\frac{5}{3}}$. **24.** $4.2^{\frac{1}{4}}$. **25.** $3.3^{-\frac{1}{4}}$.

26. $\frac{2}{3}(\frac{2}{3})^{-\frac{2}{3}}$. **27.** $\frac{1}{2}(\frac{3}{4})^{-\frac{2}{3}}$. **28.** $2\sqrt{a}$. **29.** $7a\sqrt{(2x)}$. **30.** $a(ab)^{-1}$.

31. $(a+b)(a^2-b^2)^{-\frac{1}{2}}$. **32.** $(a-b)(a^3-b^3)^{-1}$. **33.** $x\sqrt[n]{(y^2z)}$.

34. $a\sqrt{\frac{2b}{a}}$. **35.** $3ax\sqrt{\frac{2a}{3x}}$. **36.** $\frac{2a}{3b}\sqrt[4]{\frac{3b}{2a}}$. **37.** $\frac{2a}{3}\sqrt[3]{\frac{9}{4a^2}}$.

38. $\frac{3x}{2}\sqrt[4]{\left(\frac{400y^2}{81x^2}\right)}$. (C. E. 1873). **39.** $(a+x)\sqrt{\left(\frac{a-x}{a+x}\right)}$.

40. $\frac{x^3}{y}\sqrt{\left(\frac{y^4z^2}{x^8}\right)}$. **41.** $\frac{a+x}{\sqrt{(a^2-x^2)}}$. **42.** $\frac{a}{b}\sqrt[3]{\left(\frac{b^2}{a^2}\right)}$.

Reduce to their simplest forms :—

43. $\sqrt{45}$. **44.** $\sqrt{125}$. **45.** $3\sqrt{432}$. **46.** $\sqrt[3]{135}$. **47.** $3\sqrt[3]{432}$.

48. $\sqrt{\frac{3}{2}}$. **49.** $2\sqrt[3]{\frac{3}{2}}$. **50.** $3\sqrt[4]{\frac{2}{3}}$. **51.** $4\sqrt[3]{3\frac{3}{8}}$. **52.** $8^{\frac{3}{4}}$.

53. $32^{\frac{2}{5}}$. **54.** $72^{\frac{2}{3}}$. **55.** $(1\frac{1}{3})^{-\frac{1}{2}}$. **56.** $(20\frac{1}{4})^{-\frac{3}{4}}$. **57.** $(30\frac{3}{8})^{-\frac{2}{3}}$.

58. $\frac{3}{2}\sqrt{\frac{12}{7}}$. **59.** $5\sqrt[3]{4\frac{1}{20}}$. **60.** $\frac{5}{2}\sqrt[4]{9\frac{3}{5}}$. **61.** $\sqrt{(a^5b^3)}$. **62.** $\sqrt[3]{(a^7b^5c^6)}$.

Express as fractions with the surd part integral :-

63. $\sqrt{\frac{6}{7}}$. **64.** $\sqrt{\frac{3}{5}}$. **65.** $\sqrt{\frac{3}{8}}$. **66.** $\sqrt[3]{\frac{3}{4}}$. **67.** $\sqrt[3]{\frac{5}{6}}$.

Express as surds of the same order :—

68. $\sqrt{5}$ and $\sqrt[3]{11}$. **69.** $\sqrt[3]{7}$ and $\sqrt[4]{9}$. **70.** $\sqrt[4]{4}$ and $\sqrt[6]{5}$.

Which is the greater ?

71. $6\sqrt{3}$ or $4\sqrt{7}$. **72.** $3\sqrt[3]{3}$ or $2\sqrt[3]{10}$.

73. $\sqrt{5}$ or $\sqrt[3]{11}$. **74.** $\frac{1}{2}\sqrt{2}$ or $\frac{1}{3}\sqrt[4]{27}$.

Which is the greatest?

75. $2\sqrt[3]{15}$, $4\sqrt[3]{2}$ or $3\sqrt[3]{5}$.

76. $\sqrt{5}$, $2\sqrt[3]{\frac{3}{2}}$ or or $3(4\frac{1}{2})^{-\frac{1}{6}}$.

77. Shew that $\sqrt{12}$, $3\sqrt{75}$, $\frac{1}{2}\sqrt{147}$, $\frac{2}{3}\sqrt{\frac{4}{75}}$, $\sqrt[4]{\frac{9}{16}}$ and $(144)^{-\frac{1}{4}}$ are similar surds.

78. Given $\sqrt{5}=2\cdot236068\ldots$, find the value of $\frac{3}{\sqrt{5}}$.

79. Given $\sqrt{6}=2\cdot449489\ldots\ldots$, find the value of $\sqrt{\frac{2}{3}}$.

80. Given $\sqrt{7}=2\cdot645751\ldots$, find the value of $\sqrt{\frac{9}{7}}$.

II. Addition and Subtraction of Surds.

315. To **add** or **subtract** surds, reduce them, when similar, to the same surd factor, and add or subtract their rational factors.

Ex. 1. $\sqrt{8}+\sqrt{50}-\sqrt{18}=\sqrt{(4\times2)}+\sqrt{(25\times2)}-\sqrt{(9\times2)}$
$=2\sqrt{2}+5\sqrt{2}-3\sqrt{2}=(2+5-3)\sqrt{2}=4\sqrt{2}.$

Ex. 2. $4a\sqrt[3]{(a^3b^4)}+b\sqrt[3]{(8a^6b)}-\sqrt[3]{(125a^6b^4)}$
$=4a\sqrt[3]{(a^3b^3\times b)}+b\sqrt[3]{(2^3a^6\times b)}-\sqrt[3]{(5^3a^6b^3\times b)}$
$=4a^2b\sqrt[3]{b}+2a^2b\sqrt[3]{b}-5a^2b\sqrt[3]{b}=(4a^2b+2a^2b-5a^2b)\sqrt[3]{b}$
$=a^2b\sqrt[3]{b}.$

Dissimilar surds can only be connected by their signs.

Ex. 3. $\sqrt{32}+\sqrt[3]{16}-\sqrt[4]{64}=\sqrt{(16\times2)}+\sqrt[3]{(8\times2)}-\sqrt[4]{(16\times4)}$
$=4\sqrt{2}+2\sqrt[3]{2}-2\sqrt{2}=(4-2)\sqrt{2}+2\sqrt[3]{2}$
$=2\sqrt{2}+2\sqrt[3]{2}.$

III. Multiplication and Division of Surds.

316. To **multiply** surds, reduce them by Art. 312 to the same surd-index, and multiply separately the rational and surd-factors, retaining the same surd-index for the product of the latter.

Ex. 1. $\sqrt{8}\times3\sqrt{2}=3\sqrt{16}=3\times4=12.$

Ex. 2. $2\sqrt{3}\times3\sqrt{10}\times4\sqrt{6}=24\sqrt{180}=24\sqrt{(36\times5)}$
$=24\times6\sqrt{5}=144\sqrt{5}.$

Ex. 3. $2\sqrt{3}\times3\sqrt[3]{2}=2\sqrt[6]{27}\times3\sqrt[6]{4}=6\sqrt[6]{108}.$

317. **Compound surd quantities** are multiplied according to the method of rational quantities.

Ex. 1. $(2+\sqrt{3})(3-\sqrt{2})=6+3\sqrt{3}-2\sqrt{2}-\sqrt{6}$.

Ex. 2. $(2+\sqrt{3})(2-\sqrt{3})=4-3=1$.

Ex. 3. $(2\pm\sqrt{3})^2=4\pm4\sqrt{3}+3=7\pm4\sqrt{3}$.

Ex. 4. $(1+\sqrt{2})^4=1+4\sqrt{2}+12+8\sqrt{2}+4=17+12\sqrt{2}$.

318. **Division** of surds is performed, when the divisor is a simple quantity, by a process similar to that for Multiplication.

Ex. 1. $6\sqrt{3}\div5\sqrt{2}=\dfrac{6\sqrt{3}}{5\sqrt{2}}=\frac{6}{5}\sqrt{\frac{3}{2}}=\frac{6}{5}\sqrt{\frac{6}{4}}=\frac{6}{5}\times\dfrac{\sqrt{6}}{2}=\dfrac{3\sqrt{6}}{5}$.

Ex. 2. $(8\sqrt{2}-12\sqrt{3}+3\sqrt{6}-4)\div2\sqrt{6}=4\sqrt{\frac{2}{6}}-6\sqrt{\frac{3}{6}}+\frac{3}{2}-\dfrac{2}{\sqrt{6}}$

$$=4\sqrt{\tfrac{1}{3}}-6\sqrt{\tfrac{1}{2}}+\tfrac{3}{2}-\frac{2}{\sqrt{6}}=\tfrac{4}{3}\sqrt{3}-3\sqrt{2}+\tfrac{3}{2}-\tfrac{1}{3}\sqrt{6}.$$

Ex. 3. $(2\sqrt{3}-6\sqrt[3]{2})\div\sqrt{6}=2\sqrt{\frac{3}{6}}-6\sqrt[6]{\frac{4}{216}}=2\sqrt{\frac{1}{2}}-6\sqrt[6]{\frac{1}{54}}$
$=\sqrt{2}-\sqrt[6]{864}$.

Exercise CXIII.

Simplify

1. $\sqrt{128}-2\sqrt{50}+\sqrt{72}-\sqrt{18}$. 2. $\sqrt[3]{40}-\frac{1}{2}\sqrt[3]{320}+\sqrt[3]{135}$.

3. $8\sqrt{\frac{3}{4}}-\frac{1}{2}\sqrt{12}+4\sqrt{27}-2\sqrt{\frac{3}{16}}$. 4. $\sqrt[3]{72}-3\sqrt[3]{\frac{1}{3}}+6\sqrt[3]{21\frac{1}{3}}$.

5. $\sqrt{18}+\dfrac{8}{\sqrt{2}}+\dfrac{\sqrt{24}}{3\sqrt{3}}$. 6. $\sqrt[3]{\left(\dfrac{16a^4x}{3b^4}\right)}+\sqrt[3]{\left(\dfrac{27axy^3}{81bc^3}\right)}-\sqrt[3]{\left(\dfrac{ax^4}{12b^4}\right)}$.

Multiply

7. $3\sqrt{8}$ by $2\sqrt{6}$. 8. $3\sqrt{15}$ by $4\sqrt{20}$. 9. $2\sqrt[3]{4}$ by $3\sqrt[3]{54}$.

10. $3\sqrt{3}+2\sqrt{2}$ by $\sqrt{3}-\sqrt{2}$. 11. $2\sqrt{15}-\sqrt{6}$ by $\sqrt{5}+2\sqrt{2}$.

12. $\sqrt{2}+\sqrt{3}+\sqrt{\frac{2}{3}}$ by $\sqrt{6}-\sqrt{2}$. 13. $\sqrt{3}+\sqrt{2}$ by $\dfrac{1}{\sqrt{3}}+\dfrac{1}{\sqrt{2}}$.

Find the continued product of

14. $3\sqrt{8}$, $2\sqrt[3]{6}$ and $3\sqrt[4]{54}$. 15 $2\sqrt{24}$, $3\sqrt[4]{18}$ and $4\sqrt[6]{24}$.

16. $4+2\sqrt{2}$, $1-\sqrt{3}$, $4-2\sqrt{2}$, $\sqrt{2}+\sqrt{3}$, $1+\sqrt{3}$ and $\sqrt{2}-\sqrt{3}$.

17. $x-1+\sqrt{2}$, $x-1-\sqrt{2}$, $x+2+\sqrt{3}$ and $x+2-\sqrt{3}$.

Divide

18. $6\sqrt{7}$ by $5\sqrt{3}$ 19. $3\sqrt{5}$ by $7\sqrt{12}$. 20. $5\sqrt[3]{6}$ by $3\sqrt{10}$.

21. $2\sqrt{3}+3\sqrt{2}+\sqrt{30}$ by $3\sqrt{6}$. 22. $2\sqrt{3}+3\sqrt[3]{2}+\sqrt[4]{30}$ by $3\sqrt{2}$.

23. $x^2+2xy+y^2+4x+4y+16$ by $x+y-2\sqrt{(x+y)}+4$.

24. Prove that $(\sqrt{5}+\sqrt{3}+\sqrt{2}+1)^2+(\sqrt{5}-\sqrt{3}-\sqrt{2}+1)^2$
$+(\sqrt{5}+\sqrt{3}-\sqrt{2}-1)^2+(\sqrt{5}-\sqrt{3}+\sqrt{2}-1)^2=44$.

IV. Rationalization of Surds.

319. But, if the divisor be **compound**, the division is not easily performed. The form, however, in which compound surds usually occur, is that of a *binomial quadratic surd*, *i. e.*, a binomial, one or both of whose terms are surds, in which the *square* root is to be taken, such as $3+2\sqrt{5}$, $2\sqrt{3}-3\sqrt{5}$, or, generally, $\sqrt{a}\pm\sqrt{b}$, where one or both terms may be irrational · and it will be easy, in such a case, to convert the operation of division into one of multiplication, by putting the dividend and divisor in the form of a fraction, and multiplying both numerator and denominator by that quantity, which is obtained by changing the sign between the two terms of the denominator. By this means the denominator will be made **rational** : thus, if it be originally of the form $\sqrt{a}\pm\sqrt{b}$, it will become a rational quantity, $a-b$, when both numerator and denominator are multiplied by $\sqrt{a}\mp\sqrt{b}$ This process is called **rationalizing the denominator of a fraction.**

Ex. 1. $$\frac{2+\sqrt{3}}{3+\sqrt{3}}=\frac{(2+\sqrt{3})(3-\sqrt{3})}{(3+\sqrt{3})(3-\sqrt{3})}=\frac{6+3\sqrt{3}-2\sqrt{3}-3}{9-3}$$
$$=\frac{3+\sqrt{3}}{6}.$$

Ex. 2. $$\frac{1}{2\sqrt{2}-\sqrt{3}}=\frac{2\sqrt{2}+\sqrt{3}}{(2\sqrt{2}-\sqrt{3})(2\sqrt{2}+\sqrt{3})}=\frac{2\sqrt{2}+\sqrt{3}}{8-3}$$
$$=\frac{2\sqrt{2}+\sqrt{3}}{5}.$$

If, however, we had required the value of $\frac{1}{2\sqrt{2}-\sqrt{3}}$ to three places of decimals, we take the form $\frac{2\sqrt{2}+\sqrt{3}}{5}$,

$$\text{the answer}=\frac{2\times 1\cdot41421\ldots+1\cdot73205\ldots}{5}$$
$$=\frac{2\cdot82842\ldots+1\cdot73205\ldots}{5}=\frac{4\cdot56047\ldots}{5}=\cdot91209\ldots$$

320. When two quadratic surds differ only in the sign between their two terms, they are said to be **conjugate**.

Thus, $2\sqrt{3}+3\sqrt{2}$ and $2\sqrt{3}-3\sqrt{2}$ are *conjugate*.

In general $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$ are *conjugate*.

321. If there be three terms in the denominator it will be necessary, in general, to perform two such multiplications as above. We always multiply in such a case by a quantity which differs from the denominator in the sign of one of its terms.

Ex. Find the value of $\dfrac{1}{2+\sqrt{2}-\sqrt{3}}$.

If we multiply numerator and denominator by $2+\sqrt{2}+\sqrt{3}$, the expression $=\dfrac{1}{(2+\sqrt{2}-\sqrt{3})(2+\sqrt{2}+\sqrt{3})}=\dfrac{1}{(2+\sqrt{2})^2-3}=\dfrac{1}{3+4\sqrt{2}}$

$=\dfrac{3-4\sqrt{2}}{(3+4\sqrt{2})(3-4\sqrt{2})}=\dfrac{3-4\sqrt{2}}{9-32}=\dfrac{4\sqrt{2}-3}{23}$.

Exercise CXIV.

Divide

1. $2+4\sqrt{7}$ by $2\sqrt{7}-1$. 2. $3+2\sqrt{5}$ by $2\sqrt{5}-1$.

3. $5-2\sqrt{6}$ by $6-2\sqrt{6}$.

Rationalize the denominators of :—

4. $\dfrac{1}{2\sqrt{2}+\sqrt{3}}$. 5. $\dfrac{4}{\sqrt{5}-1}$. 6. $\dfrac{3}{\sqrt{5}+\sqrt{2}}$. 7. $\dfrac{8+5\sqrt{2}}{3-2\sqrt{2}}$.

8. $\dfrac{\sqrt{3}+\sqrt{2}}{2\sqrt{3}+\sqrt{2}}$. 9. $\dfrac{3+\sqrt{5}}{3-\sqrt{5}}$. 10. $\dfrac{10\sqrt{3}+3\sqrt{7}}{7\sqrt{7}-5\sqrt{3}}$. 11. $\dfrac{4\sqrt{7}+3\sqrt{2}}{5\sqrt{2}+2\sqrt{7}}$.

Find the value correct to four places of decimals of :—

12. $\dfrac{4}{3-2\sqrt{2}}$. 13. $\dfrac{2+4\sqrt{7}}{2\sqrt{7}+1}$. 14. $\dfrac{4\sqrt{7}+3\sqrt{2}}{5\sqrt{2}-2\sqrt{7}}$. 15. $\dfrac{7}{5\sqrt{3}-2\sqrt{2}}$.

Find the value of :—

16. $\left(\dfrac{10+9\sqrt{5}}{9+2\sqrt{5}}\right)^2$. 17. $\dfrac{5}{\sqrt{15}+\sqrt{6}}-\dfrac{1}{\sqrt{60}-\sqrt{24}}$. 18. $\dfrac{2(1+\sqrt{3})}{1-\sqrt{2}+\sqrt{3}}$.

19. $\left(\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}\right)^2-\left(\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}\right)^2$. 20. $\dfrac{1-\sqrt{2}+\sqrt{3}}{1+\sqrt{2}+\sqrt{3}}-\dfrac{1-\sqrt{2}-\sqrt{3}}{1+\sqrt{2}-\sqrt{3}}$.

21. $\dfrac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}-\sqrt{(a-x)}}$. 22. $\dfrac{1}{a-\sqrt{(a^2-x^2)}}-\dfrac{1}{a+\sqrt{(a^2-x^2)}}$.

23. $\dfrac{x+\sqrt{(x^2-1)}}{x-\sqrt{(x^2-1)}}-\dfrac{x-\sqrt{(x^2-1)}}{x+\sqrt{(x^2-1)}}$. (B. M. 1863).

24. $\dfrac{\sqrt{(x^2+1)}+\sqrt{(x^2-1)}}{\sqrt{(x^2+1)}-\sqrt{(x^2-1)}}+\dfrac{\sqrt{(x^2+1)}-\sqrt{(x^2-1)}}{\sqrt{(x^2+1)}+\sqrt{(x^2-1)}}$.

25. $\dfrac{c\sqrt{(ab)}-ac}{bc-c\sqrt{(ab)}}$. 26. $\dfrac{1}{4(1+\sqrt{x})}+\dfrac{1}{4(1-\sqrt{x})}+\dfrac{1}{2(1+x)}$.

27. Prove that $\dfrac{3\sqrt{8}+2\sqrt{7}}{\sqrt{8}+\sqrt{7}}=2{\cdot}51\ldots\ldots$(C. F. A. 1877).

28. Simplify $\dfrac{1}{1+\sqrt{2}+\sqrt{3}}+\dfrac{1}{1+\sqrt{2}-\sqrt{3}}+\dfrac{1}{1-\sqrt{2}+\sqrt{3}}$

$$+\frac{1}{\sqrt{2}+\sqrt{3}-1}.$$

29. Find the value of $\dfrac{x^2-x-2}{x^2-3x+2}+\dfrac{2x^2+x-3}{2x^2+5x+3}-2$, when $x=1+\sqrt{3}$.

30. Find the value of $\dfrac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}-\sqrt{(a-x)}}$, when $x=\dfrac{24}{25}a$.

CHAPTER XII.

HARDER FORMULÆ AND TRANSFORMATIONS.

I. HARDER FORMULAE.

322. There are other results in Multiplication which are not quite so important as the **Formulæ** (general results expressed in symbols) given in Arts. 102 and 170, 172, 173, but which are deserving of notice. We give them here in order that the student may be able to refer to them when they are required; they can be easily verified by actual multiplication.

1. $(a+b)^2+(a-b)^2=2a^2+2b^2.$

2. $(a+b)^2-(a-b)^2=4ab.$

3. $(a+b)^3+(a-b)^3=2a^3+6ab^2.$

4. $(a+b)^3-(a-b)^3=6a^2b+2b^3.$

5. $(b-c)+(c-a)+(a-b)=0.$

6. $a(b-c)+b(c-a)+c(a-b)=0.$

7. $(b-c)^2+(c-a)^2+(a-b)^2=2(a^2+b^2+c^2-bc-ca-ab).$

8. $(a+b+c)(a^2+b^2+c^2-bc-ca-ab)=a^3+b^3+c^3-3abc.$

9. $(bc+ca+ab)^2=b^2c^2+c^2a^2+a^2b^2+2abc(a+b+c).$

10.
$$\left.\begin{aligned}
&(a+b+c)(bc+ca+ab)\\
&\quad=(b+c)(c+a)(a+b)+abc \ldots\ldots\ldots\ldots(1)\\
&\quad=bc(b+c)+ca(c+a)+ab(a+b)+3abc \ldots(2)\\
&\quad=a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc \ldots(3)\\
&\quad=a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+3abc \ldots(4)
\end{aligned}\right\}$$

11.
$$\left.\begin{aligned}
&(b+c)(c+a)(a+b)\\
&\quad=a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+2abc \ldots\ldots(1)\\
&\quad=a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc \ldots\ldots(2)\\
&\quad=bc(b+c)+ca(c+a)+ab(a+b)+2abc \ldots\ldots(3)\\
&\quad=(a+b+c)(bc+ca+ab)-abc \ldots\ldots\ldots\ldots(4)
\end{aligned}\right\}$$

12.
$$\left.\begin{aligned}
-(b-c)(c-a)(a-b)&=a^2(b-c)+b^2(c-a)+c^2(a-b) \ldots\ldots(1)\\
&=bc(b-c)+ca(c-a)+ab(a-b) \ldots\ldots(2)\\
&=-\{a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)\} \ldots(3)
\end{aligned}\right\}$$

Ex 1. Simplify $(a+b+c)^2+(b+c-a)^2+(c+a-b)^2+(a+b-c)^2$.

The Exp. $=\{(b+c)+a\}^2+\{(b+c)-a\}^2+\{a-(b-c)\}^2+\{a+(b-c)\}^2$
$$=2(b+c)^2+2a^2+2a^2+2(b-c)^2, \quad (\text{F. I})$$
$$=4a^2+2\{(b+c)^2+(b-c)^2\}=4a^2+4b^2+4c^2.$$
$$=4(a^2+b^2+c^2).$$

Ex. 2. Simplify
$$(a-b)(x-a)(x-b)+(b-c)(x-b)(x-c)+(c-a)(x-c)(x-a).$$

The Exp. $=(a-b)\{x^2 \quad (a+b)x+ab\}+(b-c)\{x^2-(b+c)x+bc\}+$
$$(c-a)\{x^2-(c+a)x+ca\}$$
$$=x^2\{(a-b)+(b-c)+c-a)\}-x\{(a+b)(a-b)+(b+c)(b-c)+$$
$$(c-a)(c+a)\}+ab(a-b)+bc(b-c)+ca(c-a)$$
$$=x^2\times 0+x\{(a^2-b^2)+(b^2-c^2)+(c^2-a^2)\}+ab(a-b)+$$
$$bc(b-c)+ca(c-a)$$
$$=ab(a-b)+bc(b-c)+ca(c-a).$$
$$=-(b-c)(c-a)a-b).$$

Ex. 3. Simplify
$$(a+b+c)(a+b+a)+(a+c+d)(b+c+d)-(a+b+c+d)^2.$$

Let $a+b+c+d=x$, then

$$\text{The Exp} =(x-d)(x-c)+(x-b)(x-a)-x^2$$
$$=x^2-(c+d)x+cd+x^2-(a+b)x+ab-x^2$$
$$=x^2-(a+b+c+d)x+ab+cd$$
$$=ab+cd, \text{ for } a+b+c+d=x.$$

Ex. 4. Multiply $x^2+(3a+4b)x+12ab$ by $x^2-(3a+4b)x+12ab$.

We have $x^2+(3a+4b)x+12ab=(x+3a)(x+4b)$.
and $x^2-(3a+4b)x+12ab=(x-3a)(x-4b)$.

$$\therefore \text{ Product}=(x+3a)(x-3a)(x+4b)(x-4b)$$
$$=(x^2-9a^2)(x^2-16b^2)$$
$$=x^4-(9a^2+16b^2)x^2+144a^2b^2.$$

Ex. 5. Find the value of

$$(x+2y+z)(x^2+4y^2+z^2-2yz-zx-2xy).$$

$$\text{The Exp.}=\{(x+2y)+z\}[\{(x+2y)^2-z(x+2y)+z^2\}-6xy]$$
$$=(x+2y)^3+z^3-6xy(x+2y+z)$$
$$=x^3+8y^3+6xy(x+2y)+z^3-6xy(x+2y)-6xyz$$
$$=x^3+8y^3+z^3-6xyz$$

Exercise CXV.

Simplify

1. $(a+b)^2+(b+c)^2+(c+a)^2-(a+b+c)^2$.
2. $(a+b+c)^2+(a+b-c)^2-(c+a-b)^2-(b+c-a)^2$.
3. $(a+b+c+d)^2+(a-b+c-d)^2+(a+b-c-d)^2+(a-b-c+d)^2$.
4. $(a-b)(x+a)(x+b)+(b-c)(x+b)(x+c)+(c-a)(x+c)(x+a)$.
5. $(a^2+b^2+c^2)^2+(a+b+c)(b+c-a)(a+c-b)(a+b-c)$.
6. $(a^2+b^2+c^2)^2-(a+b+c)(b+c-a)(a+c-b)(a+b-c)$.
7. $(a+b+c)^3-(b+c-a)^3-(c+a-b)^3-(a+b-c)^3$.
8. $(3a+2b+5c)^3-(3a+2b-5c)^3-30c\{(3a+2b)^2-25c^2\}$.
9. $(b+c)(c+a)(a+b)-(a+b+c)(bc+ca+ab)+2abc$.
10. $(16x^5-20x^3+5x)^2+(1-x^2)\{16(1-x^2)^2-20(1-x^2)+5\}^2$.
    (C. E. 1889).
11. $(a+b+c)^3+(a+b-c)^3+(a-b+c)^3+(c+a-b)^3$.
12. $(a+b+c)^3-(b+c)^3-(c+a)^3-(a+b)^3+a^3+b^3+c^3$.
13. $(a+b+c)^3-a^3-b^3-c^3-3(b+c)(c+a)(a+b)$.

14. $(a+b+2c)(a+2b+c)(2a+b+c)-(b+c)(c+a)(a+b)$.

15. $(a+b+c)(x+y+z)+(a+b-c)(x+y-z)+(b+c-a)(y+z-x)$
$+(c+a-b)(z+x-y)$.

16. $(a-b+2)(a^2+b^2+4+ab-2a+2b)$.

17. $(x-2y-3)(x^2+y^2+9-6y+3x+2xy)$.

18. $(b-c)(b+c-a)+(c-a)(c+a-b)+(a-b)(a+b-c)$.

19. $(b-c)(1+ab)(1+ca)+(c-a)(1+bc)(1+ba)+(a-b)(1+ca)(1+cb)$

20. $(1-a^2)(1-b^2)(1-c^2)+(a-bc)(b-ca)(c-ab)$.

21. $(x-y)^3+(x+y)^3+3(x-y)^2(x+y)+3(x-y)(x+y)^2$. (C. E. 1876).

II. TRANSFORMATIONS.

323. The following **Transformations** (changes of form) of algebraical expressions are deserving of attention. They can easily be verified by actual multiplication.

1. (i) $\boldsymbol{a^2+b^2=(a+b)^2-2ab}$ or $\boldsymbol{=(a-b)^2+2ab}$.

(ii) $\boldsymbol{(a+b)^2=(a-b)^2+4ab}$.

(iii) $\boldsymbol{(a-b)^2=(a+b)^2-4ab}$.

Ex. 1. Find the value of x^2+y^2, when $x+y=8$ and $xy=15$.

$x^2+y^2=(x+y)^2-2xy=8^2-2\times 15=64-30=34$.

Ex. 2. Find the value of $(a-b)^2$, when $a+b=9$ and $ab=20$.

$(a-b)^2=(a+b)^2-4ab=9^2-4\times 20=81-80=1$.

Ex. 3. Find the value of a^2+b^2, when $a-b=5$ and $ab=14$.

$a^2+b^2=(a-b)^2+2ab=5^2+2\times 14=25+28=53$.

Ex. 4. Express $(a-b)^2+4(a-c)(b-c)$ as a square.

The Exp. $=(a-b)^2+4\{ab-(a+b)c+c^2\}$
$=\{(a-b)^2+4ab\}-4(a+b)c+4c^2$
$=(a+b)^2-4(a+b)c+4c^2=\{(a+b)-2c\}^2=(a+b-2c)^2$.

2. (i) $\boldsymbol{(a+b)(a-b)=a^2-b^2}$.

(ii) $\boldsymbol{ab=\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2}$.

Ex. 5. Express $(x+7a)(x+9a)$ as the difference of two squares.

Here, $x+7a=(x+8a)-a$ and $x+9a=(x+8a)+a$.

Hence the Exp. $=\{(x+8a)-a\}\{(x+8a)+a\}=(x+8a)^2-a^2$.

Ex. 6. Express $(x^2+4x+3)(x^2-6x-1)$ as the difference of two squares.

Let $a=x^2+4x+3$ and $b=x^2-6x-1$, then

$$\frac{a+b}{2}=\tfrac{1}{2}\{(x^2+4x+3)+(x^2-6x-1)\}=x^2-x+1,$$

and $$\frac{a-b}{2}=\tfrac{1}{2}\{(x^2+4x+3)-(x^2-6x-1)\}=5x+2:$$

Hence the Exp. $=(x^2-x+1)^2-(5x+2)^2$, by substitution.

Exercise CXVI.

1. Find the value of a^2+b^2, having given

(i) $a+b=12$, $ab=35$. (ii) $a+b=13$, $ab=30$. (iii) $a-b=5$, $ab=36$.

2. Find the value of $(x+y)^2$, having given

(i) $x-y=9$, $xy=15$. (ii) $x-y=5$, $xy=4$. (iii) $x-y=8$, $xy=12$.

3. Find the value of $(a-b)^2$, having given

(i) $a+b=7$, $ab=9$. (ii) $a^2+b^2=37$, $ab=12$. (iii) $a+b=18$, $ab=72$.

4. Find the value of a^2+b^2, having given

(i) $a-b=14$, $ab=25$. (ii) $a+b=10$, $ab=47$. (iii) $a-b=17$, $ab=23$

Express the following as squares :—

5. $(a-8b)^2+4(2a-3b)(a+5b)$. 6. $(3a+2b)^2-4(a+3b)(2a-b)$.

Express the following as the difference of two squares :—

7. $(x+1)(x+2)(x+3)(x+4)$. 8. $(x^2+7x+9)(x^2+3x+5)$.

9. $(6x^2-5x+3)(2x^2+x-5)$. 10. $(x+3a)(x+5a)(x+7a)(x+9a)$. (C. E. 1887).

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3. (i) $a^3+b^3=(a+b)^3-3ab(a+b)$.

(ii) $a^3-b^3=(a-b)^3+3ab(a-b)$.

Ex. 1. Find the value of x^3+y^3, when $x+y=5$ and $xy=9$.

$$x^3+y^3=(x+y)^3-3xy(x+y)=5^3-3\times9\times5=125-135=-10.$$

Ex. 2. Find the value of x^3-y^3, when $x-y=3$ and $xy=5$.

$$x^3-y^3=(x-y)^3+3xy(x-y)=3^3+3\times5\times3=27+45=72.$$

Ex. 3. If $a+\frac{1}{a}=10$, find the value of $a^3+\frac{1}{a^3}$.

$$a^3+\frac{1}{a^3}=\left(a+\frac{1}{a}\right)^3-3.a.\frac{1}{a}\left(a+\frac{1}{a}\right)$$

$$=\left(a+\frac{1}{a}\right)^3-3\left(a+\frac{1}{a}\right)=10^3-3\times 10=970.$$

4. (i) $a^2+b^2+c^2=(a+b+c)^2-2(bc+ca+ab)$.

(ii) $a^3+b^3+c^3=(a+b+c)^3-3(b+c)(c+a)(a+b)$.

Ex. 1. Find the value of $a^2+b^2+c^2$, when $a+b+c=10$ and $bc+ca+ab=27$.

$$a^2+b^2+c^2=(a+b+c)^2-2(bc+ca+ab)=10^2-2\times 27=46.$$

Ex. 2. Find the value of $a^3+b^3+c^3$, when $b+c=7$, $c+a=5$ and $a+b=6$.

Here, $a+b+c=\frac{1}{2}\{(b+c)+(c+a)+(a+b)\}=\frac{1}{2}(7+5+6)=9.$

Hence $a^3+b^3+c^3=(a+b+c)^3-3(b+c)(c+a)(a+b)$

$$=9^3-3\times 7\times 5\times 6=729-630=99.$$

Exercise CXVII.

1. Find the value of a^3+b^3, when

(i) $a+b=7$, $ab=3$. (ii) $a+b=12$, $ab=15$. (iii) $a+b=10$, $ab=13$

2. Find the value of a^3-b^3, when

(i) $a-b=5$, $ab=9$. (ii) $a-b=7$, $ab=4$. (iii) $a-b=12$, $ab=75$

3. If $x-\frac{1}{x}=7$, find the value of $x^3-\frac{1}{x^3}$.

4. If $a+\frac{1}{a}=6$, find the value of $a^3+\frac{1}{a^3}$.

5. Find the value of $a^2+b^2+c^2$, having given

(i) $a+b+c=7$, $bc+ca+ab=20$. (ii) $a+b+c=15$, $bc+ca+ab=125$.

6. Find the value of $a^3+b^3+c^3$, having given

(i) $b+c=3$, $c+a=5$, $a+b=6$. (ii) $b+c=10$, $c+a=15$, $a+b=17$

7. Find the value of $a^2+b^2+c^2+2bc+2ca+2ab$,

when $a=x+y$, $b=z+x$ and $c=-(y+z)$.

8. Find the value of $(b+c-a)^3+(c+a-b)^3+(a+b-c)^3+24abc$.

(B. M. 1859).

CHAPTER XIII.

HARDER FACTORS AND IDENTITIES.

I. HARDER FACTORS.

324. We have in Art. 124 restricted to the consideration of factors, which are free from terms involving square or other roots, which cannot be exactly obtained. Here we propose to extend the formula, to resolve into factors such expressions for which no factors could be found with the restrictions. The following Examples will illustrate the subject in question.

Ex. 1. $a-b=(\sqrt{a})^2-(\sqrt{b})^2=(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$.

Ex. 2. $x^2-5a^2=x^2-(\sqrt{5}a)^2=(x+\sqrt{5}a)(x-\sqrt{5}a)$, for $(\sqrt{5})^2=5$.

Ex. 3. $x^4+a^4=(x^4+2a^2x^2+a^4)-2a^2x^2=(x^2+a^2)^2-(\sqrt{2}ax)^2$
$$=(x^2+a^2+\sqrt{2}ax)(x^2+a^2-\sqrt{2}ax).$$

325. Any expression containing the second power of x is called a **quadratic expression** in x. The general form of a *quadratic expression* is ax^2+bx+c.

326. To resolve ax^2+bx+c into factors, by expressing it as the difference of two squares.

We have $ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$

$$=a(x^2+px+q),\ \text{if}\ \frac{b}{a}=p\ \text{and}\ \frac{c}{a}=q$$

$$=a(x^2+px+\tfrac{1}{4}p^2-\tfrac{1}{4}p^2+q)$$

$$=a\{(x+\tfrac{1}{2}p)^2-\tfrac{1}{4}(p^2-4q)\}\quad \text{Art. 123.}$$

$$=a\left[\left(x+\frac{p}{2}\right)^2-\left\{\frac{\sqrt{(p^2-4q)}}{2}\right\}^2\right]$$

$$=a\left\{x+\frac{p}{2}+\frac{\sqrt{(p^2-4q)}}{2}\right\}\left\{x+\frac{p}{2}-\frac{\sqrt{(p^2-4q)}}{2}\right\}\quad \text{Art. 124.}$$

$$=a\left\{x+\frac{p+\sqrt{(p^2-4q)}}{2}\right\}\left\{x+\frac{p-\sqrt{(p^2-4q)}}{2}\right\}$$

$$=a\left\{x+\frac{b+\sqrt{(b^2-4ac)}}{2a}\right\}\left\{x+\frac{b-\sqrt{(b^2-4ac)}}{2a}\right\}.$$

Ex. 1. Resolve $x^2+7x-44$ into factors.

$$x^2+7x-44=x^2+7x+(\tfrac{7}{2})^2-(\tfrac{7}{2})^2-44=x^2+7x+(\tfrac{7}{2})^2-\tfrac{225}{4}$$
$$=(x+\tfrac{7}{2})^2-(\tfrac{15}{2})^2=(x+\tfrac{7}{2}+\tfrac{15}{2})(x+\tfrac{7}{2}-\tfrac{15}{2})$$
$$=(x+11)(x-4).$$

Ex. 2. Resolve $6x^2+17x+12$ into factors.

$$6x^2+17x+12=6(x^2+\tfrac{17}{6}x+2)=6\{x^2+\tfrac{17}{6}x+(\tfrac{17}{12})^2-(\tfrac{17}{12})^2+2\}$$
$$=6\{x^2+\tfrac{17}{6}x+(\tfrac{17}{12})^2-\tfrac{1}{144}\}=6\{(x+\tfrac{17}{12})^2-(\tfrac{1}{12})^2\}$$
$$=6\{x+\tfrac{17}{12}+\tfrac{1}{12}\}\{x+\tfrac{17}{12}-\tfrac{1}{12}\}$$
$$=6(x+\tfrac{3}{2})(x+\tfrac{4}{3})=(2x+3)(3x+4).$$

Ex 3. Resolve $21x^2+xy-10y^2$ into factors.

$$21x^2+xy-10y^2=21(x^2+\tfrac{1}{21}xy-\tfrac{10}{21}y^2)$$
$$=21\{x^2+\tfrac{1}{21}xy+(\tfrac{1}{42}y)^2-(\tfrac{1}{42}y)^2-\tfrac{10}{21}y^2\}$$
$$=21\{x^2+\tfrac{1}{21}xy+(\tfrac{1}{42}y)^2-\tfrac{841}{1764}y^2\}$$
$$=21\{(x+\tfrac{1}{42}y)^2-(\tfrac{29}{42}y)^2\}$$
$$=21\{x+\tfrac{1}{42}y+\tfrac{29}{42}y\}\{x+\tfrac{1}{42}y-\tfrac{29}{42}y\}$$
$$=21(x+\tfrac{5}{7}y)(x-\tfrac{2}{3}y)=(7x+5y)(3x-2y).$$

Exercise CXVIII.

Resolve into factors, by the method of Art. 326, expressing as the difference of two squares :—

1. $x^2-12x+32.$ **2.** $x^2+3x-40.$ **3.** $x^2-103x+102$

4. $x^2+10x+21.$ **5.** $x^2-12x+27.$ **6.** $6x^2+x-22.$

7. $21x^2-13x-84.$ **8.** $25x^2-7x-86.$ **9.** $10x^2-13x-9.$

10. $7x^2+32x-15.$ **11.** $30x^2+23x-143.$ **12.** $63x^2+132x-35$

13. $2x^2+3xy-5y^2.$ **14.** $x^2-9ax-190a^2.$ **15.** $8x^2+6xy-27y^2.$

16. $24a^2+37ax-72x^2.$ **17.** $2(x+y)^2-9(x+y)(a+b)+4(a+b)^2.$

18. $4x^4+4x^2y^2-3y^4.$ **19.** $6x^4-x^2y^2-y^4.$ **20.** $x^2y^4-xy^2+272.$

Resolve into factors :—

21. $x^2-a.$ **22.** $x^2-2a^2.$ **23.** $a^4-a^2b^2+b^4.$ **24.** x^4-3a^4

327. *To prove that*

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$$
$$=\tfrac{1}{2}(a+b+c)\{(b-c)^2+(c-a)^2+(a-b)^2\}.$$

Since $a^3+b^3=(a+b)^3-3ab(a+b)$, we have

$$a^3+b^3+c^3-3abc=(a+b)^3+c^3-3ab(a+b)-3abc$$
$$=(a+b+c)\{(a+b)^2-(a+b)c+c^2\}-3ab(a+b+c). \quad \text{Art. 132}$$
$$=(a+b+c)\{(a+b)^2-(a+b)c+c^2-3ab\}$$
$$=(a+b+c)(a^2+2ab+b^2-ac-bc+c^2-3ab)$$
$$=(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$$
$$=\tfrac{1}{2}(a+b+c)(2a^2+2b^2+2c^2-2bc-2ca-2ab)$$
$$=\tfrac{1}{2}(a+b+c)\{(b-c)^2+(c-a)^2+(a-b)^2\}. \quad \text{Art. 322 (7)}$$

Ex. 1. Factorize $a^3+b^3-c^3+3abc$.

The Exp. $=a^3+b^3+(-c)^3-3ab(-c)$
$$=\{a+b+(-c)\}\{a^2+b^2+(-c)^2-b(-c)-(-c)a-ab\}$$
$$=(a+b-c)(a^2+b^2+c^2+bc+ca-ab).$$

Ex. 2. Factorize $x^3-y^3-3xy-1$.

The Exp. $=x^3+(-y)^3+(-1)^3-3x.(-y).(-1)$
$$=\{x+(-y)+(-1)\}\{x^2+(-y)^2+(-1)^2-x(-y)$$
$$-x(-1)-(-y)(-1)\}$$
$$=(x-y-1)(x^2+y^2+1+xy+x-y)$$
$$=(x-y-1)(x^2+xy+y^2+x-y+1).$$

Ex. 3. Factorize a^6+4a^3-1.

The Exp. $=a^6+a^3-1+3a^3=(a^2)^3+a^3+(-1)^3-3.a^2.a.(-1)$
$$=\{a^2+a+(-1)\}\{(a^2)^2+a^2+(-1)^2-a^2.a-a^2.(-1)-$$
$$a(-1)\}$$
$$=(a^2+a-1)(a^4+a^2+1-a^3+a^2+a)$$
$$=(a^2+a-1)(a^4-a^3+2a^2+a+1).$$

Exercise CXIX.

Factorize

1. $a^3-b^3+c^3+3abc$.
2. $a^3-b^3-c^3-3abc$.
3. $x^3-y^3+3xy+1$.
4. $x^3+y^3-3xy+1$.
5. $x^3-8y^3+27z^3+18xyz$.
6. $8a^3-27b^3-1-18ab$.
7. $a^3-b^3+8+6ab$.
8. $8a^3+b^3-1+6ab$.
9. $a^3+8b^3+27c^3-18abc$.
10. $2x^3+y^3-3x^2y$.
11. $x^3+8y^3-27z^3+18xyz$.
12. $14x^3-4y^3+9x^2y$.
13. $a^3+27-5b(25b^2-9a)$.
14. a^6+32a^3-64.
15. $(a-b)^3-(b-c)^3+(c-a)^3+3(b-c)(c-a)(a-b)$.

328. *To prove that*

$$(a+b+c)(bc+ca+ab)-abc=(b+c)(c+a)(a+b).$$

Putting x for $a+b+c$, we have

$$\begin{aligned}(a+b+c)(bc+ca+ab)-abc&=x(bc+ca+ab)-abc\\&=x^3-x^2(a+b+c)+x(bc+ca+ab)-abc, \text{ for } x^3-x^3=0\\&=(x-a)(x-b)(x-c), \text{ Art. 98.}\\&=(a+b+c-a)(a+b+c-b)(a+b+c-c), \left\{\begin{array}{l}\text{writing } a+b+c\\ \quad\text{for } x.\end{array}\right.\\&=(b+c)(c+a)(a+b).\end{aligned}$$

329. If the expression $a^2b+ab^2+a^2c+ac^2+b^2c+bc^2$, which may be written in any one of the equivalent forms

$$\left.\begin{array}{l}a^2(b+c)+b^2(c+a)+c^2(a+b)\ldots\ldots\ldots\ldots(1)\\ bc(b+c)+ca(c+a)+ab(a+b)\ldots\ldots\ldots\ldots(2)\\ a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)\ldots\ldots\ldots\ldots(3)\end{array}\right\}$$

be denoted by P, then

1. $P+2abc=(b+c)(c+a)(a+b)$.
2. $P+3abc=(a+b+c)(bc+ca+ab)$.

1. Taking the first value of $\mathbf{P}$, we have

$$\begin{aligned}&a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc\\&\quad=a^2(b+c)+a(b^2+c^2+2bc)+b^2c+bc^2 \quad \left\{\begin{array}{l}\text{multiplying and}\\ \text{re-arranging}\end{array}\right.\\&\quad=a^2(b+c)+a(b+c)^2+bc(b+c)\\&\quad=(b+c)\{a^2+a(b+c)+bc\}\\&\quad=(b+c)(a+b)(a+c)=(b+c)(c+a)(a+b).\end{aligned}$$

2. Taking the second value of $\mathbf{P}$, we have

$$\begin{aligned}&bc(b+c)+ca(c+a)+ab(a+b)+3abc\\&\quad=bc(b+c)+ca(c+a)+ab(a+b)+abc+abc+abc\\&\quad=bc(b+c)+abc+ca(c+a)+abc+ab(a+b)+abc\\&\quad=bc(b+c+a)+ca(c+a+b)+ab(a+b+c)\\&\quad=(a+b+c)(bc+ca+ab).\end{aligned}$$

(a) Since $(a+b+c)(bc+ca+ab)=\mathbf{P}+3abc$

and $(b+c)(c+a)(a+b) \qquad =\mathbf{P}+2abc$

$\therefore$ by subtraction,

$$\begin{aligned}&(a+b+c)(bc+ca+ab)-(b+c)(c+a)(a+b)\\&\qquad\qquad=(\mathbf{P}+3abc)-(\mathbf{P}+2abc)=abc.\end{aligned}$$

$\therefore$ by transposition, we have

$$(a+b+c)(bc+ca+ab)-abc=(b+c)(c+a)(a+b),$$

which is the Formula of Art. 328.

330. *To prove that*

$$(a+b+c)^3-a^3-b^3-c^3=3(b+c)(c+a)(a+b).$$

Since $(a+b+c)^3=a^3+b^3+c^3+3a^2(b+c)+3b^2(c+a)+3c^2(a+b)$
$+6abc$, Art. 172

$$=a^3+b^3+c^3+3(\mathrm{P}+2abc)$$
$$=a^3+b^3+c^3+3(b+c)(c+a)(a+b),$$

$\therefore$ by transposition, we have

$$(a+b+c)^3-a^3-b^3-c^3=3(b+c)(c+a)(a+b).$$

331. *To prove that*

$$2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$$
$$=(a+b+c)(a+b-c)(a+c-b)(b+c-a).$$

Left side $=4b^2c^2-(a^4+b^4+c^4-2a^2b^2-2a^2c^2+2b^2c^2)$

$=(2bc)^2-(a^2-b^2-c^2)^2$, Art. 128

$=\{2bc+(a^2-b^2-c^2)\}\{2bc-(a^2-b^2-c^2)\}$, Art. 124

$=\{a^2-(b^2+c^2-2bc)\}\{(b^2+c^2+2bc)-a^2\}$

$=\{a^2-(b-c)^2\}\{(b+c)^2-a^2\}$ Art. 123

$=\{a+(b-c)\}\{a-(b-c)\}\{(b+c)+a\}\{(b+c)-a\}$, Art. 124

$=(a+b-c)(a-b+c)(b+c+a)(b+c-a)$

$=(a+b+c)(a+b-c)(a+c-b)(b+c-a).$

Exercise CXX.

Resolve into factors :—

1. $(1-x-y)(xy-x-y)-xy.$
2. $(a^3+a^{-1}+a^{-2})(a^2+a+a^{-3})-1.$
3. $(a+b+c-d)(ab+ac-ad+bc-bd)-ab(c-d).$
4. $a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc.$
5. $bc(b+c)+ca(c+a)+ab(a+b)+2abc.$
6. $b^2(a-c)+c^2(a-b)-a^2(b+c)+3abc.$
7. $a(b^2+c^2)-b(c^2+a^2)-c(a^2+b^2)+2abc.$
8. $a^2(b+3c)+b^2(3c+a)+9c^2(a+b)+6abc.$
9. $a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+3abc.$
10. $x(y+z)^2+y(z+x)^2+z(x+y)^2-4xyz.$
11. $a(b-c)^2+b(c-a)^2+c(a-b)^2+9abc.$
12. $a(b+c)^2+b(c+a)^2+c(a+b)^2-3abc.$
13. $a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+a^3+b^3+c^3.$

14. $bc(b+c)+ca(c+a)+ab(a+b)-a^3-b^3-c^3-2abc.$

15. $8(a+b+c)^3-(b+c)^3-(c+a)^3-(a+b)^3.$

16. $a^4-(b^2-c^2)^2+b^4-(c^2-a^2)^2+c^4-(a^2-b^2)^2.$

332. If the expression $a^2b-ab^2+b^2c-bc^2+c^2a-ca^2$, which may be written in any one of the equivalent forms

$$\left.\begin{aligned} &a^2(b-c)+b^2(c-a)+c^2(a-b) \ldots\ldots \quad \ldots\ldots (1)\\ &bc(b-c)+ca(c-a)+ab(a-b) \ldots\ldots\ldots (2)\\ &-\{a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)\} \quad \ldots\ldots (3)\end{aligned}\right\}$$

be denoted by **Q**, then we have

$$Q=-(b-c)(c-a)(a-b).$$

Taking the first form of **Q**, we have

$$\begin{aligned} &a^2(b-c)+b^2(c-a)+c^2(a-b)\\ &=a^2(b-c)-ab^2+ac^2+b^2c-bc^2, \\ &=a^2(b-c)-a(b^2-c^2)+bc(b-c) \end{aligned} \left\{\begin{array}{l}\text{multiplying}\\ \text{and re-arranging}\end{array}\right.$$

$$\begin{aligned} &=(b-c)\{a^2-a(b+c)+bc\}\\ &=(b-c)(a-b)(a-c)\\ &=(b-c)(a-b)\times-(c-a)=-(b-c)(c-a)(a-b).\end{aligned}$$

Ex. 1. Factorize $a^3(b-c)+b^3(c-a)+c^3(a-b)$. (C. E. 1898).

The Exp. $=a^3(b-c)-ab^3+ac^3+b^3c-bc^3$

$$\begin{aligned} &=a^3(b-c)-a(b^3-c^3)+bc(b^2-c^2)\\ &=(b-c)\{a^3-a(b^2+bc+c^2)+bc(b+c)\}, \text{ Arts. 124 \& 132.}\\ &=(b-c)\{(a^3-ab^2)-abc+b^2c-ac^2+bc^2\} \\ &=(b-c)\{a(a^2-b^2)-bc(a-b)-c^2(a-b)\}\end{aligned} \left\{\begin{array}{l}\text{multiplying and}\\ \text{re-arranging.}\end{array}\right.$$

$$\begin{aligned} &=(b-c)(a-b)\{a(a+b)-bc-c^2\}, \text{ Art. 124.}\\ &=(b-c)(a-b)\{(a^2-c^2)+ab-bc\} \\ &=(b-c)(a-b)\{(a^2-c^2)+b(a-c)\}\end{aligned} \left\{\begin{array}{l}\text{multiplying and}\\ \text{re-arranging.}\end{array}\right.$$

$$\begin{aligned} &=(b-c)(a-b)(a-c)(a+c+b)\\ &=-(a+b+c)(b-c)(c-a)(a-b).\end{aligned}$$

Ex. 2. Factorize $a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)$.

In the *Formula* for **Q**, writing a^2, b^2, c^2 for a, b, c respectively, we have

$$\begin{aligned}\text{the Exp.}&=-(b^2-c^2)(c^2-a^2)(a^2-b^2)\\ &=-(b-c)(c-a)(a-b)(b+c)(c+a)(a+b).\end{aligned}$$

Ex. 3. Resolve the following into factors :—

$$(b-c)(x^2+ax+a^2)+(c-a)(x^2+bx+b^2)+(a- \ldots$$

Here, $(b-c)(x^2+ax+a^2)=(b-c)x^2+(b-c)ax+a^2(b-c)$,

$(c-a)(x^2+bx+b^2)=(c-a)x^2+(c-a)bx+b^2(c-a)$,

$(a-b)(x^2+cx+c^2)=(a-b)x^2+(a-b)cx+c^2(a-b)$.

Adding vertically the columns containing x^2 and x,

1st column $=x^2\{(b-c)+(c-a)+(a-b)\}=0$.

2nd column $=x\{a(b-c)+b(c-a)+c(a-b)\}=0$.

Hence, the given expression $=a^2(b-c)+b^2(c-a)+c^2(a-b)$

$=-(b-c)(c-a)(a-b)$. Art. 332.

Exercise CXXI.

Resolve into factors :—

1. $bc(b-c)+ca(c-a)+ab(a-b)$. (P. E. 1893).
2. $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$.
3. $bc(b^2-c^2)+ca(c^2-a^2)+ab(a^2-b^2)$.
4. $a(b^3-c^3)+b(c^3-a^3)+c(a^3-b^3)$.
5. $b^2c^2(b-c)+c^2a^2(c-a)+a^2b^2(a-b)$.
6. $b^2c^2(b^2-c^2)+c^2a^2(c^2-a^2)+a^2b^2(a^2-b^2)$.
7. $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$. (M. M. 1899).
8. $bc(b^3-c^3)+ca(c^3-a^3)+ab(a^3-b^3)$.
9. $b^3c^3(b-c)+c^3a^3(c-a)+a^3b^3(a-b)$.
10. $a^4(b-c)+b^4(c-a)+c^4(a-b)$.
11. $a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)$.
12. $a^2(b^4-c^4)+b^2(c^4-a^4)+c^2(a^4-b^4)$.
13. $a(b^5-c^5)+b(c^5-a^5)+c(a^5-b^5)$.
14. $a^5(b-c)+b^5(c-a)+c^5(a-b)$.
15. $a(b-c)^3+b(c-a)^3+c(a-b)^3$.
16. $a^2(b-c)^3+b^2(c-a)^3+c^2(a-b)^3$.
17. $(b-c)^5+(c-a)^5+(a-b)^5$. (P. E. 1893).
18. $(a^2+1)(b-c)+(b^2+1)(c-a)+(c^2+1)(a-b)$.
19. $(a+1)^2(b-c)+(b+1)^2(c-a)+(c+1)^2(a-b)$.
20. $(a+1)^3(b-c)+(b+1)^3(c-a)+(c+1)^3(a-b)$.
21. $(a^2+a+1)(b-c)+(b^2+b+1)(c-a)+(c^2+c+1)(a-b)$.
22. $(x^2-bc)(b-c)+(x^2-ca)(c-a)+(x^2-ab)(a-b)$.

23. $(a-1)(a^2+a+1)(b-c)+(b-1)(b^2+b+1)(c-a)+$
$(c-1)(c^2+c+1)(a-b)$.

24. $bc(b-c)(x-a)^2+ca(c-a)(x-b)^2+ab(a-b)(x-c)^2$.

25. $(b-c)(b+c)^3+(c-a)(c+a)^3+(a-b)(a+b)^3$.

333. Any expression containing integral powers of x arranged in order of the powers of x, has $x\pm 1$ for a factor, when the sum of the cœfficients in the odd places, is equal to that of the coefficients in the even places irrespective of the signs, *like*, when the factor is $x+1$, and *unlike*, when the factor is $x-1$. Also, if the sum of all the coefficients be equal to zero, the expression is divisible by $x-1$.

Note. If both $x+1$ and $x-1$ measure the given expression, the sum of the coefficients in each case will be zero.

Ex. 1. Resolve $2x^4+x^3-9x^2-13x-5$ into factors.

$$\left.\begin{aligned}\text{Here } 2-9-5&=-12\\ \text{and } 1-13\quad&=-12\end{aligned}\right\}\ \text{Hence } x+1 \text{ is a factor.}$$

$$\begin{aligned}\text{The Exp.}&=2x^3(x+1)-x^2(x+1)-8x(x+1)-5(x+1)\\ &=(x+1)(2x^3-x^2-8x-5).\end{aligned}$$

Again, by a similar process $2x^3-x^2-8x-5$ may be resolved into $(x+1)(2x^2-3x-5)$, and $2x^2-3x-5$ into $(x+1)(2x-5)$.

Hence, the given expression$=(x+1)^2(2x-5)$.

Ex. 2. Resolve $3x^3-8x^2+7x-2$ into factors.

$$\left.\begin{aligned}\text{Here } 3+7&=10\\ \text{and } -8-2&=-10\\ \text{or } 3-8+7-2&=0\end{aligned}\right\}\ \text{Hence } x-1 \text{ is a factor.}$$

The Exp.$=3x^2(x-1)-5x(x-1)+2(x-1)=(x-1)(3x^2-5x+2)$.

Again, by a similar process $3x^2-5x+2=(x-1)(3x-2)$.

Hence, the given expression$=(x-1)^2(3x-2)$.

Ex. 3. Resolve $2x^4-7x^3+4x^2+7x-6$ into factors.

$$\left.\begin{aligned}\text{Here } 2+4-6&=0\\ \text{and } -7+7\quad&=0\end{aligned}\right\}\ \text{Hence } x+1 \text{ and } x-1 \text{ are both factors.}$$

The Exp.$=2x^2(x^2-1)-7x(x^2-1)+6(x^2-1)=(x^2-1)(2x^2-7x+6)$.

Again, $2x^2-7x+6=(2x-3)(x-2)$.

$$\begin{aligned}\text{Hence, the given expression}&=(x^2-1)(x-2)(2x-3)\\ &=(x+1)(x-1)(x-2)(2x-3).\end{aligned}$$

334. To resolve a **homogeneous** expression of two dimensions into factors, proceed as in the following Example.

Ex. Resolve $2a^2-ab-6b^2+ac+19bc-15c^2$ into factors.

If $c=0$, the expression reduces to

$$2a^2-ab-6b^2, \text{ which } =(2a+3b)(a-2b)\ldots\ldots\ldots\ldots(1)$$

If $b=0$, the expression reduces to

$$2a^2+ac-15c^2, \text{ which } =(2a-5c)(a+3c)\ldots\ldots\ldots\ldots(2)$$

If $a=0$, the expression reduces to

$$-6b^2+19bc-15c^2, \text{ which } =(3b-5c)(-2b+3c)\ldots\ldots\ldots(3)$$

Now, comparing (1) and (2) and testing the result by (3),

we have the given Exp. $=(2a+3b-5c)(a-2b+3c)$.

335. To resolve a **reciprocal** or **recurring** expression into factors, proceed as in the following Examples.

Def. When an algebraical expression has the coefficients of its terms equidistant from the beginning and end the same, it is called a **reciprocal** or **recurring** expression.

Ex. 1. Resolve $2x^4-5x^3+6x^2-5x+2$ into factors.

The given Exp. $=x^2\left(2x^2-5x+6-\dfrac{5}{x}+\dfrac{2}{x^2}\right)$

$$=x^2\left\{2\left(x^2+\frac{1}{x^2}\right)-5\left(x+\frac{1}{x}\right)+6\right\}$$

$$=x^2\{2(y^2-2)-5y+6\}, \text{ if } x+\frac{1}{x}=y$$

$$=x^2(2y^2-5y+2)=x^2(2y-1)(y-2)$$

$$=x^2\left(2x+\frac{2}{x}-1\right)\left(x+\frac{1}{x}-2\right), \text{ restoring } y$$

$$=(2x^2+2-x)(x^2+1-2x)$$

$$=(2x^2-x+2)(x^2-2x+1)=(2x^2-x+2)(x-1)^2.$$

Ex. 2. Resolve $x^6-4x^4-13x^3+13x^2+4x-1$ into factors.

Here, $1-4-13+13+4-1=0$; $\therefore$ $x-1$ is a factor.

The given Exp. $=x^4(x-1)-3x^3(x-1)-16x^2(x-1)$

$$-3x(x-1)+(x-1)$$

$$=(x-1)(x^4-3x^3-16x^2-3x+1).$$

Now, to resolve $x^4-3x^3-16x^2-3x+1$ into factors.

$$\text{The Exp.}=x^2\left(x^2-3x-16-\frac{3}{x}+\frac{1}{x^2}\right)$$
$$=x^2\left\{\left(x^2+\frac{1}{x^2}\right)-3\left(x+\frac{1}{x}\right)-16\right\}$$
$$=x^2\{(y^2-2)-3y-16\},\ \text{if } x+\frac{1}{x}=y$$
$$=x^2(y^2-3y-18)=x^2(y-6)(y+3)$$
$$=x^2\left(x+\frac{1}{x}-6\right)\left(x+\frac{1}{x}+3\right),\ \text{restoring } y$$
$$=(x^2+1-6x)(x^2+1+3x)=(x^2-6x+1)(x^2+3x+1).$$

Hence the given Exp. $=(x-1)(x^2-6x+1)(x^2+3x+1)$.

Exercise CXXII.

Resolve into factors :

1. $x^4-9x^3+3x^2+37x+24$.

2. $x^4-7x^3-25x^2+67x-36$.

3. $3x^5+2x^4-28x^3+42x^2-23x+4$.

4. $3x^5-10x^3+15x+8$.

5. $4x^4-23x^3+30x^2-7x-4$.

6. $2x^4+5x^3-2x^2-11x-6$.

7. $x^5-8x^3+6x^2+7x-6$.

8. $x^4-ax^3+(b-1)x^2+ax-b$.

9. $6a^2+7ab+2b^2+11ac+7bc+3c^2$.

10. $2a^2+6ab+5ac+4b^2+6bc+2c^2$.

11. $2a^2+9ac-10ab+4c^2+2bc-12b^2$.

12. $a^2-3ab+2b^2-2bc-4c^2$.

13. $2a^2+ab-3b^2-ac-4bc-c^2$.

14. $a^2-10ab-15bc+21b^2+5ac$.

15. $4a^2-4ab-3b^2+12bc-9c^2$.

16. $x^4+x^3-10x^2+x+1$.

17. $x^4-5x^3-10x^2-10x+4$

18. $x^4-4x^3-6x^2-4x+1$.

19. $x^6-5x^4-5x^2+1$.

20. $x^8+3x^6-8x^4+3x^2+1$.

21. $x^8-5x^6-12x^4-5x^2+1$.

22. $x^5+3x^4+2x^3-2x^2-3x-1$.

23. $12x^4-5x^3-26x^2+5x+12$.

24. $x^5-ax^4+a^2x^3-a^3x^2+a^4x-a^5$.

336. Sometimes by suitable arrangement and grouping of terms, algebraical expressions may be resolved into factors, as shewn in the following Examples.

Ex. 1. Resolve x^3+4x^2-5 into factors.

$$\text{The Exp.}=(x^3-x^2)+5x^2-5=x^2(x-1)+5(x+1)(x-1)$$
$$=(x-1)\{x^2+5(x+1)\}=(x-1)(x^2+5x+5).$$

Ex. 2. Resolve $x^3+8x^2+24x+27$ into factors.

The Exp. $=(x^3+27)+(8x^2+24x)=(x^3+3^3)+8x(x+3)$
$=(x+3)(x^2-3x+9)+8x(x+3)$
$=(x+3)\{(x^2-3x+9)+8x\}=(x+3)(x^2+5x+9)$.

Ex. 3. Resolve $8x^3+4x-3$ into factors.

The Exp. $=(8x^3-1)+(4x-2)=(2x-1)(4x^2+2x+1)+2(2x-1)$
$=(2x-1)\{(4x^2+2x+1)+2\}=(2x-1)(4x^2+2x+3)$.

Ex. 4. Resolve $x^4-6x^3+7x^2+6x-8$ into factors.

The Exp. $=(x^4-6x^3+9x^2)-2x^2+6x-8$
$=(x^2-3x)^2-2(x^2-3x)-8$
$=a^2-2a-8$, putting a for x^2-3x
$=(a-4)(a+2)=(x^2-3x-4)(x^2-3x+2)$
$=(x+1)(x-4)(x-1)(x-2)$.

Ex. 5. Resolve $2x^3+5x^2-4x-3$ into factors.

The Exp. $=x(2x^2+5x+2)-3(2x+1)=x(2x+1)(x+2)-3(2x+1)$
$=(2x+1)\{x(x+2)-3\}=(2x+1)(x^2+2x-3)$
$=(2x+1)(x-1)(x+3)$.

Ex. 6. Resolve $(x+1)(x+3)(x+5)(x+7)+15$ into factors.

Here, $(x+1)(x+7)=x^2+8x+7$ and $(x+3)(x+5)=x^2+8x+15$.

Hence the Exp. $=(x^2+8x+7)(x^2+8x+15)+15$
$=(a+7)(a+15)+15$, putting a for x^2+8x
$=a^2+22a+105+15=a^2+22a+120$
$=(a+10)(a+12)$
$=(x^2+8x+10)(x^2+8x+12)$.

Ex. 7. Resolve $(a+b)^3+a^3-9b^3$ into factors.

The Exp. $=\{(a+b)^3-8b^3\}+(a^3-b^3)=\{(a+b)^3-(2b)^3\}+(a^3-b^3)$
$=(a+b-2b)\{(a+b)^2+2b(a+b)+4b^2\}$
$\quad+(a-b)(a^2+ab+b^2)$
$=(a-b)(a^2+4ab+7b^2)+(a-b)(a^2+ab+b^2)$
$=(a-b)(a^2+4ab+7b^2+a^2+ab+b^2)$
$=(a-b)(2a^2+5ab+8b^2)$.

Ex. 8. Resolve $a^3 - a^2 - a - 15$ into factors.

$$\text{The Exp.} = (a^3 - 27) - (a^2 - 9) - (a - 3) = (a^3 - 3^3) - (a^2 - 3^2) - (a - 3)$$
$$= (a - 3)(a^2 + 3a + 9) - (a + 3)(a - 3) - (a - 3)$$
$$= (a - 3)\{(a^2 + 3a + 9) - (a + 3) - 1\}$$
$$= (a - 3)(a^2 + 2a + 5).$$

Ex. 9. Resolve $(a^2 - b^2)^2 - 2c^2(a^2 + b^2) + c^4$ into factors.

$$\text{The Exp.} = \{(a^2 + b^2)^2 - 4a^2b^2\} - 2c^2(a^2 + b^2) + c^4$$
$$= \{(a^2 + b^2)^2 - 2c^2(a^2 + b^2) + c^4\} - 4a^2b^2$$
$$= \{(a^2 + b^2) - c^2\}^2 - (2ab)^2$$
$$= (a^2 + b^2 - c^2 + 2ab)(a^2 + b^2 - c^2 - 2ab)$$
$$= \{(a + b)^2 - c^2\}\{(a - b)^2 - c^2\}$$
$$= (a + b + c)(a + b - c)(a - b + c)(a - b - c).$$

Ex. 10. Resolve $2(a + b + c)^3 + (b + c)(c + a)(a + b) + 2abc$ into factors.

Assume $x = a + b + c$, then $b + c = x - a$, $c + a = x - b$, $a + b = x - c$.

$$\text{Hence, the Exp.} = 2x^3 + (x - a)(x - b)(x - c) + 2abc$$
$$= 2x^3 + x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc + 2abc$$
$$= 2(a + b + c)x^2 + x^3 - (a + b + c)x^2 + (ab + ac + bc)x + abc$$
$$\{\text{for } 2x^3 = 2x^2.x = 2x^2(a + b + c)\}$$
$$= x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc$$
$$= (x + a)(x + b)(x + c)$$
$$= (2a + b + c)(a + 2b + c)(a + b + 2c).$$

Exercise CXXIII.

Resolve into factors :—

1. $x^3 - 3x - 4$. **2.** $x^3 - x^2 - 4$. **3.** $x^3 - 3x + 2$.

4. $x^3 - 19x + 30$. **5.** $x^3 + 2x^2 - 3$. **6.** $x^3 - 7x - 6$.

7. $x^3 - 7x^2 - 80x + 576$. **8.** $x^3 - 8x^2 - 12x + 144$.

9. $x^3 - 2ax^2 - 5a^2x - 12a^3$. **10.** $2x^3 + 9x^2 + 4x - 15$.

11. $3x^3 - 17x^2 + 19x + 11$. **12.** $4x^3 + 8x^2 + 3x + 20$.

13. $x^4 - 10x^3 + 35x^2 - 50x + 24$. **14.** $x^4 - 2x^3 + x - 132$.

15. $x^4 - 6x^3 + 6x^2 + 9x - 4$. **16.** $x^4 + 12x^3 + 4x^2 - 192x - 320$.

17. $x^4 - 16x^3 + 35x^2 + 232x + 180$. **18.** $x^4 - 9x^2 + 30x - 25$.

19. $(x + 1)(x + 3)(x - 4)(x - 6) + 24$. **20.** $(x + 2)(x + 3)(x + 4)(x + 5) - 35$.

21. $x(2x+1)(x-2)(2x-3)-48.$ **22.** $4x^4-20x^3+24x^2+6x-9.$

23. $x^2+(a+b-2c)x+(2a-b-c)(2b-a-c).$

24. $a^3-9b(2a+3b^2)-8.$ **25.** $a^3+a^2+a-84.$

26. $(ab+1)^4-4ab(ab+1)^2-(a^2-b^2)^2.$ (M. M. 1899).

27. $a(b^2+c^2-a^2)+b(c^2+a^2-b^2).$ (M. M. 1865).

28. $a^2(b^2-c^2)+4abc-b^2+c^2.$

29. $bcx^3-(ac-b^2)x^2-2abx+a^2.$ **30.** $x^6+x^4+1.$

Miscellaneous Factors. (Harder).

Resolve into factors :—

1. $(x^2+a-1)^2-a^2x^2.$

2. $(2a+2b-ab)^2-(b^2-4a)(a^2-4b).$ (M. M. 1877).

3. $4a^2+17ab+5ac+5bc+4b^2+c^2.$

4. $(1+y)^2-2(1+y^2)x^2+(1-y)^2x^4.$ (M. M. 1893).

5. $(x-1)(x-2)-2(y-1)(x-2)+(y-1)(y-2).$ (M. M. 1896).

6. $(a^2+b^2)^2-(a^2-b^2)^2-(a^2+b^2-c^2)^2.$ (M. M. 1898).

7. $a^3(a+1)+b^3(b+1)-ab(a-b)^2.$ (M. M. 1898).

8 $x^2+(a-1)xy-ay^2+(a-1)x+(a^2+1)y-a.$

9. $(a^2-b^2)(a+b)+(b^2-c^2)(b+c)+(c^2-a^2)(c+a).$ (M. M. 1899).

10. $(a^2-b^2)^2+(c^2-d^2)^2-(a+b)^2(c-d)^2-(a-b)^2(c+d)^2.$ (M. M. 1876).

11. $(b+c)^2-2(b^2+c^2)a^2+(b-c)^2a^4.$ (M. M. 1899).

12. $x^2-\left(a+\frac{1}{a}\right)x+1.$ (C. E. 1885).

13. $a^2x^3-\frac{8a^2}{y^3}-x^3+\frac{8}{y^3}.$ (P. E. 1889).

14. $x^3-2x^2-23x+60.$ (B. M. 1887).

15. $x^5+7x^3-5x^2-35.$ (B. M. 1889).

16. $(x^2+4x)^2-2(x^2+4x)-15.$ (B. M. 1889).

17. $8x^3-5x+3.$ (B. M. 1894).

18. $a(b-c)^2+b(c-a)^2+c(a-b)^2+8abc.$ (B. M. 1890).

19. $(2b-a)^3+(2a-b)^3-(a+b)^3.$ (B. M. 1900).

20. $(a+b+c)^3-a^3-b^3-c^3.$ **21.** $x^4-5x^3+9x^2-7x+2.$ (B. M. 1901).

22. $x^4-5x^3+x^2+21x-18.$ (B. M. 1902).

23. $a^3b+b^3c+c^3a-(ab^3+bc^3+ca^3).$ (M. M. 1892).

24. $(a+b+c)^3-3(a+b+c)+2.$ **25.** $(a+3)^4+(a+2)^2-1.$

II. HARDER IDENTITIES.

337. We shall in this Section consider some important Identities of a somewhat harder type than those considered in Section III. of Chap. IV., and establish their truth with the help of the preceding Formulae. The following are illustrative Examples.

Ex. 1. Prove that $a(b+c)^2+b(c+a)^2+c(a+b)^2-(b+c)(a-b)\times(a-c)-(c+a)(b-a)(b-c)-(a+b)(c-a)(c-b)=12abc.$

We have, $a(b+c)^2+b(c+a)^2+c(a+b)^2$

$$=a(b^2+2bc+c^2)+b(c^2+2ca+a^2)+c(a^2+2ab+b^2)$$
$$=a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+6abc$$
$$=\mathbf{P}+6abc. \text{ (Art. 329).}$$

Again, $(b+c)(a-b)(a-c)=(b+c)\{a^2-(b+c)a+bc\}$
$$=a^2(b+c)-a(b+c)^2+bc(b+c).$$
Similarly, $(c+a)(b-a)(b-c)=b^2(c+a)-b(c+a)^2+ca(c+a),$
and $(a+b)(c-a)(c-b)=c^2(a+b)-c(a+b)^2+ab(a+b).$

$\therefore$ their sum (adding vertical columns)
$$=\mathbf{P}-(\mathbf{P}+6abc)+\mathbf{P}=\mathbf{P}-6abc. \text{ (Art. 329).}$$

Hence, the Exp $=(\mathbf{P}+6abc)-(\mathbf{P}-6abc)=12abc.$

Ex. 2. Prove that $a^3+b^3+c^3+24abc$
$$=(a+b+c)^3-3\{a(b-c)^2+b(c-a)^2+c(a-b)^2\}.$$

Since $a(b-c)^2+b(c-a)^2+c(a-b)^2$
$$=a(b^2-2bc+c^2)+b(c^2-2ca+a^2)+c(a^2-2ab+b^2)$$
$$=a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)-6abc$$
$$=\mathbf{P}-6abc. \text{ (Art. 329).}$$

and $(a+b+c)^3=a^3+b^3+c^3+3a^2(b+c)+3b^2(c+a)+3c^2(a+b)+6abc$
$$=a^3+b^3+c^3+3\mathbf{P}+6abc. \text{ (Art. 329).}$$

$\therefore$ Second side $=a^3+b^3+c^3+3\mathbf{P}+6abc-3(\mathbf{P}-6abc)$
$$=a^3+b^3+c^3+24abc.$$

Ex. 3. Prove that $(b+c)^2(2a+b+c)+(c+a)^2(2b+c+a)+(a+b)^2\times(2c+a+b)+2(b+c)(c+a)(a+b)=(2a+b+c)(2b+c+a)(2c+a+b).$

Putting x for $b+c$, y for $c+a$ and z for $a+b$,

The first side reduces to
$$x^2(y+z)+y^2(z+x)+z^2(x+y)+2xyz$$
$$=\mathbf{P}+2xyz=(y+z)(z+x)(x+y). \text{ Art. 329.}$$
$$=(2a+b+c)(2b+c+a)(2c+a+b).$$

Ex. 4. Prove that $(a+b+c)^3-27abc = \frac{1}{2}\{(b+c+7a)(b-c)^2+(c+a+7b)(c-a)^2+(a+b+7c)(a-b)^2\}$.

Second side $=\frac{1}{2}[\{(b+c+a)+6a\}(b-c)^2+\{(c+a+b)+6b\}(c-a)^2\}$
$+\{(a+b+c)+6c\}(a-b)^2]$

$$=\tfrac{1}{2}(a+b+c)\{(b-c)^2+(c-a)^2+(a-b)^2\}+3\{a(b-c)^2+b(c-a)^2+c(a-b)^2\}$$

$$=a^3+b^3+c^3-3abc+3(\mathbf{P}-6abc).$$

Arts. 327 and 329.

$$=(a^3+b^3+c^3+3\mathbf{P}+6abc)-27abc$$

$$=(a+b+c)^3-27abc. \quad \text{Art. } 172.$$

Ex. 5. Prove that $(2a+b+c)^2(b-c)+(2b+c+a)^2(c-a)+(2c+a+b)^2(a-b)=-(b-c)(c-a)(a-b)$.

Putting x for $a+b+c$, the expression reduces to

$$(x+a)^2(b-c)+(x+b)^2(c-a)+(x+c)^2(a-b)$$

$$=(x^2+2ax+a^2)(b-c)+(x^2+2bx+b^2)(c-a)+(x^2+2cx+c^2)(a-b)$$

$$=x^2\{(b-c)+(c-a)+(a-b)\}+2x\{a(b-c)+b(c-a)+c(a-b)\}+a^2(b-c)+b^2(c-a)+c^2(a-b)$$

$=a^2(b-c)+b^2(c-a)+c^2(a-b)$, for 1st and 2nd group $=0$

Art. 322 (5 & 6)

$=-(b-c)(c-a)(a-b)$. Art. 332.

338. We shall now consider certain important general **Conditional** Identities, and shew how to employ them in establishing the truth of other Identities. Each of these results should be carefully committed to memory.

339. In connection with conditional Identities, the following important results should be carefully noticed.

If $a+b+c=0$, we may obtain by transposition of terms:

(i) $b+c=-a$			$a=-(b+c)$
(ii) $c+a=-b$		and	$b=-(c+a)$
(iii) $a+b=-c$			$c=-(a+b)$

340. If $a+b+c=0$, prove that

1. $a^2+b^2+c^2=-2(bc+ca+ab)$.

Since $(a+b+c)^2=a^2+b^2+c^2+2bc+2ca+2ab$. Art. 93.

$$\therefore 0^2=a^2+b^2+c^2+2bc+2ca+2ab.$$

Hence, by transposition, we obtain

$$a^2+b^2+c^2=-2(bc+ca+ab).$$

2. $\boldsymbol{a^3+b^3+c^3=3abc}$ by four different methods.

(i) Since $a+b=-c$,

Cubing, $a^3+3ab(a+b)+b^3=-c^3$. Art. 100

Transposing, $a^3+b^3+c^3=-3ab(a+b)$

$$=-3ab\times -c=3abc.$$

(ii) Since $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$, Art. 327

$$=0\times(a^2+b^2+c^2-bc-ca-ab)=0,$$

$\therefore$ by transposition, $a^3+b^3+c^3=3abc$.

(iii) Since $a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc$

$$=(a+b+c)(bc+ca+ab). \quad \text{Art. } 329$$

$\therefore$ $a^2(-a)+b^2(-b)+c^2(-c)+3abc=0\times(bc+ca+ab)$

or $-a^3-b^3-c^3+3abc=0$.

Hence, by transposition, $a^3+b^3+c^3=3abc$.

(iv) Since $(a+b+c)^3=a^3+b^3+c^3+3(b+c)(c+a)(a+b)$. Art. 172

$\therefore$ $0^3=a^3+b^3+c^3+3(-a)(-b)(-c)$

$$=a^3+b^3+c^3-3abc.$$

Hence, by transposition, $a^3+b^3+c^3=3abc$.

3. $\boldsymbol{a^4+b^4+c^4=2(b^2c^2+c^2a^2+a^2b^2)}$ by two different methods

(i) Since $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$

$$=(a+b+c)(a+b-c)(a+c-b)(b+c-a). \quad \text{Art. } 331$$

$$=0\times(a+b-c)(a+c-b)(b+c-a)=0,$$

$\therefore$ by transposition, $a^4+b^4+c^4=2(b^2c^2+c^2a^2+a^2b^2)$.

(ii) Since $a+b=-c$, Squaring $a^2+2ab+b^2=c^2$.

Transposing, $a^2+b^2-c^2=-2ab$.

Squaring, $a^4+b^4+c^4+2a^2b^2-2a^2c^2-2b^2c^2=4a^2b^2$.

Transposing, $a^4+b^4+c^4=2(b^2c^2+c^2a^2+a^2b^2)$.

4. $\boldsymbol{a^5+b^5+c^5=-5abc(bc+ca+ab)=\tfrac{5}{2}abc(a^2+b^2+c^2)}$.

Since $a+b=-c$, Raising both sides to the 5th power,

$$a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5=-c^5.$$

Transposing, $a^5+b^5+c^5=-5ab(a^3+2a^2b+2ab^2+b^3)$

$=-5ab(a+b)(a^2+ab+b^2)$

$=5abc(a^2+ab+b^2)$, for $a+b=-c$.

$=5abc\{(a+b)^2-ab\}$

$=5abc\{(a+b)\times -c-ab\}$

$=-5abc(bc+ca+ab)$

$=\frac{5}{2}abc(a^2+b^2+c^2)$. Art. 340. **1.**

5. $\boldsymbol{a^7+b^7+c^7=7abc(bc+ca+ab)^2}$.

Since $a+b=-c$, Raising both sides to the 7th power,

$$a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7=-c^7.$$

Transposing, $a^7+b^7+c^7=-7ab(a^5+3a^4b+5a^3b^2+5a^2b^3+3ab^4+b^5)$

$=-7ab(a+b)(a^2+ab+b^2)^2$

$=7abc(a^2+ab+b^2)^2$, for $a+b=-c$

$=7abc(bc+ca+ab)^2$. (as above).

Ex. 1. If $a+b+c=0$, shew that

$$(bc+ca+ab)^2=b^2c^2+c^2a^2+a^2b^2=\tfrac{1}{4}(a^2+b^2+c^2)^2.$$

We have, $(bc+ca+ab)^2=b^2c^2+c^2a^2+a^2b^2+2abc(a+b+c)$. Art. 93

$=b^2c^2+c^2a^2+a^2b^2$, for $a+b+c=0$.

$=\frac{1}{4}(a^2+b^2+c^2)^2$. Art. 340, **1.**

Ex. 2. If $a+b+c+d=0$, prove that

$$a^3+b^3+c^3+d^3=3(bcd+cda+dab+abc).$$

We have, $a+b=-(c+d)$. Cubing both sides,

$$a^3+3ab(a+b)+b^3=-\{c^3+3cd(c+d)+d^3\}.$$

Transposing, $a^3+b^3+c^3+d^3=-3ab(a+b)-3cd(c+d)$

$=-3ab\times-(c+d)-3cd\times-(a+b)$

$=3(abc+abd+acd+bcd)$.

Ex. 3. Prove that $(b-c)^3+(c-a)^3+(a-b)^3=3(b-c)(c-a)(a-b)$.
(C. E. 1866 & B. M. 1895).

Let $b-c=x$, $c-a=y$ and $a-b=z$.

then $x+y+z=b-c+c-a+a-b=0$.

$\therefore$ $x^3+y^3+z^3=3xyz$. Art. 340. **2.**

Restoring values, we obtain

$$(b-c)^3+(c-a)^3+(a-b)^3=3(b-c)(c-a)(a-b).$$

Ex. 4. Prove that

$$\{(b-c)^2+(c-a)^2+(a-b)^2\}^2=2\{(b-c)^4+(c-a)^4+(a-b)^4\}.$$

(B.M. 1866).

Let $b-c=x$, $c-a=y$ and $a-b=z$,

then $x+y+z=b-c+c-a+a-b=0$.

Now, since $x^4+y^4+z^4=2y^2z^2+2z^2x^2+2x^2y^2$. Art. 340, **3**.

Add $x^4+y^4+z^4$ to both sides,

$$\therefore\ 2(x^4+y^4+z^4)=x^4+y^4+z^4+2y^2z^2+2z^2x^2+2x^2y^2$$
$$=(x^2+y^2+z^2)^2. \quad \text{Art. 128}$$

Restoring values, we obtain

$$2\{(b-c)^4+(c-a)^4+(a-b)^4\}=\{(b-c)^2+(c-a)^2+(a-b)^2\}^2.$$

Ex. 5. If $x=b+c-a$, $y=c+a-b$, $z=a+b-c$, prove that

$$x^3+y^3+z^3-3xyz=4(a^3+b^3+c^3-3abc).$$

From the given relations, we have

$$x+y+z=(b+c-a)+(c+a-b)+(a+b-c)=a+b+c.$$
$$y-z=(c+a-b)-(a+b-c)=2(c-b).$$
$$z-x=(a+b-c)-(b+c-a)=2(a-c)$$

and $x-y=(b+c-a)-(c+a-b)=2(b-a)$.

Now, $x^3+y^3+z^3-3xyz=\frac{1}{2}(x+y+z)\{(y-z)^2+(z-x)^2+(x-y)^2\}$,

Art. 327.

$$=\tfrac{1}{2}(a+b+c)\{4(c-b)^2+4(a-c)^2+4(b-a)^2\}$$
$$=4(a+b+c)\{a^2+b^2+c^2-bc-ca-ab)$$
$$=4(a^3+b^3+c^3-3abc).$$

Ex. 6. If $2s=a+b+c$, prove that

$$(s-a)^3+(s-b)^3+(s-c)^3+3abc=s^3.$$

We have $s=3s-2s=3s-(a+b+c)$

$$\therefore\ =(s-a)+(s-b)+(s-c)$$

Cubing both sides of the above equality, we get

$$s^3=(s-a)^3+(s-b)^3+(s-c)^3+3(2s-b-c)(2s-a-c)(2s-a-b),$$

Art. 172.

$$=(s-a)^3+(s-b)^3+(s-c)^3+3abc, \text{ for } 2s=a+b+c.$$

Ex. 7. Prove that $(b-c)(1+ab)(1+ac)+(c-a)(1+bc)(1+ba)+(a-b)(1+ca)(1+cb)=(b-c)(c-a)(a-b)$.

$$\text{1st Exp. on the left}=(b-c)\{1+a(b+c)+a^2bc\}$$
$$=(b-c)+a(b^2-c^2)+a^2bc(b-c).$$

$$\text{Similarly, 2nd Exp.}=(c-a)+b(c^2-a^2)+ab^2c(c-a).$$
$$\text{3rd Exp.}=(a-b)+c(a^2-b^2)+abc^2(a-b).$$

Hence the Exp. on the left (adding the columns vertically)

$$=(b-c+c-a+a-b)+a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$$
$$+abc\{a(b-c)+b(c-a)+c(a-b)\}$$
$$=a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2),\ \text{Art. 322 (5 \& 6)}$$
$$=(b-c)(c-a)(a-b).\quad\text{Art. 332.}$$

Ex. 8. Prove that $a^2(b-c)^3+b^2(c-a)^3+c^2(a-b)^3$
$$=(bc+ca+ab)(b-c)(c-a)(a-b).$$

$$\text{1st Exp. on the left}=a^2\{b^3-3bc(b-c)-c^3\}$$
$$=a^2(b^3-c^3)-3a^2bc(b-c).$$

$$\text{Similarly, 2nd Exp.}=b^2(c^3-a^3)-3ab^2c(c-a).$$
$$\text{3rd Exp.}=c^2(a^3-b^3)-3abc^2(a-b).$$

Hence the Exp. on the left (adding the columns vertically)

$$=a^2(b^3-c^3)+b^2(c^3-a^3)+c^2(a^3-b^3)$$
$$-3abc\{a(b-c)+b(c-a)+c(a-b)\}$$
$$=a^2(b^3-c^3)+b^2(c^3-a^3)+c^2(a^3-b^3),\ \text{Art. 322 (6)}$$
$$=a^2(b^3-c^3)-b^2c^2(b-c)-a^3(b^2-c^2),\ \text{(re-arranging)}$$
$$=(b-c)\{a^2(b^2+bc+c^2)-b^2c^2-a^3(b+c)\}$$
$$=(b-c)\{a^2(b+c)^2-b^2c^2-a^2(bc+ca+ab)\}$$
$$=(b-c)(bc+ca+ab)\{a(b+c)-bc-a^2\}$$
$$=(b-c)(bc+ca+ab)\times-\{a^2+bc-a(b+c)\}$$
$$=(b-c)(bc+ca+ab)\times-(a-b)(a-c)$$
$$=(bc+ca+ab)(b-c)(c-a)(a-b).$$

Exercise CXXIV.

Establish the following Identities :—

1. $a(b+c)(b^2+c^2-a^2)+b(c+a)(c^2+a^2-b^2)+c(a+b)(a^2+b^2-c^2)$
$$=2abc(a+b+c).$$

2. $4(x-y)^3-(x-4y)(2x+y)^2=27xy^2.$

3. $(x-y)(x-y-z)(x+2y-2z)+y(y-z)(3x-2y-2z)$
$=x(x-z(x-2z)$. (P. E. 1891).

4. $(x-a)^2+(y-b)^2+(a^2+b^2-1)(x^2+y^2-1)$
$=(ax+by-1)^2+(ay-bx)^2$.

5. $(1-a^2)(1-b^2)(1-c^2)-(a+bc)(b+ca)(c+ab)$
$=(1+abc)(1-a^2-b^2-c^2-2abc)$.

6. $(a-x)(a-2x)(a-3x)+9x(a-x)(a-2x)+18x^2(a-x)+6x^3$
$=a(a+x)(a+2x)$.

7. $x^3+6(y+z)x^2+12(y+z)^2x+8(y+z)^3$
$=4(3x+2y+6z)y^2+(x+6y+2z)(x+2z)^2$. (M. M. 1881).

8. $a(a-2b)^3-b(b-2a)^3=(a-b)(a+b)^3$.

9. $a(a+2b)^3-b(b+2a)^3=(a+b)(a-b)^3$.

10. $4(a^2+ab+b^2)^3-(a-b)^2(a+2b)^2(2a+b)^2=27a^2b^2(a+b)^2$.
(M. M. 1888).

11. $a(b+c-a)^2+b(c+a-b)^2+c(a+b-c)^2+$
$(b+c-a)(c+a-b)(a+b-c)=4abc$.

12. $a(b-c)^2+b(c-a)^2+c(a-b)^2+9abc=(a+b+c)(bc+ca+ab)$.

13. $x(y+z)^2+y(z+x)^2+z(x+y)^2-4xyz=(y+z)(z+x)(x+y)$.

14. $8(a+b+c)^3-(b+c)^3-(c+a)^3-(a+b)^3$
$=3(2a+b+c)(a+2b+c)(a+b+2c)$. (M. M. 1881).

15. $a^2+b^2+c^2-3ab-bc-2ac+a(a+b+c)-(b-c)^2=(a-b)(2a-c)$.

16. $a^2(b-c)+b^2(c-a)+c^2(a-b)+(b-c)(c-a)(a-b)=0$.

17. $a^3(b-c)+b^3(c-a)+c^3(a-b)+(a+b+c)(b-c)(c-a)(a-b)=0$.

18. $(b-c)(b+c)^2+(c-a)(c+a)^2+(a-b)(a+b)^2=-(b-c)(c-a)(a-b)$

19. $(b-c)(b+c)^3+(c-a)(c+a)^3+(a-b)(a+b)^3$
$=-2(a+b+c)(b-c)(c-a)(a-b)$.

20. $(c+2a+3b)^2(b+c-2a)+(a+2b+3c)^2(c+a-2b)+$
$(b+2c+3a)^2(a+b-2c)+(b+c-2a)(c+a-2b)(a+b-2c)=0$.

21. $(b-c)(x-b)(x-c)+(c-a)(x-c)(x-a)+(a-b)(x-a)(x-b)$
$=-(b-c)(c-a)(a-b)$.

22. $a(b-c)(x-b)(x-c)+b(c-a)(x-c)(x-a)+c(a-b)(x-a)(x-b)$
$=-x(b-c)(c-a)(a-b)$.

23. $a(b-c)(1+ab)(1+ac)+b(c-a)(1+bc)(1+ba)$
$$+c(a-b)(1+ca)(1+cb)=-abc(b-c)(c-a)(a-b).$$

24. $a(b-c)^3+b(c-a)^3+c(a-b)^3=(a+b+c)(b-c)(c-a)(a-b).$

25. $a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3=3abc(b-c)(c-a)(a-b).$

26. $(b+c-2a)^3+(c+a-2b)^3+(a+b-2c)^3$
$$=3(b+c-2a)(c+a-2b)(a+b-2c).$$

27. $(b-c)(b+c-2a)^3+(c-a)(c+a-2b)^3+(a-b)(a+b-2c)^3=0.$

28. $(x-a)^3(b-c)^3+(x-b)^3(c-a)^3+(x-c)^3(a-b)^3$
$$=3(x-a)(x-b)(x-c)(b-c)(c-a)(a-b).$$

29. $9(a^3+b^3+c^3)-(a+b+c)^3=(4b+4c+a)(b-c)^2$
$$+(4c+4a+b)(c-a)^2+(4a+4b+c)(a-b)^2.$$

30. $(y-z)^4+(z-x)^4+(x-y)^4$
$$=2\{(y-z)^2(z-x)^2+(z-x)^2(x-y)^2+(x-y)^2(y-z)^2\}$$
$$=2(x^2+y^2+z^2-yz-zx-xy)^2.$$

31. If $a+b+c=0$, prove that

(1) $b^2-ca=c^2-ab=a^2-bc=\frac{1}{2}(a^2+b^2+c^2).$

(2) $b^2+bc+c^2=c^2+ca+a^2=a^2+ab+b^2=\frac{1}{2}(a^2+b^2+c^2).$

(3) $a(b^2+c^2-a^2)=b(c^2+a^2-b^2)=c(a^2+b^2-c^2)=-2abc.$

(4) $(bc+ca+ab)^2=b^2c^2+c^2a^2+a^2b^2=\frac{1}{4}(a^2+b^2+c^2)^2.$

(5) $a(a+b)(a+c)=b(b+a)(b+c)=c(c+a)(c+b).$

(6) $bc(b+c)+ca(c+a)+ab(a+b)+3abc=0.$

(7) $a(b-c)^2+b(c-a)^2+c(a-b)^2+9abc=0.$

(8) $(a^2+b^2+c^2)^2=2(a^4+b^4+c^4)=4(bc+ca+ab)^2.$

(9) $6(a^5+b^5+c^5)=5(a^2+b^2+c^2)(a^3+b^3+c^3).$

(10) $25(a^3+b^3+c^3)(a^7+b^7+c^7)=21(a^5+b^5+c^5)^2.$

(11) $(b+c-a)^3+(c+a-b)^3+(a+b-c)^3$
$$=3(b+c-a)(c+a-b)(a+b-c)=-24abc.$$

(12) $a(b-c)^3+b(c-a)^3+c(a-b)^3=0.$ (M.M. 1878).

32. If $a+b+c+d=0$, prove that

(1) $(a+b)(a+c)(a+d)=(b+c)(b+d)(b+a)$
$$=(c+a)(c+b)(c+d)=(d+a)(d+b)(d+c).$$

(2) $a^3+b^3+c^3+d^3+3(b+c)(c+a)(a+b)=0.$

33. If $a=x^2-yz$, $b=y^2-zx$ and $c=z^2-xy$, prove that
$$c^2-ab=z(ax+by+cz).$$

34. Show that the expression $a^2+b^2+c^2-bc-ca-ab$ is not altered by adding the same quantity to a, to b and to c, or by subtracting the same quantity from each.

35. Prove that the expression $b^2c+c^2a+a^2b-bc^2-ca^2-ab^2$ is exactly divisible by the difference of any two of the quantities a, b, c.

36. If $x=a^2-bc$, $y=b^2-ca$ and $z=c^2-ab$, shew that
$$ax+by+cz=(a+b+c)(x+y+z).$$

37. If $x=b+c$, $y=c+a$ and $z=a+b$, prove that
$$x^3+y^3+z^3-3xyz=2(a^3+b^3+c^3-3abc).$$

38. If $a=x^2-yz$, $b=y^2-zx$ and $c=z^2-xy$, prove that
$$a^3+b^3+c^3-3abc=(x^3+y^3+z^3-3xyz)^2.$$

39. If $x=2a+b+c$, $y=2b+c+a$ and $z=2c+a+b$, shew that
$$x^3+y^3+z^3-3xyz=4(a^3+b^3+c^3-3abc).$$

40. If $2s=a+b+c$, prove that :—

(1) $s(s-c)+(s-a)(s-b)=ab$.

(2) $2(s-b)(s-c)+2(s-c)(s-a)+2(s-a)(s-b)=2s^2-a^2-b^2-c^2$.

(3) $\{(s-a)+(s-b)\}^3=(s-a)^3+(s-b)^3+3(s-a)(s-b)c$.

(4) $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4=16s(s-a)(s-b)(s-c)$.

(C. E. 1867).

41. If $2S=a^2+b^2+c^2$ and $2s=a+b+c$, prove that
$$(S-a^2)(S-b^2)+(S-b^2)(S-c^2)+(S-c^2)(S-a^2)$$
$$=4s(s-a)(s-b)(s-c).$$

42. If $s=a+b+c$, prove that
$$(s-3a)^3+(s-3b)^3+(s-3c)^3-3(s-3a)(s-3b)(s-3c)=0.$$

43. If $3s=a+b+c$, prove that
$$(s-a)^4+(s-b)^4+(s-c)^4$$
$$=2(s-b)^2(s-c)^2+2(s-c)^2(s-a)^2+2(s-a)^2(s-b)^2.$$

44. Prove that $(x^2+2yz)^3+(y^2+2zx)^3+(z^2+2xy)^3$
$$-3(x^2+2yz)(y^2+2zx)(z^2+2xy)=(x^3+y^3+z^3-3xyz)^2.$$

45. Prove that $2\{(b+c-2a)^4+(c+a-2b)^4+(a+b-2c)^4\}$
$$=\{(b+c-2a)^2+(c+a-2b)^2+(a+b-2c)^2\}^2.$$ (C.E. 1896).

46. Prove that $25\{(b-c)^7+(c-a)^7+(a-b)^7\}\{(b-c)^3+(c-a)^3+(a-b)^3\}$
$=21\{(b-c)^5+(c-a)^5+(a-b)^5\}^2.$

47. Prove that $(b-c)^6+(c-a)^6+(a-b)^6$
$=5(b-c)(c-a)(a-b)(a^2+b^2+c^2-bc-ca-ab).$

48. Prove that $(b+2c-3a)^3+(c+2a-3b)^3+(a+2b-3c)^3$
$=3(b+2c-3a)(c+2a-3b)(a+2b-3c).$

49. If $X=ax+by+cz$, $Y=bx+cy+az$ and $Z=cx+ay+bz$, prove that

(1) $X^2+Y^2+Z^2-YZ-ZX-XY$
$=(a^2+b^2+c^2-bc-ca-ab)(x^2+y^2+z^2-yz-zx-xy).$

(2) $X^3+Y^3+Z^3-3XYZ$
$=(a^3+b^3+c^3-3abc)(x^3+y^3+z^3-3xyz).$

50. If $a+b+c=p$, $bc+ca+ab=q$ and $abc=r$, find the values of

(1) $(b+c)(c+a)(a+b).$ (2) $a^2(b+c)+b^2(c+a)+c^2(a+b).$

(3) $a^3+b^3+c^3.$ (4) $b^3c^3+c^3a^3+a^3b^3.$

CHAPTER XIV.

HARDER FRACTIONS.

I. REDUCTION OF FRACTIONS.

341. To simplify a fraction by resolving its terms into factors.

Ex. 1. Simplify $\dfrac{a^4-a^3b-ab^3+b^4}{a^4+3a^3b+4a^2b^2+3ab^3+b^4}$.

$$\text{Numr.}=a^3(a-b)-b^3(a-b)=(a-b)(a^3-b^3)$$
$$=(a-b)(a-b)(a^2+ab+b^2)=(a-b)^2(a^2+ab+b^2).$$

$$\text{Denr.}=(a^4+a^2b^2+b^4)+3ab(a^2+ab+b^2)$$
$$=(a^2+ab+b^2)(a^2-ab+b^2)+3ab(a^2+ab+b^2)$$
$$=(a^2+ab+b^2)\{(a^2-ab+b^2)+3ab\}$$
$$=(a^2+ab+b^2)(a^2+2ab+b^2)=(a^2+ab+b^2)(a+b)^2.$$

$$\therefore \text{ Fraction}=\left(\frac{a-b}{a+b}\right)^2.$$

Ex. 2. Simplify $\dfrac{(ab+1)^2+(a-b+2)(a-b-2ab)}{(ab-1)^2-(a-b)^2}$.

$$\text{Numr.}=(ab+1)^2+\{(a-b)+2\}\{(a-b)-2ab\}$$
$$=(ab+1)^2+(a-b)^2-2(a-b)(ab-1)-4ab$$
$$=\{(ab+1)^2-4ab\}+(a-b)^2-2(a-b)(ab-1)$$
$$=(ab-1)^2+(a-b)^2-2(a-b)(ab-1)=\{(ab-1)-(a-b)\}^2$$

$$\text{Denr.}=\{(ab-1)+(a-b)\}\{(ab-1)-(a-b)\}.$$

$$\therefore\ \text{Fraction}=\frac{(ab-1)-(a-b)}{(ab-1)+(a-b)}=\frac{b(a+1)-(a+1)}{b(a-1)+(a-1)}=\frac{(a+1)(b-1)}{(a-1)(b+1)}.$$

Ex. 3. Simplify $\dfrac{(y+z)^3(y-z)+(z+x)^3(z-x)+(x+y)^3(x-y)}{(y+z)^2(y-z)+(z+x)^2(z-x)+(x+y)^2(x-y)}$.

Let $y+z=a$, $z+x=b$ and $x+y=c$; then $b-c=(z+x)-(x+y)=z-y$, $c-a=(x+y)-(y+z)=x-z$, and $a-b=(y+z)-(z+x)=y-x$.

$$\text{Now, Numr.}=-a^3(b-c)-b^3(c-a)-c^3(a-b)$$
$$=-\{a^3(b-c)+b^3(c-a)+c^3(a-b)\}$$
$$=(a+b+c)(b-c)(c-a)(a-b).\quad \text{Art. 332, Ex. 1.}$$

$$\text{and Denr.}=-a^2(b-c)-b^2(c-a)-c^2(a-b)$$
$$=-\{a^2(b-c)+b^2(c-a)+c^2(a-b)\}$$
$$=(b-c)(c-a)(a-b).\quad \text{Art. 332.}$$

$$\therefore\ \text{Fraction}=a+b+c=2(x+y+z).$$

Exercise CXXV.

Simplify the following expressions :—

1. $\dfrac{(a+b)\{(a+b)^2-c^2\}}{4b^2c^2-(a^2-b^2-c^2)^2}$.

2. $\dfrac{(x+y)^7-x^7-y^7}{(x+y)^5-x^5-y^5}$.

3. $\dfrac{e^{2x}x^3+e^{2x}-x^3-1}{e^{3x}x^2+2e^xx^2-e^{2x}-2e^x+x^2-1}$. (C. F. A. 1863).

4. $\dfrac{(x^2-xy+y^2)^3+(x^2+xy+y^2)^3}{2(x^2+y^2)}$. (B. M. 1891).

5. $\dfrac{xy+2x^2-3y^2+4yz-xz-z^2}{2x^2-9xz-5xy+4z^2-8yz-12y^2}$. (M. M. 1883).

6. $\dfrac{(a^4-b^4)^2+2a^6b^2+5a^4b^4+2a^2b^6}{(a^2+ab+b^2)^2(a^2-ab+b^2)^2}$. (M. M. 1880).

7. $\dfrac{(y-z)(y+z)^3+(z-x)(z+x)^3+(x-y)(x+y)^3}{(y+z)(y-z)^3+(z+x)(z-x)^3+(x+y)(x-y)^3}$.

(M. M. 1892 & B. M. 1888).

8. $\dfrac{(b+c-2a)^3+(c+a-2b)^3+(a+b-2c)^3}{(b+c-2a)(c+a-2b)(a+b-2c)}$.

9. $\dfrac{a^3+b^3+c^3+3(b+c)(c+a)(a+b)}{a^3+b^3+c^3+3ab(a+b)}$.

10. $\dfrac{ax^m-bx^{m+1}}{a^2bx-b^3x^3}$.

11. $\dfrac{8a^2b^2c^2+(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)}{(a+b+c)(a+b-c)(a-b+c)(b+c-a)}$.

12. $\dfrac{2(a^2-1)(b^2-1)ab+4a^2b^2-(a^2+b^2)(1+a^2b^2)}{2(a^2+1)(b^2+1)ab-4a^2b^2-(a^2+b^2)(1+a^2b^2)}$.

13. $\dfrac{(1-a^2)(1-b^2)(1-c^2)-(a+bc)(b+ca)(c+ab)}{1-a^2-b^2-c^2-2abc}$.

14. $\dfrac{3x^2-(4a+2b)x+a^2+2ab}{x^3-(2a+b)x^2+(a^2+2ab)x-a^2b}$. (B. M. 1896).

15. $\dfrac{(b+c-2a)^3-(c+a-2b)^3}{(c+a-2b)^3-(a+b-2c)^3}$. (M. M 1897).

16. $\dfrac{a^3+b^3+c^3-3abc}{(a-b)^2+(b-c)^2+(c-a)^2}$. (A. E. 1902).

17 $\dfrac{8(a+b+c)^3-(b+c)^3-(c+a)^3-(a+b)^3}{3(2a+b+c)(a+2b+c)(a+b+2c)}$. (M. M. 1881).

18. $\dfrac{(a+b\sqrt{c})^2+(a-b\sqrt{c})^2-(2b\sqrt{ac})^2}{(a+b\sqrt{c})^2+(a-b\sqrt{c})^2}$. (P. E. 1896).

19. $\dfrac{bc(b-c)(b^2+c^2)+ca(c-a)(c^2+a^2)+ab(a-b)(a^2+b^2)}{b^2c^2(b-c)+c^2a^2(c-a)+a^2b^2(a-b)}$.

20. $\dfrac{a^m(b-c)(c-d)+c^m(a-b)(a-d)}{b^m(a-d)(c-d)+d^m(a-b)(b-c)}$, when $m=0$, 1 or 2.

II. ADDITION AND SUBTRACTION OF FRACTIONS

342. The following are illustrative Examples.

Ex. 1. Simplify

$$\frac{b+c}{2bc}(b^2+c^2-a^2)+\frac{c+a}{2ca}(c^2+a^2-b^2)+\frac{a+b}{2ab}(a^2+b^2-c^2).\ \text{(M.M. 1877)}.$$

$$\text{1st fraction}=\tfrac{1}{2}\left(\frac{1}{b}+\frac{1}{c}\right)(b^2+c^2-a^2)=\tfrac{1}{2}\left(b+\frac{c^2}{b}-\frac{a^2}{b}+\frac{b^2}{c}+c-\frac{a^2}{c}\right).$$

$$\text{2nd fraction}=\tfrac{1}{2}\left(\frac{1}{c}+\frac{1}{a}\right)(c^2+a^2-b^2)=\tfrac{1}{2}\left(c+\frac{a^2}{c}-\frac{b^2}{c}+\frac{c^2}{a}+a-\frac{b^2}{a}\right).$$

$$\text{3rd fraction}=\tfrac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)(a^2+b^2-c^2)=\tfrac{1}{2}\left(a+\frac{b^2}{a}-\frac{c^2}{a}+\frac{a^2}{b}+b-\frac{c^2}{b}\right).$$

$$\therefore\ \text{sum}=a+b+c.$$

Ex. 2. Simplify $\dfrac{1}{1+x^{n-m}}+\dfrac{1}{1+x^{m-n}}$.

$$\text{1st fraction}=\frac{x^m}{x^m(1+x^{n-m})}=\frac{x^m}{x^m+x^n}.$$

$$\text{2nd fraction}=\frac{x^n}{x^n(1+x^{m-n})}=\frac{x^n}{x^n+x^m}.$$

$$\therefore\ \text{sum}=\frac{x^m+x^n}{x^m+x^n}=1.$$

Ex. 3. Simplify $\dfrac{x^{3n}}{x^n-1}-\dfrac{x^{2n}}{x^n+1}-\dfrac{1}{x^n-1}+\dfrac{1}{x^n+1}$. (C. E. 1885).

$$\text{The Exp.}=\frac{x^{3n}-1}{x^n-1}-\frac{x^{2n}-1}{x^n+1}$$

$$=\frac{(x^n-1)(x^{2n}+x^n+1)}{x^n-1}-\frac{(x^n+1)(x^n-1)}{x^n+1}$$

$$=(x^{2n}+x^n+1)-(x^n-1)=x^{2n}+2.$$

Ex. 4. Simplify $\dfrac{a^2}{(x-a)^n}+\dfrac{2a}{(x-a)^{n-1}}+\dfrac{1}{(x-a)^{n-2}}$.

(A. E. 1898 & M. M. 1865).

$$\text{The Exp.} = \frac{a^2}{(x-a)^n} + \frac{2a(x-a)}{(x-a)^n} + \frac{(x-a)^2}{(x-a)^n}$$

$$= \frac{a^2 + 2a(x-a) + (x-a)^2}{(x-a)^n}$$

$$= \frac{\{a+(x-a)\}^2}{(x-a)^n} = \frac{x^2}{(x-a)^n}.$$

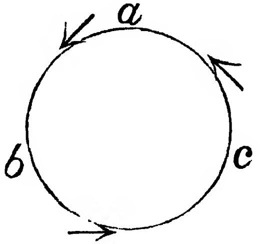
Ex. 5. Simplify $\dfrac{3x-12}{x^2-5x+6} + \dfrac{5x-3}{x^2-2x-3} - \dfrac{x-13}{x^2-5x-6}$.

$$\text{The Exp.} = \frac{3x-12}{(x-2)(x-3)} + \frac{5x-3}{(x+1)(x-3)} - \frac{x-13}{(x+1)(x-6)}$$

$$= \left(\frac{6}{x-2} - \frac{3}{x-3}\right) + \left(\frac{3}{x-3} + \frac{2}{x+1}\right) - \left(\frac{2}{x+1} - \frac{1}{x-6}\right)$$

$$= \frac{6}{x-2} + \frac{1}{x-6} = \frac{6x-36+x-2}{(x-2)(x-6)} = \frac{7x-38}{(x-2)(x-6)}.$$

343. There is a certain peculiarity in the arrangement of the three letters a, b, c, known as **Cyclic Order.** In this order, if we arrange them round the circumference of a circle, thus (as shewn in the diagram) they are said to be taken in *Cyclic Order*, if starting at any letter and moving in the direction of the arrows, we take them in the order in which we meet them.



Thus, starting from a, we come next to b, and thus $a-b$ or $a+b$ are in Cyclic Order. After b we come to c, and thus $b-c$ or $b+c$ are in Cyclic Order. After c we come back to a, and thus $c-a$ or $c+a$ are in Cyclic Order.

The following are important Examples.

Ex. 1. Simplify $\dfrac{bc}{(a-b)(a-c)} + \dfrac{}{(b-c)(b-a)} + \dfrac{ab}{(c-a)(c-b)}$.

Considering the first denominator, we see that one factor $a-c$ is not in Cyclic Order. But $a-c=-(c-a)$, so that $(a-b)(a-c)=-(a-b)(c-a)$, and thus

$$\frac{bc}{(a-b)(a-c)}=-\frac{bc}{(a-b)(c-a)}.$$

Similarly, we have

$$\frac{ca}{(b-c)(b-a)}=-\frac{ca}{(b-c)(a-b)} \text{ and } \frac{ab}{(c-a)(c-b)}=-\frac{ab}{(c-a)(b-c)}.$$

Now, L. C. M. of the denrs. $=(b-c)(c-a)(a-b)$.

Thus, the whole expression

$$=\frac{-bc(b-c)-ca(c-a)-ab(a-b)}{(b-c)(c-a)(a-b)}$$

$$=\frac{(b-c)(c-a)(a-b)}{(b-c)(c-a)(a-b)}=1. \quad \text{(Art. 322 ; 12)}$$

Ex. 2. Simplify

$$\frac{1}{(a-b)(a-c)(x-a)}+\frac{1}{(b-c)(b-a)(x-b)}+\frac{1}{(c-a)(c-b)(x-c)}$$

The expression in Cyclic Order

$$=-\frac{1}{(a-b)(c-a)(x-a)}-\frac{1}{(b-c)(a-b)(x-b)}-\frac{1}{(c-a)(b-c)(x-c)}$$

The L. C. D. is now $=(b-c)(c-a)(a-b)(x-a)(x-b)(x-c)$.

$\therefore$ Fraction $=$

$$\frac{-(b-c)(x-b)(x-c)-(c-a)(x-c)(x-a)-(a-b)(x-a)(x-b)}{(b-c)(c-a)(a-b)(x-a)(x-b)(x-c)}.$$

$$\begin{aligned}
\text{Numr.} &= \left.\begin{array}{l}-(b-c)\{x^2-(b+c)x+bc\}\\ -(c-a)\{x^2-(c+a)x+ca\}\\ -(a-b)\{x^2-(a+b)x+ab\}\end{array}\right\}\\
&= \left.\begin{array}{l}-(b-c)x^2+(b^2-c^2)x-bc(b-c)\\ -(c-a)x^2+(c^2-a^2)x-ca(c-a)\\ -(a-b)x^2+(a^2-b^2)x-ab(a-b)\end{array}\right\}\\
&= -\{(b-c)+(c-a)+(a-b)\}x^2+\{(b^2-c^2)+(c^2-a^2)+(a^2-b^2)\}x\\
&\quad -\{bc(b-c)+ca(c-a)+ab(a-b)\}\\
&= -\{bc(b-c)+ca(c-a)+ab(a-b)\} \quad \text{Art. 322 (5 \& 6).}\\
&= (b-c)(c-a)(a-b). \quad \text{Art. 332.}
\end{aligned}$$

$\therefore$ Fraction $=\dfrac{\quad}{(x-a)(x-b)(x-c)}.$

344. The following results, if carefully committed to memory, will be of great service in simplifying Harder Examples in Fractions.

1. If $\dfrac{1}{(a-b)(a-c)}=X$, $\dfrac{1}{(b-c)(b-a)}=Y$

and $\dfrac{1}{(c-a)(c-b)}=Z$;

then, (i) $X+Y+Z=0$. (ii) $aX+bY+cZ=0$.

(iii) $a^2X+b^2Y+c^2Z=1$. (iv) $bcX+caY+abZ=1$.

(v) $a^3X+b^3Y+c^3Z=a+b+c$.

(vi) $a^4X+b^4Y+c^4Z=a^2+b^2+c^2+bc+ca+ab$.

2. If $\dfrac{1}{(a-b)(a-c)(x\pm a)}=P$, $\dfrac{1}{(b-a)(b-c)(x\pm b)}=Q$,

$\dfrac{1}{(c-a)(c-b)(x\pm c)}=R$, and $\dfrac{1}{(x\pm a)(x\pm b)(x\pm c)}=S$;

then, (i) $P+Q+R=S$. (ii) $aP+bQ+cR=Sx$.

(iii) $a^2P+b^2Q+c^2R=Sx^2$.

These results can be easily verified.

345. The following Examples illustrate the use of the above formulae.

Ex. 1. Simplify

$$\frac{pa^2+qa+r}{(a-b)(a-c)}+\frac{pb^2+qb+r}{(b-c)(b-a)}+\frac{pc^2+qc+r}{(c-a)(c-b)}.$$

The expression $=$

$$p\left\{\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)}\right\}$$

$$+q\left\{\frac{a}{(a-b)(a-c)}+\frac{b}{(b-c)(b-a)}+\frac{c}{(c-a)(c-b)}\right\}$$

$$+r\left\{\frac{1}{(a-b)(a-c)}+\frac{1}{(b-c)(b-a)}+\frac{1}{(c-a)(c-b)}\right\}$$

$$=p\times 1+q\times 0+r\times 0=p+0+0=p.$$

Ex. 2. Simplify

$$\frac{pa^2+qa+r}{(a-b)(a-c)(x+a)}+\frac{pb^2+qb+r}{(b-c)(b-a)(x+b)}+\frac{pc^2+qc+r}{(c-a)(c-b)(x+c)}.$$

The expression =

$$p\left\{\frac{a^2}{(a-b)(a-c)(x+a)}+\frac{b^2}{(b-c)(b-a)(x+b)}+\frac{c^2}{(c-a)(c-b)(x+c)}\right\}$$

$$+q\left\{\frac{a}{(a-b)(a-c)(x+a)}+\frac{b}{(b-c)(b-a)(x+b)}+\frac{c}{(c-a)(c-b)(x+c)}\right\}$$

$$+r\left\{\frac{1}{(a-b)(a-c)(x+a)}+\frac{1}{(b-c)(b-a)(x+b)}+\frac{1}{(c-a)(c-b)(x+c)}\right\}$$

$$=\frac{px^2}{(x+a)(x+b)(x+c)}+\frac{qx}{(x+a)(x+b)(x+c)}+\frac{r}{(x+a)(x+b)(x+c)}$$

$$=\frac{px^2+qx+r}{(x+a)(x+b)(x+c)}.$$

Exercise CXXVI

Simplify the following :—

1. $\frac{b+c}{2bc}(b+c-a)+\frac{c+a}{2ca}(c+a-b)+\frac{a+b}{2ab}(a+b-c).$

2. $\frac{x^2+x+1}{x^2-x+1}-\frac{x^2-x+1}{x^2+x+1}-\frac{x-1}{x+1}+\frac{x+1}{x-1}.$

3. $\frac{x^2-x+1}{x^2+x+1}+\frac{2x(x-1)^2}{x^4+x^2+1}+\frac{2x^2(x^2-1)^2}{x^8+x^4+1}.$

4. $\left(\frac{a+6b^3}{6b^2}\right)^3+\left(\frac{a-6b^3}{6b^2}\right)^3-2\left(\frac{a}{6b^2}\right)^3.$

5. $\frac{b+c-2a}{b-a}-\frac{(b-c)^2}{(a-b)(a-c)}+\frac{b+c-2a}{c-a}.$ (M. M. 1868).

6. $\frac{2}{b-c}+\frac{2}{c-a}+\frac{2}{a-b}+\frac{(b-c)^2+(c-a)^2+(a-b)^2}{(b-c)(c-a)(a-b)}.$ (P. E. 1888).

7. $\frac{(a+b)^2}{(x-a)(x+a+b)}-\frac{a+2b+x}{2(x-a)}+\frac{(a+b)x}{x^2+bx-a^2-ab}+\frac{1}{2}.$ (M. M. 1879).

8. $\frac{a^4+b^4+ab(a^2+b^2)}{(a+b)^2}-\frac{a^4+b^4-ab(a^2+b^2)}{(a-b)^2}+\frac{12a^2b^2}{(a+b)^2-(a-b)^2}.$ (M. M. 1882).

9. $\frac{\sqrt{x}}{\sqrt{x}-\sqrt{a}}-\frac{\sqrt{a}}{\sqrt{x}+\sqrt{a}}-\frac{x-a}{x+a}.$ (P. E. 1896).

10. $\dfrac{ac-1}{(a-b)(1+ax)}+\dfrac{bc-1}{(b-a)(1+bx)}$. (M. M. 1870).

11. $\left\{\sqrt{\left(\dfrac{a+x}{x}\right)}-\sqrt{\left(\dfrac{x}{a+x}\right)}\right\}^2-\left\{\sqrt{\dfrac{x}{a}}-\sqrt{\dfrac{a}{x}}\right\}^2+\dfrac{x^2}{a(a+x)}$.

(B. M. 1876).

12. $\dfrac{a^3}{(x+a)^n}-\dfrac{3a^2}{(x+a)^{n-1}}+\dfrac{3a}{(x+a)^{n-2}}-\dfrac{1}{(x+a)^{n-3}}$.

13. $\dfrac{(ac+bd)^3-(ad+bc)^3}{(a-b)(c-d)}-\dfrac{(ac+bd)^3+(ad+bc)^3}{(a+b)(c+d)}$. (M. M. 1874).

14. $\dfrac{1}{1+x^{q-r}+x^{q-p}}+\dfrac{1}{1+x^{r-p}+x^{r-q}}+\dfrac{1}{1+x^{p-q}+x^{p-r}}$. (P. E. 1903).

15. $\dfrac{b+c}{b-c}+\dfrac{c+a}{c-a}+\dfrac{a+b}{a-b}+\dfrac{(b+c)(c+a)(a+b)}{(b-c)(c-a)(a-b)}$.

16. $\dfrac{b-c}{b+c}+\dfrac{c-a}{c+a}+\dfrac{a-b}{a+b}+\dfrac{(b-c)(c-a)(a-b)}{(b+c)(c+a)(a+b)}$.

17. $\dfrac{x^3+3x^2+5x+15}{x^3+2x^2+5x+10}+\dfrac{x^4+x^3+3x^2+x-2}{x^4+2x^3+3x^2+4x-4}$. (M. M. 1878).

18. $\dfrac{1}{x^3-x^2+x-1}+\dfrac{3}{2x^2-x-1}+\dfrac{1-3x}{2x^3+x^2+2x+1}$. (M. M. 1898).

19. $\dfrac{2}{b-c}+\dfrac{b-c}{(c-a)(a-b)}+\dfrac{2}{c-a}+\dfrac{c-a}{(a-b)(b-c)}+\dfrac{2}{a-b}+\dfrac{a-b}{(b-c)(c-a)}$.

(A. E. 1893)

20. $\dfrac{x^4-(x-1)^2}{(x^2+1)^2-x^2}+\dfrac{x^2-(x^2-1)^2}{x^2(x+1)^2-1}+\dfrac{x^2(x-1)^2-1}{\quad}$. (M. M. 1871).

21. $\dfrac{b+c}{2bc}(b^3+c^3-a^3)+\dfrac{c+a}{2ca}(c^3+a^3-b^3)+\dfrac{a+b}{2ab}(a^3+b^3-c^3)$.

Simplify the following :—

22. $\dfrac{a^m}{(a-b)(a-c)}+\dfrac{b^m}{(b-c)(b-a)}+\dfrac{c^m}{(c-a)(c-b)}$.

when $m=0$, 1, 2, 3 and 4. when $m=3$. (C. E. 1865-87).

23. $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-c)(b-a)}+\dfrac{1}{c(c-a)(c-b)}$. (C. E. 1865-72).

24. $\dfrac{1}{abx}+\dfrac{1}{a(a-b)(x-a)}+\dfrac{1}{b(b-a)(x-b)}$. (C. E. 1881).

25. $\frac{p+q}{(q-r)(r-p)}+\frac{q+r}{(r-p)(p-q)}+\frac{r+p}{(p-q)(q-r)}$. (A. E. 1892).

26. $\frac{m-n}{(x-m)(x-n)}+\frac{n-p}{(x-n)(x-p)}+\frac{p-m}{(x-p)(x-m)}$.

27. $\frac{x^2-yz}{(x-y)(x-z)}+\frac{y^2+zx}{(y+z)(y-x)}+\frac{z^2+xy}{(z-x)(z+y)}$. (C. E. 1865).

28. $\frac{a^2+a+1}{(a-b)(a-c)}+\frac{b^2+b+1}{(b-c)(b-a)}+\frac{c^2+c+1}{(c-a)(c-b)}$. (A. E. 1892).

29. $\frac{(a+1)^2}{(a-b)(a-c)}+\frac{(b+1)^2}{(b-c)(b-a)}+\frac{(c+1)^2}{(c-a)(c-b)}$. (M. M. 1887).

30. $\frac{(x-b)(x-c)}{(a\text{[illegible]}-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}+\frac{(x-a)(x-b)}{(c-a)(c-b)}$. (C. E. 1901).

[illegible] $\frac{a^2-\text{[illegible]}c)^2}{(a-b)(a-c)}+\frac{b^2-(c-a)^2}{(b-c)(b-a)}+\frac{c^2-(a-b)^2}{(c-a)(c-b)}$. (M. M. 1890).

32. $\frac{\text{[illegible]}a)}{\text{[illegible]}-c)}+\frac{ca(x-b)}{(b-c)(b-a)}+\frac{ab(x-c)}{(c-a)(c-b)}$.

[illegible] $\frac{b^2+c^2-2a^2}{\text{[illegible]})}+\frac{c^2+a^2-2b^2}{(b-c)(b-a)}+\frac{a^2+b^2-2c^2}{(c-a)(c-b)}$. (M. M. 1889).

3[illegible] [illegible] $\frac{(c+a)(x^2+b^2)}{(a-b)(b-c)}+\frac{(a+\text{[illegible]})}{(b-c)(c-a)}$. (P[illegible]

35. $\frac{3a+2b+2c}{(a-b)(a-c)}+\frac{3b\text{[illegible]}}{\text{[illegible]})(b\text{[illegible]}}+\frac{\text{[illegible]}2a+2b}{\text{[illegible]})(c-b)}$. (B. M. 1901).

36. $\frac{bc(x-a)\text{[illegible]}}{(a-b)(a-c)}+\frac{\text{[illegible]}-b)^2}{(b-c)(b-a)}+\frac{\text{[illegible]}}{(c-a)(c-b)}$.

[illegible] $\frac{1}{\left(1-\frac{b}{a}\right)\left(1-\frac{c}{a}\right)}+\frac{1}{\left(1-\frac{c}{b}\right)\left(1-\frac{a}{b}\right)}+\frac{1}{\left(1-\frac{a}{c}\right)\left(1-\frac{b}{c}\right)}$. (B. M. 1897).

38. $\frac{a^2}{\left(\frac{1}{a}-\frac{1}{b}\right)\left(\frac{1}{a}-\frac{1}{c}\right)}+\frac{b^2}{\left(\frac{1}{b}-\frac{1}{c}\right)\left(\frac{1}{b}-\frac{1}{a}\right)}+\frac{c^2}{\left(\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{c}-\frac{1}{b}\right)}$. (B[illegible] 1900).

3[illegible] $a\cdot\frac{(x-b)(x-c)}{(a-b)(a-c)}+b\cdot\frac{(x-c)(x-a)}{(b-c)(b-a)}+c\cdot\frac{(x-a)(x-b)}{(c-a)(c-b)}$.

$bc\,\frac{a+d}{(a-b)(a-c)}+ca\cdot\frac{b+d}{(b-a)(b-c)}+ab\cdot\frac{c+d}{(c-a)(c-b)}$.

41. $\dfrac{a^m}{(a-b)(a-c)(x+a)}+\dfrac{b^m}{(b-a)(b-c)(x+b)}+\dfrac{c^m}{(c-a)(c-b)(x+c)}$,
when $m=1$, 0 and 2.

42. $\dfrac{a^m}{(a-b)(a-c)(x-a)}+\dfrac{b^m}{(b-a)(b-c)(x-b)}+\dfrac{c^m}{(c-a)(c-b)(x-c)}$,
when $m=1$, 0 and 2.

43. $\dfrac{p^2+p+1}{(a-b)(a-c)(x+a)}+\dfrac{q^2+q+1}{(b-a)(b-c)(x+b)}+\dfrac{r^2+r+1}{(c-a)(c-b)(x+c)}$.

44. $\dfrac{b+c-a}{(b+c)(c-a)(a-b)}+\dfrac{c+a-b}{(c+a)(a-b)(b-c)}+\dfrac{a+b-c}{(a+b)(b-c)(c-a)}$.

45. $bc.\dfrac{(a-p)(a-q)}{(a-b)(a-c)}+ca.\dfrac{(b-p)(b-q)}{(b-a)(b-c)}+ab.\dfrac{(c-p)(c-q)}{(c-b)(c-a)}$.

III. SIMPLIFICATION OF FRACTIONS.

346. The following are illustrative Examples.

Ex. 1. Simplify $\dfrac{\dfrac{a^3}{b^3}-\dfrac{b^3}{a^3}}{\left(\dfrac{a}{b}-\dfrac{b}{a}\right)\left(\dfrac{a}{b}+\dfrac{b}{a}-1\right)}\div\dfrac{\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{ab}}{\dfrac{1}{b}-\dfrac{1}{a}}$.

First fraction $=\dfrac{\dfrac{a^6-b^6}{a^3b^3}}{\dfrac{a^2-b^2}{ab}\cdot\dfrac{a^2+b^2-ab}{ab}}=\dfrac{a^6-b^6}{a^3b^3}\times\dfrac{a^2b^2}{(a^2-b^2)(a^2-ab+b^2)}$

$=\dfrac{(a^2-b^2)(a^2+ab+b^2)(a^2-ab+b^2)}{a^3b^3}\times\dfrac{a^2b^2}{(a^2-b^2)(a^2-ab+b^2)}$

$=\dfrac{a^2+ab+b^2}{ab}$.

2nd fraction $=\dfrac{\dfrac{b^2+a^2+ab}{a^2b^2}}{\dfrac{a-b}{ab}}=\dfrac{a^2+b^2+ab}{a^2b^2}\times\dfrac{ab}{a-b}=\dfrac{a^2+ab+b^2}{ab(a-b)}$.

$\therefore$ The Exp. $=\dfrac{a^2+ab+b^2}{ab}\div\dfrac{a^2+ab+b^2}{ab(a-b)}=\dfrac{a^2+ab+b^2}{ab}\times\dfrac{ab(a-b)}{a^2+ab+b^2}$

Ex. 2. Simplify $\dfrac{\dfrac{a}{a-x}+\dfrac{b}{b-x}+\dfrac{c}{c-x}}{\dfrac{3}{x}-\dfrac{1}{x-a}-\dfrac{1}{x-b}-\dfrac{1}{x-c}}$.

$$\text{Denr.}=\left(\frac{1}{x}-\frac{1}{x-a}\right)+\left(\frac{1}{x}-\frac{1}{x-b}\right)+\left(\frac{1}{x}-\frac{1}{x-c}\right)$$

$$=\frac{-a}{x(x-a)}+\frac{-b}{x(x-b)}+\frac{-c}{x(x-c)}=\frac{1}{x}\left\{\frac{a}{a-x}+\frac{b}{b-x}+\frac{c}{c-x}\right\}.$$

$$\therefore \text{ The Exp.}=\left(\frac{a}{a-x}+\frac{b}{b-x}+\frac{c}{c-x}\right)\div\frac{1}{x}\left(\frac{a}{a-x}+\frac{b}{b-x}+\frac{c}{c-x}\right)=x.$$

Exercise CXXVII.

Simplify the following :-

1. $\dfrac{a+b}{a-b+\dfrac{b^2}{a+b}}-\dfrac{a-b}{a+b+\dfrac{b^2}{a-b}}$. (M. M. 1897).

2. $\left\{\dfrac{\dfrac{x}{y}+2}{\dfrac{x}{y}+1}+\dfrac{x}{y}\right\}\div\left\{\dfrac{x}{y}+2-\dfrac{\dfrac{x}{y}}{\dfrac{x}{y}+1}\right\}$ (P. E. 1894).

3. $\left\{\dfrac{b+\dfrac{a-b}{1+ab}}{1-\dfrac{(a-b)b}{1+ab}}-\dfrac{a-\dfrac{a-b}{1-ab}}{1-\dfrac{a(a-b)}{1-ab}}\right\}\div\left(\dfrac{a}{b}-\dfrac{b}{a}\right)$. (P. E. 1898).

4. $\dfrac{\dfrac{1+x}{1-x}+\dfrac{4x}{1+x^2}+\dfrac{8x}{1-x^2}-\dfrac{1-x}{1+x}}{\dfrac{1+x^2}{1-x^2}+\dfrac{4x^2}{1+x^4}-\dfrac{1-x^2}{1+x^2}}$ (C. E. 1870).

5. $\dfrac{\dfrac{1}{x-a}+\dfrac{1}{x-b}+\dfrac{1}{x-c}}{\dfrac{a}{x-a}+\dfrac{b}{x-b}+\dfrac{c}{x-c}+3}$.

6. $\dfrac{\dfrac{x}{x-y}+\dfrac{y}{y-z}+\dfrac{z}{z-x}}{\dfrac{x+y}{x-y}+\dfrac{y+z}{y-z}+\dfrac{z+x}{z-x}+3}$.

7. $$\frac{\dfrac{a^3}{b^3}-\dfrac{b^3}{a^3}}{\left(\dfrac{a}{b}-\dfrac{b}{a}\right)\left(\dfrac{a}{b}+\dfrac{b}{a}-1\right)}\times\frac{\dfrac{1}{b}-\dfrac{1}{a}}{\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{ab}}.$$ (C. E. 1874).

8. $$\frac{\dfrac{a}{a-b}-\dfrac{a}{a+b}}{\dfrac{b}{a-b}-\dfrac{b}{a+b}}+\frac{\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}\times\frac{a^2}{a^2+b^2}.$$ (C. E. 1876).

9. $$\frac{\dfrac{a+b+c}{a+b-c}+\dfrac{a+c-b}{b+c-a}}{\dfrac{a+b-c}{a+c-b}+\dfrac{b+c-a}{a+b+c}}\div\frac{\dfrac{a+b+c}{a+b-c}}{\dfrac{b+c-a}{a+c-b}}.$$ (M. M. 1875).

10. $$\frac{\dfrac{1+\dfrac{a-bx}{c+bx}}{x-2(a+c)}}{1+\dfrac{x}{a+c-x}}-\frac{1+\dfrac{a-bx}{c+bx}}{\dfrac{x-2(a+c)}{1+\dfrac{x}{a+c-x}}}.$$ (M. M. 1871).

11. $$\frac{a^2\left(\dfrac{1}{b}-\dfrac{1}{c}\right)+b^2\left(\dfrac{1}{c}-\dfrac{1}{a}\right)+c^2\left(\dfrac{1}{a}-\dfrac{1}{b}\right)}{a\left(\dfrac{1}{b}-\dfrac{1}{c}\right)+b\left(\dfrac{1}{c}-\dfrac{1}{a}\right)+c\left(\dfrac{1}{a}-\dfrac{1}{b}\right)}.$$ (C. F. A. 1880).

12. $$\frac{1}{(4x^3-3x)^2}-\left\{\frac{\dfrac{3\sqrt{(1-x^2)}}{x}-\dfrac{(1-x^2)^{\frac{3}{2}}}{x^3}}{1-3\left(\dfrac{1-x^2}{x^2}\right)}\right\}^2.$$ (C. E. 1878).

13. $$\left\{\frac{1}{\dfrac{4}{(1+t^2)^{\frac{3}{2}}}-\dfrac{3}{(1+t^2)^{\frac{1}{2}}}}\right\}^2-\left(\frac{3t-t^3}{1-3t^2}\right)^2+(1+t^2)^2-2t^2-2.$$ (C. E. 1895).

14. $$\tfrac{1}{2}\cdot\frac{\sqrt{(x^2-1)}}{x+\sqrt{(x^2-1)}-1}\cdot\frac{1+\sqrt{\left(\dfrac{x-1}{x+1}\right)}}{1-\left(\dfrac{x-1}{x+1}\right)}+\tfrac{1}{4}\cdot\frac{\sqrt{(x+1)}-\sqrt{(x-1)}}{x-\sqrt{(x^2-1)}}\times$$

$$\frac{\sqrt{(x-1)}}{\sqrt{\left(\dfrac{x+1}{x-1}\right)}+1}.$$ (P. E. 1899).

IV. EVALUATION OF FRACTIONS.

347. The following are illustrative Examples.

Ex. 1. Find the value of $\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}$, when $x=\frac{4ab}{a+b}$.

(C. E. 1865, B. M. 1883 & P. E. 1899).

$$\text{The Exp.}=\left(\frac{x+2a}{x-2a}-1\right)+\left(\frac{x+2b}{x-2b}-1\right)+2$$

$$=\frac{4a}{x-2a}+\frac{4b}{x-2b}+2=4\left\{\frac{a(x-2b)+b(x-2a)}{(x-2a)(x-2b)}\right\}+2$$

$$=4\left\{\frac{(a+b)x-4ab}{x^2-2(a+b)x+4ab}\right\}+2=4\times 0+2=2.$$

[for $(a+b)x-4ab=0$].

Ex. 2. Evaluate $\frac{x^3-3abx-2b^3}{x^2-ab}+\frac{x^2-4ab}{x-2a}$, when $x=a+b$.

(B. M. 1890).

$$\text{The Exp.}=\frac{(a+b)^3-3ab(a+b)-2b^3}{(a+b)^2-ab}+\frac{(a+b)^2-4ab}{(a+b)-2a}$$

$$=\frac{a^3-b^3}{a^2+ab+b^2}+\frac{(a-b)^2}{b-a}=\frac{(a-b)(a^2+ab+b^2)}{a^2+ab+b^2}-\frac{(a-b)^2}{a-b}$$

$$=(a-b)-(a-b)=0.$$

Ex. 3. Find the value of $\left(\frac{2x-a}{2x-b}\right)^2-\frac{a-x}{b-x}$, when $x=\frac{ab}{a+b}$.

$$\text{Here, }\frac{2x-a}{2x-b}=\left(\frac{2ab}{a+b}-a\right)\div\left(\frac{2ab}{a+b}-b\right)=\frac{a(b-a)}{a+b}\div\frac{b(a-b)}{a+b}$$

$$=-\frac{a(a-b)}{a+b}\times\frac{a+b}{b(a-b)}=-\frac{a}{b}.$$

$$\text{And }\frac{a-x}{b-x}=\left(a-\frac{ab}{a+b}\right)\div\left(b-\frac{ab}{a+b}\right)=\frac{a^2}{a+b}\div\frac{b^2}{a+b}$$

$$=\frac{a^2}{a+b}\times\frac{a+b}{b^2}=\frac{a^2}{b^2}.$$

$$\text{Hence the Exp.}=\left(-\frac{a}{b}\right)^2-\frac{a^2}{b^2}=\frac{a^2}{b^2}-\frac{a^2}{b^2}=0.$$

Exercise CXXVIII.

Find the value of :—

1. $\frac{x-a}{b}-\frac{x-b}{a}$, when $x=\frac{a^2}{a-b}$.

2. $\frac{x+2a}{2b-x}+\frac{x-2a}{2b+x}+\frac{4ab}{x^2-4b^2}$, when $x=\frac{ab}{a+b}$.

3. $\frac{x^2-y^2+x}{y^2-x^2+y}$, when $x=\frac{a-b}{a+b}$ and $y=\frac{a+b}{a-b}$. (C. E. 1884).

4. $\left(\frac{x}{x+1}\right)^2+\left(\frac{x}{x-1}\right)^2$, when $x=\sqrt{\left(\frac{n-1}{n+1}\right)}$. (C. E. 1885).

5. $\left(\frac{x-a}{x-b}\right)^3-\frac{x-2a+b}{x+a-2b}$, when $x=\frac{a+b}{2}$.

6. $\frac{1}{a+bc}+\frac{1}{b+ca}+\frac{1}{c+ab}$.

7. $\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}+\frac{1}{(a+b)^2}$.

when $a+b+c=3$; $bc+ca+ab=4$; and $abc=5$.

8. $\frac{1}{x-a}+\frac{1}{x-b}-\frac{1}{a}$, when $x=\frac{2ab}{a+b}$.

9. $\frac{x+y-1}{x-y+1}$, when $x=\frac{a+1}{ab+1}$ and $y=\frac{a(b+1)}{ab+1}$.

10. $\frac{a^n}{2na^n-2nx}+\frac{b^n}{2nb^n-2nx}$, when $x=\frac{a^n+b^n}{2}$.

V. FRACTIONAL IDENTITIES.

348. The following are illustrative Examples.

Ex. 1. Shew that

$$1-\left(\frac{b^2+c^2-a^2}{2bc}\right)^2=\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2}. \text{ (C.E. 1868).}$$

$$\text{Left side}=\left(1+\frac{b^2+c^2-a^2}{2bc}\right)\left(1-\frac{b^2+c^2-a^2}{2bc}\right), \text{ Art. 124.}$$

$$=\frac{2bc+b^2+c^2-a^2}{2bc}\times\frac{2bc-b^2-c^2+a^2}{2bc}$$

$$=\frac{(b+c)^2-a^2}{2bc}\times\frac{a^2-(b-c)^2}{2bc}$$

$$=\frac{(b+c+a)(b+c-a)}{2bc}\times\frac{(a+b-c)(a-b+c)}{2bc}$$

$$=\frac{(a+b+c)(b+c-a)(a+b-c)(c+a-b)}{4b^2c^2}.$$

Ex. 2. Shew that

$$\frac{(a+b)^3-(b+c)^3+(c+d)^3-(d+a)^3}{(a+b)^2-(b+c)^2+(c+d)^2-(d+a)^2}=\tfrac{3}{2}(a+b+c+d). \quad \text{(M. M. 1873).}$$

Since $\{(a+b)+(c+d)\}^3=\{(b+c)+(d+a)\}^3$;

Expanding each side, we obtain

$$(a+b)^3+(c+d)^3+3(a+b)(c+d)(a+b+c+d)$$
$$=(b+c)^3+(d+a)^3+3(b+c)(d+a)(a+b+c+d).$$

By transposition,

$$(a+b)^3-(b+c)^3+(c+d)^3-(d+a)^3$$
$$=3(a+b+c+d)\{(b+c)(d+a)-(a+b)(c+d)\}.$$

Again, since $\{(a+b)+(c+d)\}^2=\{(b+c)+(d+a)\}^2$;

Expanding each side, we obtain

$$(a+b)^2+(c+d)^2+2(a+b)(c+d)=(b+c)^2+(d+a)^2+2(b+c)(d+a).$$

By transposition,

$$(a+b)^2-(b+c)^2+(c+d)^2-(d+a)^2=2\{(b+c)(d+a)-(a+b)(c+d)\}.$$

Hence, left side $=\dfrac{3(a+b+c+d)\{(b+c)(d+a)-(a+b)(c+d)\}}{2\{(b+c)(d+a)-(a+b)(c+d)\}}$

$$=\tfrac{3}{2}(a+b+c+d).$$

Ex. 3. Shew that $\dfrac{a}{a^2-1}+\dfrac{a^2}{a^4-1}+\dfrac{a^4}{a^8-1}=\dfrac{1}{2}\left(\dfrac{a+1}{a-1}-\dfrac{a^8+1}{a^8-1}\right)$.

We have $\dfrac{a}{a^2-1}=\dfrac{(a+1)-1}{a^2-1}=\dfrac{a+1}{a^2-1}-\dfrac{1}{a^2-1}=\dfrac{1}{a-1}-\dfrac{1}{a^2-1}$;

Similarly, $\dfrac{a^2}{a^4-1}=\dfrac{1}{a^2-1}-\dfrac{1}{a^4-1}$, and $\dfrac{a^4}{a^8-1}=\dfrac{1}{a^4-1}-\dfrac{1}{a^8-1}$.

Hence, adding and cancelling like terms, we have

$$\frac{a}{a^2-1}+\frac{a^2}{a^4-1}+\frac{a^4}{a^8-1}=\frac{1}{a-1}-\frac{1}{a^8-1}$$

$$=\frac{1}{2}\left(\frac{2}{a-1}-\frac{2}{a^8-1}\right)$$

$$=\frac{1}{2}\left\{\left(1+\frac{2}{a-1}\right)-\left(1+\frac{2}{a^8-1}\right)\right\}$$

$$=\frac{1}{2}\left(\frac{a+1}{a-1}-\frac{a^8+1}{a^8-1}\right).$$

Ex. 4. Shew that $\left(\frac{b}{c}+\frac{c}{b}\right)^2+\left(\frac{c}{a}+\frac{a}{c}\right)^2+\left(\frac{a}{b}+\frac{b}{a}\right)^2$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{c}\right)\left(\frac{a}{b}+\frac{b}{a}\right). \quad \text{(C. E. 1867).}$$

Left side $=\left(\frac{b}{c}+\frac{c}{b}\right)^2+\left(\frac{c^2}{a^2}+2+\frac{a^2}{c^2}\right)+\left(\frac{a^2}{b^2}+2+\frac{b^2}{a^2}\right)$

$$=\left(\frac{b}{c}+\frac{c}{b}\right)^2+4+\left(\frac{c^2}{a^2}+\frac{b^2}{a^2}\right)+\left(\frac{a^2}{c^2}+\frac{a^2}{b^2}\right)$$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)^2+\frac{1}{a^2}(c^2+b^2)+a^2\left(\frac{1}{c^2}+\frac{1}{b^2}\right)$$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)^2+\frac{bc}{a^2}\left(\frac{c^2+b^2}{bc}\right)+\frac{a^2}{bc}\left(\frac{b^2+c^2}{bc}\right)$$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)^2+\frac{bc}{a^2}\left(\frac{b}{c}+\frac{c}{b}\right)+\frac{a^2}{bc}\left(\frac{b}{c}+\frac{c}{b}\right)$$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{b}{c}+\frac{c}{b}+\frac{bc}{a^2}+\frac{a^2}{bc}\right)$$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left\{\left(\frac{b}{c}+\frac{a^2}{bc}\right)+\left(\frac{c}{b}+\frac{bc}{a^2}\right)\right\}$$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left\{\frac{a}{c}\left(\frac{b}{a}+\frac{a}{b}\right)+\frac{c}{a}\left(\frac{a}{b}+\frac{b}{a}\right)\right\}$$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{c}\right)\left(\frac{a}{b}+\frac{b}{a}\right).$$

Ex. 5. Prove that

$$\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}+\frac{1}{(a-b)^2}=\left(\frac{1}{b-c}+\frac{1}{c-a}+\frac{1}{a-b}\right)^2.$$

(B. P. E. 1886).

Let $\left.\begin{array}{l} b-c=x, \\ c-a=y \\ \text{and } a-b=z \end{array}\right\}$ then $x+y+z=b-c+c-a+a-b=0.$

Since $(yz+zx+xy)^2=y^2z^2+z^2x^2+x^2y^2+2xyz(x+y+z),$

$$=y^2z^2+z^2x^2+x^2y^2, \text{ for } x+y+z=0.$$

$$\therefore \left(\frac{yz+zx+xy}{xyz}\right)^2=\frac{y^2z^2+z^2x^2+x^2y^2}{x^2y^2z^2}, \text{ (dividing by } x^2y^2z^2)$$

$$\therefore \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2=\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}.$$

Now, restoring values, we obtain

$$\left(\frac{1}{b-c}+\frac{1}{c-a}+\frac{1}{a-b}\right)^2=\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}+\frac{1}{(a-b)^2}.$$

Ex. 6. If $x=\dfrac{b^2+c^2-a^2}{2bc}$, $y=\dfrac{c^2+a^2-b^2}{2ca}$, $z=\dfrac{a^2+b^2-c^2}{2ab}$,

shew that $(b+c)x+(c+a)y+(a+b)z=a+b+c.$

Let $a^2+b^2+c^2=2s^2$, then $a^2+b^2-c^2=(a^2+b^2+c^2)-2c^2=2(s^2-c^2).$

Similarly, $b^2+c^2-a^2=2(s^2-a^2)$ and $c^2+a^2-b^2=2(s^2-b^2).$

Hence, the expression

$$(b+c)\times\frac{2(s^2-a^2)}{2bc}+(c+a)\times\frac{2(s^2-b^2)}{2ca}+(a+b)\times\frac{2(s^2-c^2)}{2ab}$$

$$=\left(\frac{1}{b}+\frac{1}{c}\right)(s^2-a^2)+\left(\frac{1}{c}+\frac{1}{a}\right)(s^2-b^2)+\left(\frac{1}{a}+\frac{1}{b}\right)(s^2-c^2)$$

$$=\frac{1}{a}(2s^2-b^2-c^2)+\frac{1}{b}(2s^2-c^2-a^2)+\frac{1}{c}(2s^2-a^2-b^2)$$

(collecting the coefficients of $1/a$, $1/b$ and $1/c$)

$$=\frac{1}{a}\times a^2+\frac{1}{b}\times b^2+\frac{1}{c}\times c^2=a+b+c.$$

Ex. 7. If $a+b+c=0$, shew that

$$\frac{a^3}{2a^2+bc}+\frac{b^3}{2b^2+ca}+\frac{c^3}{2c^2+ab}=0.$$

Since $a+b+c=0$, $\therefore a=-(b+c)$

$\therefore a\times a$ or $a^2=-a(b+c)$.

$\therefore 2a^2+bc=a^2-a(b+c)+bc=(a-b)(a-c)$.

Similarly, $2b^2+ca=(b-a)(b-c)$ and $2c^2+ab=(c-a)(c-b)$.

Hence, the Exp. $=\dfrac{a^3}{(a-b)(a-c)}+\dfrac{b^3}{(b-a)(b-c)}+\dfrac{c^3}{(c-a)(c-b)}$

$=a+b+c$ (Art. 344) $=0$.

Ex. 8. If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$, shew that

$$\frac{1}{a^{2n+1}}+\frac{1}{b^{2n+1}}+\frac{1}{c^{2n+1}}=\frac{1}{a^{2n+1}+b^{2n+1}+c^{2n+1}}.$$

Since $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ or $\frac{bc+ca+ab}{abc}=\frac{1}{a+b+c}$,

$\therefore (bc+ca+ab)(a+b+c)-abc=0$.

$\therefore (b+c)(c+a)(a+b)=0$. (Art. 328).

$\therefore$ any one of these factors, say $b+c=0$.

$\therefore \frac{b+c}{bc}$ or $\frac{1}{b}+\frac{1}{c}=0$; $\therefore \frac{1}{b}=-\frac{1}{c}$

$\therefore \left(\frac{1}{b}\right)^{2n+1}$ or $\frac{1}{b^{2n+1}}=\left(-\frac{1}{c}\right)^{2n+1}$ or $-\frac{1}{c^{2n+1}}$,

since $2n+1$ is always an odd number.

$$\therefore \frac{1}{b^{2n+1}}+\frac{1}{c^{2n+1}}=0.$$

Similarly, $b^{2n+1}=(-c)^{2n+1}=-c^{2n+1}$; $\therefore b^{2n+1}+c^{2n+1}=0$.

Hence, left side $=\dfrac{1}{a^{2n+1}}=\dfrac{1}{a^{2n+1}+b^{2n+1}+c^{2n+1}}$.

Exercise CXXIX.

Prove the following Identities :—

1. $\dfrac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3}=(a+b)(b+c)(c+a).$

2. $\left(1+\dfrac{x}{y}\right)\left(1+\dfrac{y}{z}\right)\left(1+\dfrac{z}{x}\right)+1=(x+y+z)\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right)$

3. $\left(\dfrac{1}{a}-\dfrac{1}{c}\right)^2+\dfrac{4}{(a+c)^2}=\left(\dfrac{a+c}{ac}-\dfrac{2}{a+c}\right)^2.$ (M. M. 1865).

4 $\dfrac{b-c}{1+bc}+\dfrac{c-a}{1+ca}+\dfrac{a-b}{1+ab}=\dfrac{(b-c)(c-a)(a-b)}{(1+bc)(1+ca)(1+ab)}.$ (C. E. 1897).

5. $\dfrac{a-b}{m+ab}+\dfrac{b-c}{m+bc}+\dfrac{c-a}{m+ca}=m\dfrac{(a-b)(b-c)(c-a)}{(m+ab)(m+bc)(m+ca)}.$ (M. M. 1884).

If $a+b+c=0$, prove that

6. $\dfrac{a^2+b^2+c^2}{a^3+b^3+c^3}+\dfrac{2}{3}\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)=0.$ (M. M. 1877).

7. $\left(\dfrac{b-c}{a}+\dfrac{c-a}{b}+\dfrac{a-b}{c}\right)\left(\dfrac{a}{b-c}+\dfrac{b}{c-a}+\dfrac{c}{a-b}\right)=9.$

8. If $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=0$, prove that $\dfrac{1}{2}\left(\dfrac{y^2z^2}{x^2}+\dfrac{z^2x^2}{y^2}+\dfrac{x^2y^2}{z^2}\right)=(x+y+z)^2.$
(M. M. 1897).

9. Shew that $\dfrac{(a+3x+2b)(a+x)^3-(2a+3x+b)(b+x)^3}{(a+2x+b)^3}=a-b.$
(M. M. 1899).

10. Shew that $\dfrac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3}=\dfrac{(a+b)(b+c)(c+a)}{abc}.$
(B. M. 1895).

11. If $y=\dfrac{1+x}{1-x}$, prove that $\left(x-\dfrac{1}{x}\right)\left(y-\dfrac{1}{y}\right)=4\dfrac{xy+1}{x-y}$. (M. M. 1888).

12. Prove that $\dfrac{(a+2b-3c)^2}{(b+2c-3a)(c+2a-3b)}+\dfrac{(b+2c-3a)^2}{(c+2a-3b)(a+2b-3c)}$
$+\dfrac{(c+2a-3b)^2}{(a+2b-3c)(b+2c-3a)}=3.$ (M. M. 1896).

13. Prove that $\dfrac{(a-5b)(3a+b)^3+(5a-b)(a+3b)^3}{a+b}=32(a-b)^3$.

(M. M. 1898).

If $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=\dfrac{1}{a+b+c}$, prove that

14. $\dfrac{1}{a^3}+\dfrac{1}{b^3}+\dfrac{1}{c^3}=\dfrac{1}{a^3+b^3+c^3}=\dfrac{1}{(a+b+c)^3}$.

15. $\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)^7=\dfrac{1}{a^7+b^7+c^7}=\dfrac{1}{(a+b+c)^7}$.

16. $\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)^{2n+1}=\dfrac{1}{(a+b+c)^{2n+1}}$.

17. Prove that $\left(x+\dfrac{1}{x}\right)^2+\left(y+\dfrac{1}{y}\right)^2+\left(z+\dfrac{1}{z}\right)^2$

$$=4+\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)\left(z+\frac{1}{z}\right), \text{ if } xyz=1.$$

18. If $(b+c-a)x=(c+a-b)y=(a+b-c)z=2$,

prove that $\left(\dfrac{1}{y}+\dfrac{1}{z}\right)\left(\dfrac{1}{z}+\dfrac{1}{x}\right)\left(\dfrac{1}{x}+\dfrac{1}{y}\right)=abc$

19. If $\dfrac{a+b}{1-ab}=\dfrac{c+d}{cd-1}$, prove that

$$a+b+c+d=abcd\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

20. If $\dfrac{a-b}{1+ab}+\dfrac{c-d}{1+cd}=0$, shew that

$$\frac{a-d}{1+ad}=\frac{b-c}{1+bc} \text{ and } \frac{a+c}{1-ac}=\frac{b+d}{1-bd}.$$

21. If $\dfrac{a}{b}+\dfrac{c}{d}=\dfrac{b}{a}+\dfrac{d}{c}$, prove that $\dfrac{a^3}{b^3}+\dfrac{c^3}{d^3}=\dfrac{b^3}{a^3}+\dfrac{d^3}{c^3}$. (M. M. 1866).

If $2s=a+b+c$, shew that

22. $c^2-\left(\dfrac{a^2+c^2-b^2}{2a}\right)^2=\dfrac{4}{a^2}s(s-a)(s-b)(s-c)$.

23. $\dfrac{1}{s-a}+\dfrac{1}{s-b}+\dfrac{1}{s-c}-\dfrac{1}{s}=\dfrac{abc}{s(s-a)(s-b)(s-c)}$.

24. $\frac{s-a}{(s-b)(s-c)}+\frac{s-b}{(s-c)(s-a)}+\frac{s-c}{(s-a)(s-b)}=\frac{a^2+b^2+c^2-s^2}{(s-a)(s-b)(s-c)}$.

25. If $a+b+c=0$, prove that

$$\frac{a^2}{2a^2+bc}+\frac{b^2}{2b^2+ca}+\frac{c^2}{2c^2+ab}=1.$$

26. Prove that

$$\frac{bc}{(x-b)(x-c)}+\frac{ca}{(x-c)(x-a)}+\frac{ab}{(x-a)(x-b)}=0.\ \text{(M. M. 1873).}$$

$$\text{if } \frac{1}{x}=\frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

27. If $x=\frac{a+1}{a-1}$, $y=\frac{b+1}{b-1}$, $z=\frac{c+1}{c-1}$, prove that

$$\frac{(x^2+1)(y^2+1)(z^2+1)}{(yz+1)(zx+1)(xy+1)}=\frac{(a^2+1)(b^2+1)(c^2+1)}{(bc+1)(ca+1)(ab+1)}.\ \text{(M. M. 1874).}$$

28. If $\frac{y}{z}+\frac{z}{y}=a$, $\frac{z}{x}+\frac{x}{z}=b$, $\frac{x}{y}+\frac{y}{x}=c$,

prove that $a^2+b^2+c^2=4+abc$.

29. If $x+y+z=xyz$, prove that

$$\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}.\ \text{(C. E. 1898).}$$

30. If $x+y=2z$, show that $\frac{x}{x-z}+\frac{y}{y-z}=2$. (B. M. 1882).

31. Given the relation $\frac{1-2bx+b^2}{1-b^2}-\frac{1-b^2}{1+2by+b^2}$,

prove that $\frac{x-y}{1-xy}=\frac{2b}{1+b^2}$. (B. M. 1

32. If $b^2=ac$, $x=\frac{1}{2}(a+b)$ and $y=\frac{1}{2}(b+c)$, prove that $\frac{a}{x}+\frac{c}{y}=2$.

(B. M. 1893).

33. If $x+\frac{1}{y}=1$ and $y+\frac{1}{z}=1$, prove that $z+\frac{1}{x}=1$ and $xyz+1=0$.

(B. M. 1887).

34. If $\frac{a^2(b-c)}{a-d}=\frac{b^2(a-c)}{b-d}$, then will $\frac{1}{a}+\frac{1}{b}=\frac{1}{c}+\frac{1}{d}$ or $a=b$.

35. If $x+y+z=xyz$, prove that

$$\frac{x+y}{1-xy}+\frac{y+z}{1-yz}+\frac{z+x}{1-zx}=\frac{x+y}{1-xy}\cdot\frac{y+z}{1-yz}\cdot\frac{z+x}{1-zx}.$$

36. If $a+b=c+d$, prove that either of them is equal to

$$\frac{abcd}{ab+cd}\left\{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right\}. \quad \text{(B. M. 1887).}$$

37. If $\dfrac{ad-bc}{a-b-c+d}=\dfrac{ac-bd}{a-b+c-d}$, then will $a+b=c+d$; and each of the former quantities $=\frac{1}{4}(a+b+c+a)$.

38. If $bc+ca+ab=1$, prove that

$$\left\{1-\frac{a^2}{1+a^2}-\frac{b^2}{1+b^2}-\frac{c^2}{1+c^2}\right\}^2=\frac{4a^2b^2c^2}{(1+a^2)(1+b^2)(1+c^2)}. \quad \text{(M. M. 1871).}$$

39. If $y+z=ax$, $z+x=by$ and $x+y=cz$, prove that

$$a+b+c=abc-2.$$

40. Shew that

$$\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\left(\frac{y}{x}+\frac{x}{z}+\frac{z}{y}\right)=1+\left(\frac{x}{y}+\frac{y}{z}\right)\left(\frac{y}{z}+\frac{z}{x}\right)\left(\frac{z}{x}+\frac{x}{y}\right). \quad \text{(B. M. 1887).}$$

41. If $xy=ab(a+b)$ and $x^2-xy+y^2=a^3+b^3$,

prove that $\left(\dfrac{x}{a}-\dfrac{y}{b}\right)\left(\dfrac{x}{b}-\dfrac{y}{a}\right)=0$. (B. M. 1876).

42. If $x=a+b+\dfrac{(a-b)^2}{4(a+b)}$, and $y=\dfrac{a+b}{4}+\dfrac{ab}{a+b}$,

prove that $(x-a)^2-(y-b)^2=b^2$.

43. If $a+b+c=0$, prove that

$$\frac{1}{b^2+c^2-a^2}+\frac{1}{c^2+a^2-b^2}+\frac{1}{a^2+b^2-c^2}=0.$$

44. If $\dfrac{a-1}{x}-\dfrac{a-2}{y}=\dfrac{1}{b}$ and $\dfrac{b-1}{x}-\dfrac{b-2}{y}=\dfrac{1}{a}$,

shew that $\dfrac{c-1}{x}-\dfrac{c-2}{y}=\dfrac{c}{ab}$. (M. M. 1869).

45. If $x=\dfrac{b^2+c^2-a^2}{2bc}$, $y=\dfrac{c^2+a^2-b^2}{2ca}$, $z=\dfrac{a^2+b^2-c^2}{2ab}$,

shew that $a(x+yz)=b(y+zx)=c(z+xy)$. (M. M. 1871).

46. Prove that

$$\frac{a(x-b)(x-c)}{bc(a-b)(a-c)}+\frac{b(x-c)(x-a)}{ca(b-c)(b-a)}+\frac{c(x-a)(x-b)}{ab(c-a)(c-b)}=\frac{x^2}{abc}.\quad \text{(M. M. 1895).}$$

47. If $x+y+z=0$, shew that

$$\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}=\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3+\frac{3}{xyz}.\quad \text{(M. M. 1899).}$$

48. If $\frac{x-y}{x+y}=a$, $\frac{y-z}{y+z}=b$, $\frac{z-x}{z+x}=c$, shew that

$$(1-a)(1-b)(1-c)=(1+a)(1+b)(1+c).\quad \text{(A. E. 1901).}$$

49. Prove that $\frac{2}{b-c}+\frac{2}{c-a}+\frac{2}{a-b}+\frac{(b-c)^2+(c-a)^2+(a-b)^2}{(b-c)(c-a)(a-b)}=0.$

(P. E. 1888).

50. If $\frac{b^2+c^2-a^2}{2bc}+\frac{c^2+a^2-b^2}{2ca}+\frac{a^2+b^2-c^2}{2ab}=1$, shew that $(b+c-a)(c+a-b)(a+b-c)=0$, and thence prove that two of the three fractions on the left side $=1$ and the other $=-1$.

REVISION PAPERS III.

Paper I.

1. Shew that $(ay-bx)^2+(bz-cy)^2+(cx-az)^2+(ax+by+cz)^2$ is divisible by $a^2+b^2+c^2$ and $x^2+y^2+z^2$. (A. E. 1897).

2. Find the H. C. F. of $4x^4-9x^2+6x-1$ and $6x^3-7x^2+1$.

(A. E. 1897).

3. Simplify $\left(\frac{ax}{x^2-y^2}-\frac{b}{y-x}-\frac{a}{x+y}\right)\div\left(\frac{ax}{a^2-b^2}-\frac{y}{b-a}-\frac{x}{a+b}\right)$.

(A. E. 1894).

4. Extract the square root of

$$x^6+\frac{1}{x^6}+6\left(x^4+\frac{1}{x^4}\right)+15\left(x^2+\frac{1}{x^2}\right)+20.\quad \text{(M. M. 1899).}$$

5. Reduce to its lowest terms $\frac{3x^3-23x^2+43x-8}{x^4-5x^3-6x^2+35x-7}$.

(M. M. 1899).

6. Solve the equations :—

(i) $3x-4y-6z+16=0=4x-y-z-5=x-3y-4z+12$. (M.M. 1899).

(ii) $x+2y+3z=\frac{19}{6}$, $2x+3y+z=2$, $3x-4y-7z=\frac{1}{6}$. (M. M. 1899).

7. Simplify $\dfrac{3x-12}{x^2-5x+6}+\dfrac{5x+3}{x^2-2x-3}-\dfrac{x+15}{x^2-5x-6}$. (M. M. 1891).

8. If 3 be added to the numerator and denominator of a certain fraction, the fraction becomes $\frac{2}{3}$; if 5 be subtracted from the numerator and denominator, it becomes $\frac{1}{2}$. Find the fraction. (M. M. 1894).

Paper II.

1. Find the H. C. F. of x^6+11x^3-54 and $2x^6-11x^3-9$.
(M. M. 1894).

2. Simplify $\dfrac{1}{x^2+3x+2}+\dfrac{x-1}{2x^2+5x+2}-\dfrac{x}{2x^2+3x+1}$. (M. M. 1896).

3. Divide $(x+y)^3-8z^3$ by $x+y-2z$. (A. E. 1894).

4. Find the square root of

$$\frac{4a^2-12ab-6bc+4ac+9b^2+c^2}{4a^2+9c^2-12ac}. \quad \text{(C. E. 1892).}$$

5. Simplify the following fractional expression :-

$$\frac{\sqrt{(a^4-4a^3+12a^2-16a+16)}}{\sqrt[3]{(a^9+24a^6+192a^3+512)}}. \quad \text{(B. M. 1885).}$$

6. Solve the following equations :—

(i) $$\left.\begin{aligned} 2x-\frac{y+3}{4}&=7+\frac{3y-2x}{5} \\ 4y+\frac{x-2}{3}&=26\tfrac{1}{2}-\frac{2y+1}{2} \end{aligned}\right\}. \quad \text{(M. M. 1892).}$$

(ii) $$\left.\begin{aligned} &\frac{1}{x}+\frac{3}{y}-\frac{2}{z}=6, \\ &\frac{3}{x}+\frac{1}{z}=5,\quad \frac{2}{y}+\frac{5}{z}=16 \end{aligned}\right\}. \quad \text{(M. M. 1896).}$$

7. Simplify the expressions :—

(i) $\left\{\dfrac{(x^m)^{\frac{1}{r}}(x^n)^{\frac{1}{p}}}{\sqrt[n]{x^p}\,\sqrt[m]{x^r}}\right\}^{pr}$. (P. E. 1888). (ii) $\dfrac{2^{2n+1}-2^{n+2}+2}{2^{2n+1}-2^{n+1}}$. (M. M. 1870).

8. One person starts from a place **A** to walk to a place **B** and back again at the same time as another person starts from **B** to walk to **A** and back again. They meet first at a distance of 2 miles from **A** and afterwards at a distance of 4 miles from **A**. Find the distance between **A** and **B**. (M. M. 1896).

Paper III.

1. Subtract $\frac{x+5}{x^2+5x-6}$ from $\frac{x+6}{x^2+3x-10}$, and divide the difference by $1+\frac{2(x^2+4x-8)}{x^2+11x+30}$. (B. M. 1902).

2. Simplify $\frac{x^3+2x^2-29x-30}{x^3-3x^2-34x+120}$. (B. M. 1901).

3. Find the H. C. F. of $x^5+11x-12$ and x^5+11x^3+54. (P. E. 1895)

4. If $(a+b)(b+c)(c+d)(d+a)=(a+b+c+d)(bcd+cda+dab+abc)$ prove that $ac=bd$. (B. M. 1884).

5. Find in terms of a the value of the expression

$$x(y+2)+\frac{x}{y}+\frac{y}{x}, \text{ when } x=\frac{y}{y+1} \text{ and } y=\frac{a-2}{2}. \quad (\text{B.M. } 1889)$$

6 Simplify $\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\left(\frac{x}{z}+\frac{y}{x}+\frac{z}{y}\right)-\left(\frac{x}{y}+\frac{y}{z}\right)\left(\frac{y}{z}+\frac{z}{x}\right)\left(\frac{z}{x}+\frac{x}{y}\right)$. (B. M. 1887)

7. Solve the following equations .

(1) $x+2y+3z=20,\ 2x+3y-5z=-7,\ 4x-5y+7z=21$. (C.E. 1898).

(2) $2x+3y+4z=38,\ 3x-2y+5z=26,\ 4x+6y-3z=21$. (C.E. 1901)

8. A man walks one-third of the distance from **A** to **B** at the rate of a miles an hour, and the remainder at the rate of $2b$ miles an hour, and travelling back from **B** to **A** at the rate of $3c$ miles per hour, takes the same time ; prove that $1/a+1/b=1/c$. (B. M. 1885).

Paper IV.

1. Simplify by using factors :—

(i) $\frac{x^2-7xy+12y^2}{x^2+5xy+6y^2}\div\frac{x^2-5xy+4y^2}{x^2+xy-2y^2}$. (B. M. 1891).

(ii) $\dfrac{a^6 - a^4b - ab^4 + b^5}{a^4 - a^3b - a^2b^2 + ab^3}$.

2. Divide $a+b+c+3(b^{\frac{1}{3}}+c^{\frac{1}{3}})(c^{\frac{1}{3}}+a^{\frac{1}{3}})(a^{\frac{1}{3}}+b^{\frac{1}{3}})$ by $a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}}$. (P. E. 1895).

3. Prove that $\dfrac{x+y}{\sqrt{(x+y)}-y} = \dfrac{\sqrt{(x^3+x^2y-xy^2-y^3)}}{\sqrt{(x^2-y^2)}-y\sqrt{(x-y)}}$. (B. M. 1867).

4. Simplify $\left\{\dfrac{(x-a)(y-b)\sqrt{(xy)}.x^7}{\sqrt[3]{(by)}\ \sqrt[3]{(a+b)(x-y)^7}}\right\}^0 \times \dfrac{(x+y)(x-y)(x^2+y^2)}{x^4-y^4}$
$\times \sqrt{(a^2+2ab+b^2)}$. (M. M. 1860).

5. Solve the equations :—

(i) $69x - \dfrac{49}{y} = 182\frac{1}{2}$, $49x - \dfrac{69}{y} = 112\frac{1}{2}$. (P. E. 1900).

(ii) $2y+z=11$, $2z+x=12$, $2x+y=13$. (P. E. 1900).

6. Find the square root of $x+\dfrac{1}{x}+\sqrt{2}\left(\sqrt{x}+\dfrac{1}{\sqrt{x}}\right)+\frac{3}{2}$. (P.E. 1900).

7. Find the H. C. F. of $3x^3-23x^2+43x-8$ and $x^4-5x^3-6x^2+35x-7$. (P. E. 1894).

8. There is a certain number whose three digits are in descending order of magnitude and differ from each other in succession by the same amount. If the number be divided by the sum of the digits, the quotient will be 48; and if from the number 198 be subtracted, the digits of the difference will be the same as those of the original number but in the reverse order; find the number. (B. M. 1864).

Paper V.

1. Divide $x^{12}-x^{-12}+6(x^8-x^{-8})+9(x^4-x^{-4})$
by $x^6-x^{-6}+3(x^2-x^{-2})$. (P. E. 1899).

2. Express $X^3+Y^3+Z^3-3XYZ$ in terms of a, b, c, being given $X=b+c-a$, $Y=c+a-b$, $Z=a+b-c$. (P.E. 1898).

3. Find the H. C. F. of $x^3-3a^2x-2a^3$ and $x^3-ax^2-4a^3$, and the L. C. M. of $2x^3-7x-2$ and $2x^2-x-6$. (P. E. 1898).

4. If $x=a^2-bc$, $y=b^2-ca$, $z=c^2-ab$, prove that
$$bx+cy+az=0=cx+ay+bz.$$ (P. E. 1900).

5. Simplify the expression :—

$$1+\frac{a}{x-a}+\frac{bx}{(x-a)(x-b)}+\frac{cx^2}{(x-a)(x-b)(x-c)}. \quad \text{(P. E. 1897).}$$

6. Extract the square root of

$(bc+ca+ab+a^2)(bc+ca+ab+b^2)(bc+ca+ab+c^2)$. (P.E. 1897).

7. Solve the equations :—

(i) $b(a+b)x=a(a-b)y,\ \frac{a-bx}{a^2}-\frac{b-ay}{b^2}=\frac{x}{a}+\frac{y}{b}$. (B. M. 1895).

(ii) $cy+bz=bc,\ az+cx=ca,\ bx+ay=ab$. (P. E. 1897).

8. The sum of the three digits of which a number consists is 9 ; the first digit is one-eighth of the number consisting of the last two, and the last is likewise one-eighth of the number consisting of the first two ; find the number. (B. M. 1874).

Paper VI.

1 If $x=a+b-2c,\ y=b+c-2a,\ z=c+a-2b$, find the value of $x^3+y^3+z^3-3xyz$. (C. E. 1900).

2. Find the H. C. F. of $2x^4+13x^3-4x^2+6x+1$ and

$x^4+7x^3-2x^2-21x-3$. (M. M. 1899).

3. Simplify

(i) $\frac{1}{x^4+2x^3}+\frac{1}{x^4-2x^3}+\frac{2}{x^4+4x^2}$. (M. M. 1899).

(ii) $\left\{\frac{x-a}{(x+a)^2}+\frac{x+a}{(x-a)^2}\right\}\div\left\{\frac{1}{(a+x)^2}+\frac{1}{a^2-x^2}+\frac{1}{(a-x)^2}\right\}$.

(M. M. 1899)

4. Two ships 56 miles apart sail towards one another at the rates of 7 and 9 miles an hour. Find, graphically, when they meet.

5. Extract the square root of

(i) $1+(x+1)(x+2)(x+3)(x+4)$. (A. E. 1900).

(ii) $(a-b)^4-2(a^2+b^2)(a-b)^2+2(a^4+b^4)$. (A. E. 1901).

6. Solve the equations :—

(i) $a(x+y)+b(x-y)=2a,\ y(a+b)-x(a-b)=2b$. (A. E. 1902).

(ii) $(a+b)x+(a-b)y=2ac,\ (b+c)x+(b-c)y=2bc$. (A.E. 1895).

7. The expression $ax-by$ is equal to 10 when $x=2$ and $y=3$ and it is equal to 25 when $x=3$ and $y=2$, a and b being constants; find a and b. (A. E. 1900).

8. I wished to give a certain number of old men 1*a*. 8*p*. each, and I found that I had not money enough in my purse by 11 annas; so I gave them 1*a*. 5*p*. each, and then I had money enough and 3 annas 3 pies to spare. Find the number of old men. (A.E 1902).

Paper VII.

1. Reduce to their simplest forms :—

(i) $\left\{1-\dfrac{4}{x-1}+\dfrac{12}{x-3}\right\}\left\{1+\dfrac{4}{x+1}-\dfrac{12}{x+3}\right\}$. (M. M. 1895).

(ii) $\left(\dfrac{ap^2-aq^2+2bpq}{p^2+q^2}\right)^2+\left(\dfrac{bq^2-bp^2+2apq}{p^2+q^2}\right)^2$. (M. M. 1883).

2. If $x=\dfrac{a-b}{m-c}$, $y=\dfrac{b-c}{m-a}$, $z=\dfrac{c-a}{m-b}$, find the value of

$$x+y+z+xyz. \quad \text{(M. M. 1875).}$$

3. Simplify $\dfrac{\sqrt{(ax)}}{\sqrt{a}+\sqrt{x}-\sqrt{(a+x)}}-\dfrac{\sqrt{(ax)}}{\sqrt{a}+\sqrt{x}+\sqrt{(a+x)}}$.

(M. M. 1875).

4. Shew that, if $a+b+c=1$, $bc+ca+ab=\frac{1}{3}$, $abc=\frac{1}{27}$,

then $\dfrac{1}{a+bc}+\dfrac{1}{b+ca}+\dfrac{1}{c+ab}=\dfrac{27}{4}$. (M. M. 1878).

5. Plot the following points and find the equation of the graph which passes through them $(0, 1\frac{2}{3})$, $(1, 2)$, $(2, 2\frac{1}{3})$, $(3, 2\frac{2}{3})$, $(4, 3)$.

6. Solve the following equations :—

(i) $\dfrac{2x+3y}{5a+b}=\dfrac{ab}{a^2-b^2}=\dfrac{ax+by}{a^2+b^2}$. (B. M. 1901).

(ii) $\dfrac{(a-b)x+(a+b)y}{a^2-b^2}=\dfrac{ab}{a-b}=\dfrac{ab(x-y)-(a^2y-b^2x)}{2ab^2}$. (B.M. 1902).

7. Find a homogeneous and symmetrical expression of the second degree in x and y which shall be equal to 3, when x and y are each equal to unity, and shall be equal to 11, when $x=2$, $y=1$. (P. E. 1900).

8. **A** walks half a mile per hour faster than **B**, and three quarters of a mile per hour faster than **C**. To walk a certain distance **C** takes three-quarters of an hour more than **B**, and two hours more than **A**. Find the rates of walking of **A**, **B** and **C**.

Paper VIII.

1. Divide $x(1+y^2)(1+z^2)+y(1+z^2)(1+x^2)+z(1+x^2)(1+y^2)$ $+4xyz$ by $1+xy+yz+zx$. (C. E. 1878).

2. Simplify $\left(\frac{x^2-x+1}{12}\right)^2-27\left\{\frac{(x+1)(x-2)(2x-1)}{432}\right\}^2$, and extract the square root of the result. (P. E. 1891).

3. Find the H. C. F. of x^8-1 and $x^{10}-1$. (A. E. 1896).

4. Simplify

$$a^4+x^4+ax(a^2+x^2)+a^2x^2 \quad \frac{a^2+x^2+ax}{a^3-x^3}.$$ (A. E. 1897).

5. Find the G. C. M. of $x^3+6x^2+11x+6$ and $x^4+x^3-4x^2-4x$ and the L. C. M. of $x^3-x^2-14x+24$, x^3-2x^2-5x+6 and x^2-4x+3 (C. E. 1901 and 1902).

6. By performing the operation of extracting the square root, find a value of x which will make $x^4+6x^3+11x^2+3x+31$ a perfect square.

7. Solve the equations :—

(i) $(a+b)x+by=ax+(a+b)y=a^3-b^3$. (B. M. 1896).

(ii) $\frac{a_1}{x}+\frac{b_1}{y}=c_1$, $\frac{a_2}{x}+\frac{b_2}{y}=c_2$. (A. E. 1896)

(iii) $2x+\frac{3}{y}=4$, $3x+\frac{2}{y}=5$. (A. E. 1898).

8. The gross income of a certain person was *Rs.*4 more in the second of two particular years than in the first, but as he paid income-tax at the rate of 4*p.* in the rupee in the first year and at the rate of 5*p.* in the rupee in the second year, his net income in the second year was *Rs.*6½ less than his net income in the first What was his gross income in each year? (M. M. 1895.)

Paper IX.

1. Find the G. C. M. of $7x^4-2x^2-9x-2$ and $5x^3-6x^2-6x-11$. (M. M. 1892).

2. Divide $(b-c)(x-a)^3+(c-a)(x-b)^3+(a-b)(x-c)^3$ by $(b-c)(c-a)(a-b)$. (M. M. 1897).

3. Simplify

(i) $$\frac{\sqrt[3]{(64x^6-48x^4+12x^2-1)}-\sqrt{(16x^4-64x^3+24x^2+80x+25)}}{4x^2-12x-7}.$$ (B. M. 1900).

(ii) $$\frac{(x-a)^2}{(a-b)(a-c)}+\frac{(x-b)^2}{(b-a)(b-c)}+\frac{(x-c)^2}{(c-a)(c-b)}.$$ (A. E. 1900).

4. Add together the squares of $2[\sqrt{\{ab(1+a)(1+b)\}}+\sqrt{\{ab(1-a)(1-b)\}}]$ and $\{a+\sqrt{(1-a^2)}\}\{b-\sqrt{(1-b^2)}\}-\{a-\sqrt{(1-a^2)}\}\{b+\sqrt{(1-b^2)}\}$, and simplify the result. (M. M. 1875).

5. Solve the equations :

(i) $(1+p)(x-py)=2p^2\left(\dfrac{x}{1+p}+\dfrac{y}{1-p}\right)=\dfrac{2p^2}{1-p}$. (B. M. 1900)

(ii) $x+\dfrac{ay}{a-b}=b=\dfrac{ax}{a+b}+y$. (M. M. 1898).

(iii) $\dfrac{a}{x}+\dfrac{b}{y}=c$, $\dfrac{b}{x}+\dfrac{a}{y}=d$. (P. E. 1894).

6. Two passengers have together 7 maunds of luggage, and for the excess above the weight allowed free one of them is charged *Rs.*3 and the other *Rs.*5. If all the luggage had belonged to one passenger he would have been charged *Rs.*11. What amount of luggage is each passenger allowed free of charge? (B. M. 1900).

7. A straight wire joins the top ends of two vertical posts, 17 ft. and 24 ft. high respectively, 35 feet apart. By means of squared paper, without actual measurement, find the length of the wire to the nearest foot.

8. Plot the points (10, 5), (−5, 15), (10, 22) and find the area of the triangle formed by joining them.

Paper X.

1. Divide $(1-a^2)(1-b^2)(1-c^2)-(a+bc)(b+ca)(c+ab)$ by $1-a^2-b^2-c^2-2abc$. (A. E. 1900).

2. Find the first four terms of the square root of a^2+x^2, and from the result deduce the square root of 101 correct to six places of decimals. (C. E. 1877).

3. Express $\frac{(a-b)(b-c)}{(c-d)(d-a)}-\frac{(b-c)(c-d)}{(d-a)(a-b)}+\frac{(c-d)(d-a)}{(a-b)(b-c)}$ $\frac{(d-a)(a-b)}{(b-c)(c-d)}$ as a fraction whose numerator and denominator consist of four factors each. (M. M. 1894).

4. Choosing a suitable unit, draw accurately the graph of $3y=2x+7$.

5. Plot the points (0, 0), (8, 5), (12, 18), (0, 23) and find the area of the quadrilateral formed by joining them.

6. Solve the equations :—

(i) $\frac{x-y}{a}+\frac{x+y}{b}=\frac{x-y}{b}-\frac{x+y}{a}=c.$ (M. M. 1895).

(ii) $(a^2-b^2)x-(a^2-ab+c^2)y=a(a-2b)-\frac{bc^2}{a-b}$

$\frac{x}{a}+\frac{y}{b}=\frac{2a}{a^2-b^2}$ (M. M. 1880).

7. If the telegraph posts by the side of a railway be 60 yards apart, shew that twice the number passed by a train in a minute gives roughly the number of miles per hour at which the train is moving. If eleven posts be passed in a minute, in what time would the distance traversed, estimated by this rule, be one mile in error? (B. M. 1876).

8. 50 articles cost 4*s.* 10*d.* Construct a graph from which you can read off the cost (to the nearest half penny) of any number of articles up to 50. Write down the cost of 23 things, and the number you would get for 3*s.*

Paper XI.

1. Plot the points given by the table below, and deduce the equation of the graph which passes through them.

x	-5	-1	3	7	11	15
y	7	4	1	-2	-5	-8

2. Draw the graphs of $\frac{x}{10}+\frac{y}{12}=1$, and $5y=6x$. Hence solve these simultaneous equations, and verify your solution by algebra.

3. Find the H. C. F. of x^3-2x^2+1 and $2x^3+x^2+4x-7$.

(A. E. 1901).

4. The sum of two fractions, which are reciprocals of each other, is $2\frac{1}{6}$. Find their difference. (P. E. 1893).

5. Solve the equations :—

(i) $\frac{x}{4}+\frac{y}{5}+1=\frac{x}{5}+\frac{y}{4}=23$. (P. E. 1893).

(ii) $\frac{1}{x}+\frac{1}{y}=\frac{5}{6}$, $3x+2y=2xy$. (A. E. 1899).

6. Simplify $\frac{1}{1+x+x^2}-\frac{1}{1-x+x^2}+\frac{2x}{1-x^2+x^4}$. (A. E. 1901).

7. The expression $ax-3b$ is equal to 30 when x is 3, and to 42, when x is 7 ; what is its value when x is $4\dot{3}$: and for what value of x is it zero? (C. E. 1874).

8. A walks at 4 miles an hour, and 4 hours after his start **B** bicycles after him at 10 miles an hour. Find, graphically, as accurately as you can, how far from the start **B** catches **A** up.

Paper XII.

1. Plot the points (15, 0), (19, 6), (10, 14), (−14, 8) and find the area of the quadrilateral formed by joining them.

2. Draw the graphs of the equations :—

$$\frac{x}{12}+\frac{y}{16}=1,\ 4x-3y=0,\ y-x=2\ ;$$

and shew that they all pass through one point. Find also the co-ordinates of the common point.

3. Taking 7 cms. = 2·76 inches, draw a graph which will enable you to convert centimetres to inches and *vice versa*. From the graph read off the value of

(i) 3·8 cms. in inches. (ii) 2·25 in. in cms.

4. The distances through which a body would fall freely in certain times are given in the following table :—

Time in secs.	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
Distances in ft.	1	4	9	16	36	64	100	144

Draw the graph, and estimate the distance through which a body will fall in $2\frac{1}{4}$ seconds.

5. A clerk is paid at the rate of *Rs.* 1200 a year ; make a graph to determine (to the nearest rupees) his salary for any given number of weeks. Write down his salary for 23 weeks.

6. Find the area of the triangle formed by the graphs of $y=8$, $x=18$, $x-y+8=0$.

7. The price, (P) shillings, of carriage cases of length (L) inches is given in a certain price list as follows :—

L.	18	20	24	26
P.	9	10	12	13

What is the probable price for a case 22 inches long ?

8. Two cyclists **A** and **B** set out at the same time. **A** rides for 2 hours at a speed of 9 miles per hour, rests 15 minutes and then continues at 6 miles per hour. **B** rides without stopping at a speed of 7 miles per hour. When will **B** overtake **A** ?

9. Find by plotting and careful measurement the co-ordinates of the point in which the straight line $2y-3x+7=0$ meets the straight line joining the points $(6, -2)$ and $(-8, 7)$.

10. At 8 A. M. **A** starts from **P** to ride to **Q** which is 48 miles distant. At the same time **B** sets out from **Q** to meet **A**. If **A** rides at 8 miles an hour, and rests half an hour at the end of every hour, while **B** walks uniformly at 4 miles an hour, find graphically (i) the time and place of meeting ; (ii) the distance between **A** and **B** at 11 A. M. ; (iii) at what time they are 14 miles apart

CHAPTER XV.

DIVISIBILITY AND REMAINDER THEOREM.

I. DIVISIBILITY.

349. We have already considered in Art. 115 the divisibility of the expressions $a^n - b^n$ and $a^n + b^n$ by $a-b$ and $a+b$, where n is a positive integer, *even* or *odd*, in particular cases. We now proceed to establish the propositions generally. We shall have to consider four cases.

When n is a **positive integer.**

1. The expression $a^n - b^n$ is **always** exactly divisible by $a-b$, whether n be **even** or **odd**.

Divide $a^n - b^n$ by $a-b$, and let Q be the quotient and R the remainder, so that R does not contain a.

Since, Dividend = Divisor $\times$ Quotient + Remainder,

$$\therefore\ a^n - b^n = Q(a-b) + R, \text{ (identically)}.$$

Now, since R does not contain a, it remains the same whatever value be given to a.

Put $a=b$, in the last equation, and we have

$$b^n - b^n = Q'(b-b) + R, \text{ or } 0 = Q' \times 0 + R,$$

where Q′ is the value of Q when b is substituted for a.

$$\text{But } Q' \times 0 = 0\ ;\ \therefore R = 0.$$

Hence the remainder being *zero*, the truth is manifest.

Thus, $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \dots\dots\dots + b^{n-1})$.

2. The expression $a^n - b^n$ is exactly divisible by $a+b$, if n be **even**, but not if n be **odd**.

With the same notation as above, we have

$$a^n - b^n = Q(a+b) + R, \text{ (identically)}.$$

Since R does not contain a, it remains the same whatever value be given to a.

Put $a=b$ in the last equation, and we have

$$(-b)^n - b^n = Q(-b+b) + R = Q' \times 0 + R.$$

Now, when n is even, $(-b)^n - b^n = b^n - b^n = 0$,
and ....odd, $(-b)^n - b^n = -b^n - b^n = -2b^n$. } (Art. 166)

Hence $R = 0$, when n is even but not when n is odd, there being a remainder in the latter case $= -2b^n$.

$\therefore$ $a^n - b^n$ is divisible by $a+b$, when n is even, but not when n is odd.

Thus, $a^n - b^n = (a+b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \ldots\ldots - b^{n-1})$.

3. The expression $a^n + b^n$ is exactly divisible by $a+b$, if n be **odd**, but not if n be **even**.

With the same notation as before, we have

$$a^n + b^n = Q(a+b) + R, \text{ (identically).}$$

Since R does not contain a, it remains the same whatever value be given to a.

Put $a = -b$ in the last equation, and we have

$$(-b)^n + b^n = Q(-b+b) + R = Q' \times 0 + R.$$

Now, when n is odd, $(-b)^n + b^n = -b^n + b^n = 0$
and ....even, $(-b)^n + b^n = b^n + b^n = 2b^n$ } (Art. 166).

Hence $R = 0$ when n is odd, but not when n is even.

$\therefore$ $a^n + b^n$ is divisible by $a+b$ when n is odd, but not when n is even.

Thus, $a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \ldots\ldots + b^{n-1})$.

4. The expression $a^n + b^n$ is **never** divisible by $a-b$, whether n be **even** or **odd**.

With the same notation as before, we have

$$a^n + b^n = Q(a-b) + R, \text{ (identically)}$$

Since R does not contain a, it remains the same whatever value be given to a.

Put $a = b$ in the last equation, and we have

$$b^n + b^n = Q(b-b) + R = Q' \times 0 + R, \text{ or } R = 2b^n.$$

Since R does not vanish for any value of n, $a^n + b^n$ is never divisible by $a-b$.

II. REMAINDER THEOREM.

350. We have already seen in Art. 218 that the theorem is true in particular cases. We now proceed to establish the theorem *generally*.

351. *If any rational and integral expression which contains x be divided by $x-a$, the remainder is found by putting a in the place of x in the given expression.*

Let the given expression be $ax^n+bx^{n-1}+cx^{n-2}+\ldots\ldots$ Divide it by $x-a$ and let Q denote the quotient and R the remainder, so that R does not contain x.

To prove that $R=aa^n+ba^{n-1}+ca^{n-2}+\ldots\ldots$

By the nature of division, we have

$$ax^n+bx^{n-1}+cx^{n-2}+\ldots\ldots\ldots=Q(x-a)+R, \text{ (identically).}$$

Since R does not contain x, it remains the same whatever value be given to x.

Putting a for x in the above equation, we have

$$aa^n+ba^{n-1}+ca^{n-2}+\ldots\ldots\ldots=Q'(a-a)+R=Q'\times 0+R$$
$$=R,$$

(where Q' is the form of Q when a is substituted for x).

352. If a rational and integral expression which contains x vanishes identically when $x=a$, then will the expression be exactly divisible by $x-a$.

To prove that $ax^n+bx^{n-1}+cx^{n-2}+\ldots\ldots$ is exactly divisible by $x-a$, if $aa^n+ba^{n-1}+ca^{n-2}+\ldots\ldots=0$.

Since by the above Art. the remainder on division $=aa^n+ba^{n-1}+ca^{n-2}+\ldots\ldots$; hence, if the last expression be zero, the given expression will be divisible by $x-a$.

Ex. 1. Find the complete quotient of $(x^7-y^7)\div(x+y)$.

Since 7 is odd, therefore x^7-y^7 is not exactly divisible by $x+y$.

Now $x^7-y^7=(x^7+y^7)-2y^7$, and x^7+y^7 is divisible by $x+y$.

$$\therefore \frac{x^7-y^7}{x+y}=\frac{x^7+y^7}{x+y}-\frac{2y^7}{x+y}$$
$$=x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^6-\frac{2y^7}{x+y}.$$

Ex. 2. Shew that the last digit in $3^{2n+1}+2^{2n+1}$ is 5, if n be any whole number. (M. M. 1868).

Since $2n+1$ is always an odd number,

$\therefore$ $3^{2n+1}+2^{2n+1}$ is divisible by $3+2$ or 5.

Again, since 3^{2n+1} is an odd number, for all its factors are 3;
and 2^{2n+1} is an even number, for all its factors are 2;

$\therefore$ $3^{2n+1}+2^{2n+1}$ = an odd number + an even number
= an odd number.

Hence $3^{2n+1}+2^{2n+1}$ is an odd number divisible by 5.

Now, since $3^{2n+1}+2^{2n+1}$ is odd, the last digit must be one of the numbers 1, 3, 5, 7 or 9. But since it is divisible by 5, the last digit must be none of the numbers 1, 3, 7, 9 but 5 only.

Ex. 3. Shew that $x+2$ is a factor of x^3-3x+2.

Putting $x=-2$ in x^3-3x+2, we have

$$(-2)^3-3(-2)+2=-8+6+2=0.$$

Therefore x^3-3x+2 is exactly divisible by $x-(-2)$ or $x+2$.

Ex. 4. Shew that $x^n-nx+n-1$ is exactly divisible by $(x-1)^2$, when n is a positive integer.

The Exp. $=(x^n-1)-n(x-1)$

$$=(x-1)(x^{n-1}+x^{n-2}+x^{n-3}+\quad \ldots\ldots +1)-n(x-1).$$

Dividing by $x-1$, we get

$$(x^{n-1}+x^{n-2}+x^{n-3}+\ldots\ldots+x+1)-n, \text{ as quotient.}$$

Now breaking up n into $1+1+1+\quad$ *i. e.* *n ones*, we get the quotient $=(x^{n-1}-1)+(x^{n-2}-1)+\ldots\ldots\ldots\ldots+(x-1)$, every part of which enclosed by brackets is divisible by $x-1$.

Hence the expression itself is divisible by $(x-1)(x-1)$ or $(x-1)^2$.

Ex. 5. Shew that $(b-c)^5+(c-a)^5+(a-b)^5$ is divisible by each of $a-b$, $b-c$ and $c-a$.

Putting $a=b$ in the given expression, we have

$$(b-c)^5+(c-b)^5+0=(b-c)^5+\{-(b-c)\}^5$$
$$=(b-c)^5-(b-c)^5=0.$$

Therefore the expression is divisible by $a-b$.

Similarly, putting $b=c$ and $c=a$ successively, the expression may be proved to be divisible by each of $b-c$ and $c-a$.

353. Indirect Multiplication. Sometimes it is convenient to find continued products without the trouble of actual multiplication.

Ex. 1. Find the continued product $(x+a)(x^2+a^2)(x^4+a^4)(x^8+a^8)$.

Let P denote the continued product required.

Then $P=(x+a)(x^2+a^2)(x^4+a^4)(x^8+a^8)$.

Multiplying each side by $x-a$, we have

$$\begin{aligned}(x-a)P&=(x-a)(x+a)(x^2+a^2)(x^4+a^4)(x^8+a^8)\\&=(x^2-a^2)(x^2+a^2)(x^4+a^4)(x^8+a^8)\\&=x^{16}-a^{16}, \text{ by successive multiplication.}\end{aligned}$$

$$\therefore\ P=\frac{x^{16}-a^{16}}{x-a}=x^{15}+ax^{14}+a^2x^{13}+\dots\dots+a^{13}x^2+a^{14}x+a^{15}$$

354. Find the condition that ax^2+bx+c may be a perfect square.

Using the ordinary rule for square root, we have

$$\begin{array}{r|l} & ax^2+bx+c\ \big(\ \sqrt{a}.x+\dfrac{b}{2\sqrt{a}} \\ & ax^2 \\ \hline 2\sqrt{a}.x+\dfrac{b}{2\sqrt{a}} & bx+c \\ & bx+\dfrac{b^2}{4a} \\ \hline & c-\dfrac{b^2}{4a} \end{array}$$

Therefore ax^2+bx+c will be a perfect square, if $c-\dfrac{b^2}{4a}=0$.

Hence $b^2=4ac$, the condition required.

Otherwise thus: Since ax^2+bx+c is a perfect square,

$\therefore\ ax^2+bx+c=(\sqrt{a}.x+\sqrt{c})^2=ax^2+2x\sqrt{ac}+c$.

Now, comparing coefficients, we have

$$b=2\sqrt{ac}, \text{ and } \therefore\ b^2=4ac.$$

Exercise CXXX.

In each of the following examples, state whether the first expression is divisible by the second, and where it is so divisible, find the quotient :—

1. a^4+b^4 by $a+b$.	**2.** a^4-b^4 by $a+b$.	**3.** a^4+b^4 by $a-b$.
4. a^5+b^5 by $a+b$.	**5.** a^5-b^5 by $a-b$.	**6.** a^6-b^6 by $a+b$.
7. a^7+b^7 by $a-b$.	**8.** a^7-b^7 by $a+b$.	**9.** a^8+b^8 by $a-b$.

10. $a^8 - b^8$ by $a-b$. **11.** $a^{13} - b^{13}$ by $a+b$. **12.** $a^{15} - b^{16}$ by $a+b$.

13. $a^{12} - b^{12}$ by a^2+b^2. **14.** $a^{11}+b^{11}$ by $a-b$. **15.** $a^{16}+b^{10}$ by a^2+b^2.

16. Shew that $22^{2n+1}+1$ is divisible by 23, if n is any positive integer.

17. Shew that $(2x+y)^n+y^n$ is divisible by $x+y$, when n is odd.

18. Shew that $1-x-x^n+x^{n+1}$ is divisible by $(1-x)^2$, when n is any positive integer.

19. What must be the form of m in order that $a^m - x^m$ may have both a^n+x^n and a^n-x^n for divisors, n being any positive integer? (M. M. 1875).

20. Shew that $2^{4n}-1$ is divisible by 15, if n be a positive integer. (M. M. 1875).

21. Assuming that x^n-y^n is divisible by $x-y$, when n is any whole number, shew that $(ab)^n-(bc)^n+(cd)^n-(da)^n$ is always divisible by $ab-bc+cd-da$. (M. M. 1873).

22. Shew that $(x-1)^2$ is a factor of $nx^{n+1}-(n+1)x^n+1$, when n is any positive integer.

23. Shew that $(1-x)^{2n}-(4-7x-x^2)^n$ is divisible both by $x+3$ and $2x-1$, if n be any positive integer.

24. Shew that $(2a+b)^n-(a+2b)^n-a^n+b^n$ is divisible by a^2-b^2. when n is any whole number.

25. Divide $x^8+x^6y^2+x^4y^4+x^2y^6+y^8$ by $x^4-x^3y+x^2y^2-xy^3+y^4$. (C. E. 1870).

26. Divide $x^6-1-5(x-1)$ by $(x-1)^2$. (P. E. 1893).

27. If n is a positive whole number, shew that the last digit of $^{4n+1}+1$ is 0.

28. Prove that $6^{2n}+7^n+6$ is divisible by 7, n being any positiv integer.

29. Shew that $4(4^6+3^6)$ ends in two ciphers.

30. Shew that 5^6+7^6 is divisible by 12 and 4^6+3^6 by 25, without a remainder.

31. Divide $1+a+a^2+a^3+a^4+a^6+a^7+a^8+a^9+a^{15}$ by $1-a^5+a^6$. (B. M. 1888).

32. Shew that $(1-x)^2$ is a factor of $1-x-x^7+x^8$.

33. If $x^4+ax^3+bx^2+cx+d$ be a perfect square, shew that the coefficients satisfy the relations

$$8c=a(4b-a^2) \text{ and } (4b-a^2)^2=64d.$$

Shew without actual division that the following expressions are exactly divisible by each of $a-b$, $b-c$ and $c-a$.

34. $bc(b-c)+ca(c-a)+ab(a-b)$.

35. $a^3(b-c)+b^3(c-a)+c^3(a-b)$.

36. $a(b-c)^3+b(c-a)^3+c(a-b)^3$.

37. $(b-c)^7+(c-a)^7+(a-b)^7$.

38. Shew that $x^n(x-1)+y^n(y-1)$ is not divisible by $x+y$, whatever positive whole number n may be.

39. If x^n+y^n be divided by $x-y$, shew that the remainder is $2y^n$.

40. Write down the product $(1+a)(1+a^2)(1+a^4)(1+a^8)(1+a^{16})$.

CHAPTER XVI.

HARDER SIMPLE EQUATIONS.

355. Number of Roots. *A Simple Equation cannot have more than one root.*

The general form of a simple equation is $ax+b=0$.

If possible, let α and β be the two different roots of the equation $ax+b=0$.

Then, we must have identically

$$\left.\begin{aligned} a\alpha+b&=0 \\ \text{and } a\beta+b&=0 \end{aligned}\right\}\begin{aligned}&\ldots\ldots\ldots\ldots(1)\\&\ldots\ldots\ldots\ldots(2)\end{aligned}$$

By subtraction, $a(\alpha-\beta)=0$.

Now a is not zero, (by supposition)

$$\therefore\ \alpha-\beta=0 \text{ and } \therefore\ \alpha=\beta.$$

Hence α and β are not different from each other, which is contrary to the hypothesis.

Therefore a simple equation has only *one* root.

356. Principle of Identity. If a simple equation is satisfied by more than *one* value of the unknown quantity, it is an **Identity**.

Suppose the simple equation $ax+b=0$ is satisfied by two different values α and β of x; then as in the preceding Art.

$$a(\alpha-\beta)=0$$

and $\because$ $\alpha-\beta$ is not zero, $\therefore$ a is zero.

Substituting the value of a in (1) or (2) of the preceding Article, we evidently find $b=0$.

Now, because a and b are each equal to zero in the equation $ax+b=0$, $\therefore$ any value of x will make the left-hand side $=0$. Hence the equation is an *Identity*.

357. Common Factor. When an equation is reduced to the form

$$\mathsf{AX}=0,$$

where the expression X contains the unknown quantity x and the expression A does or does not contain x at all, then

1. When A contains x, the equation is satisfied by either

$$\mathsf{A}=0 \text{ or } \mathsf{X}=0,$$

from which equations, the roots of the given equation may be obtained.

2. When A does not contain x, we can divide both sides of the equation by A, and obtain

$$\mathsf{X}=0,$$

from which equation, the root or roots may be obtained.

I. EQUATIONS NOT INVOLVING FRACTIONS.

358 The following are typical examples.

Ex. 1. Solve $3(x+1)^2+4(x+3)^2=7(x+2)^2$.

Since $7(x+2)^2=3(x+2)^2+4(x+2)^2$, by transposition, we get

$$3\{(x+1)^2-(x+2)^2\}=4\{(x+2)^2-(x+3)^2\},$$

$\therefore$ $3(2x+3)\times-1=4(2x+5)\times-1$, Art. 124.

Dividing both sides by -1 and multiplying out, we have

$$6x+9=8x+20, \therefore -2x=11, \therefore x=-\tfrac{11}{2}=-5\tfrac{1}{2}.$$

Ex. 2. Solve $(x-a)^3+(x-b)^3+(x-c)^3=3(x-a)(x-b)(x-c)$.

By transposition, we have

$$(x-a)^3+(x-b)^3+(x-c)^3-3(x-a)(x-b)(x-c)=0.$$

Now, the left-hand side

$$=\tfrac{1}{2}(x-a+x-b+x-c)[\{(x-b)-(x-c)\}^2+\{(x-c)-(x-a)\}^2$$
$$+\{(x-a)-(x-b)\}^2, \text{ Art. } 327.$$
$$=\tfrac{1}{2}(3x-a-b-c)\{(b-c)^2+(c-a)^2+(a-b)^2\}.$$
$$\therefore\ \tfrac{1}{2}(3x-a-b-c)\{(b-c)^2+(c-a)^2+(a-b)^2\}=0.$$
$$\therefore\ 3x-a-b-c=0, \text{ Art. } 357, \text{ or } x=\tfrac{1}{3}(a+b+c).$$

Exercise CXXXI.

Solve the following equations :—

1. $(x+5)^2+5(x+7)^2=6(x+9)^2$. 2. $4(x+1)^2+9(x+2)^2=13(x+3)^2$.
3. $(5x+21)^2+(7x+36)^2=(7x+41)^2+(5x+13)^2$.
4. $(x-3)^3-3(x-2)^3+3(x-1)^3-x^3=9-x$.
5. $(x-a)(x-b)=(x-a-b)^2$. 6. $(x-1)^3=x(x-1)(x-2)$.
7. $(x+a)^2+(x+b)^2+(x+c)^2=(x-a)^2+(x-b)^2+(x-c)^2$.
8. $27(x-2)^3+(2x-5)^3+(3x-7)^3=9(x-2)(2x-5)(3x-7)$.
9. $(x-2a)^3+(x-2b)^3=2(x-a-b)^3$. 10. $(x-5)^3+(x-7)^3=2(x-6)^3$.
11. $(3a+3b-2x)^2+(3b-3a+2x)^2-(3b+3c-2x)^2=(3b-3c+2x)^2$.
12. $(x-2)^3+(x-5)^3+(x-7)^3=3(x-2)(x-5)(x-7)$.

II. EQUATIONS INVOLVING FRACTIONS.

359. Multiplying across. *If* $\dfrac{a}{b}=\dfrac{c}{d}$, *then will* $ad=bc$. Equations such as the following can easily be solved by the above method.

Ex. 1. Solve $\dfrac{63}{7x+3}=\dfrac{45}{9x+11}$.

Dividing both sides by 9, we have $\dfrac{7}{7x+3}=\dfrac{5}{9x+11}$.

Multiplying across, $7(9x+11)=5(7x+3)$;

or $63x+77=35x+15$; $\therefore\ 28x=-62$ and $\therefore\ x=-\frac{62}{28}=-2\frac{3}{14}$.

Ex. 2. Solve $\dfrac{3x-7}{x+5}=\dfrac{6x-1}{2x+7}$.

Multiplying across, we have

$$(3x-7)(2x+7)=(x+5)(6x-1);$$

or $6x^2+7x-49=6x^2+29x-5$.

Subtracting $6x^2$ from both sides and transposing, we get

$$7x-29x=49-5;\ \therefore\ -22x=44 \text{ and } \therefore\ x=-2.$$

360. Convenient transposition. In Equations like the following, combine by transposition of terms, the simplest fractions with like denominators.

Ex. 1. Solve $\dfrac{10x+17}{18}-\dfrac{12x+2}{11x-8}=\dfrac{5x-4}{9}$.

Transposing, $\dfrac{12x+2}{11x-8}=\dfrac{5x-4}{9}-\dfrac{10x+17}{18}$

$$=\frac{10x-8-10x-17}{18}=-\frac{25}{18}.$$

Multiplying across, $18(12x+2)=25(11x-8)$;

or $216x+36=275x-200$; $\therefore\ 216x-275x=-200-36$;

or $-59x=-236$, and $\therefore\ x=4$.

Ex. 2. Solve $\dfrac{6x-7\frac{1}{3}}{13-2x}+2x+\dfrac{16x+1}{24}=4\frac{5}{12}-\dfrac{12\frac{5}{8}-8x}{3}$.

Transposing, $\dfrac{6x-7\frac{1}{3}}{13-2x}=4\frac{5}{12}-\dfrac{12\frac{5}{8}-8x}{3}-2x-\dfrac{16x+1}{24}$

$$=\frac{6-101+64x-48x-16x-1}{24}$$

$$=\tfrac{4}{24}=\tfrac{1}{6}.$$

Multiplying across, $13-2x=6(6x-7\frac{1}{3})=36x-44$,

$\therefore\ -38x=-57$, and $\therefore\ x=\frac{57}{38}=\frac{3}{2}=1\frac{1}{2}$.

361. Reduction by Division. Sometimes it would be advantageous to bring improper fractions to mixed quantities, so as to make the integral parts on both sides of the equation the same.

Ex. 1. Solve $\frac{3x+2}{x-1}+\frac{2x-4}{x+2}=5.$

By division the equation reduces to

$$\left(3+\frac{5}{x-1}\right)+\left(2-\frac{8}{x+2}\right)=5, \text{ or } \frac{5}{x-1}-\frac{8}{x+2}=0.$$

Transposing and multiplying across, we have

$$5(x+2)=8(x-1), \text{ or } 5x+10=8x-8\,;$$

$$\therefore\ -3x=-18 \text{ and } \therefore\ x=6.$$

Ex. 2. Solve $\frac{3x-14}{x-5}-\frac{3x-8}{x-3}=\frac{3x-32}{x-11}-\frac{3x-26}{x-9}.$

By division, the equation reduces to

$$\left(3+\frac{1}{x-5}\right)-\left(3+\frac{1}{x-3}\right)=\left(3+\frac{1}{x-11}\right)-\left(3+\frac{1}{x-9}\right)$$

$$\therefore\ \frac{}{x-5}-\frac{}{x-3}=\frac{}{x-11}-\frac{}{x-9}.$$

Simplifying each side separately,

$$\frac{(x-3)-(x-5)}{(x-3)(x-5)}=\frac{(x-9)-(x-11)}{(x-9)(x-11)}, \text{ or } \frac{2}{(x-3)(x-5)}=\frac{2}{(x-9)(x-11)}.$$

Dividing both sides by 2, and multiplying up,

$$(x-9)(x-11)=(x-3)(x-5), \text{ or } x^2-20x+99=x^2-8x+15.$$

$$\therefore\ -12x=-84, \text{ and } \therefore\ x=\tfrac{84}{12}=7.$$

Ex. 3. Solve $\frac{6x+1}{3x-5}-\frac{2x-5}{3x-4}=\frac{4}{3}.$

Multiplying all the terms by 3, we get

$$\frac{18x+3}{3x-5}-\frac{6x-15}{3x-4}=4.$$

By division, the equation reduces to

$$\left(6+\frac{33}{3x-5}\right)-\left(2-\frac{7}{3x-4}\right)=4, \text{ or } \frac{33}{3x-5}+\frac{7}{3x-4}=0.$$

Transposing and multiplying across, we get

$$33(3x-4)=-7(3x-5), \text{ or } 99x-132=-21x+35\,;$$

$$\therefore\ 120x=167, \text{ and } \therefore\ x=\tfrac{167}{120}=1\tfrac{47}{120}.$$

362. Suitable transposition and grouping of terms. The following are typical examples.

Ex. 1. Solve $\frac{11}{2x+11}-\frac{9}{2x+9}=\frac{9}{2x-9}-\frac{7}{2x-7}$.

By transposition, $\frac{11}{2x+11}+\frac{7}{2x-7}=9\left\{\frac{1}{2x-9}+\frac{1}{2x+9}\right\}$..

$$\therefore \frac{11(2x-7)+7(2x+11)}{(2x+11)(2x-7)}=9\times\frac{2x+9+2x-9}{(2x-9)(2x+9)},$$

or $\frac{36x}{4x^2+8x-77}=\frac{36x}{4x^2-81}$. Hence $x=0$, Art. 357.

and $4x^2+8x-77=4x^2-81$.

Cancelling $4x^2$ from both sides, we have

$8x-77=-81$, $\therefore$ $8x=-4$ and $\therefore$ $x=-\frac{1}{2}$.

Ex. 2. Solve $\frac{3}{10x+9}+\frac{4}{45x+2}=\frac{7}{18x+5}$. (M. M. 1884)

Here, $\frac{3}{10x+9}+\frac{4}{45x+2}=\frac{7}{18x+5}=\frac{3}{18x+5}+\frac{}{18x+5}$.

By transposition, $\frac{3}{10x+9}-\frac{3}{18x+5}=\frac{4}{18x+5}-\frac{4}{45x+2}$,

Simplifying, $\frac{3(8x-4)}{(10x+9)(18x+5)}=\frac{4(27x-3)}{(18x+5)(45x+2)}$;

Multiplying by $18x+5$ and dividing by 12,

$\frac{2x-1}{10x+9}=\frac{9x-1}{45x+2}$; $\therefore$ $(2x-1)(45x+2)=(10x+9)(9x-1)$;

or $90x^2-41x-2=90x^2+71x-9$; $\therefore$ $-41x-2=71x-9$;

$\therefore$ $-112x=-7$ and $\therefore$ $x=\frac{7}{112}=\frac{1}{16}$.

363. Alternando. *If* $\frac{a}{b}=\frac{c}{d}$, *then will* $\frac{a}{c}=\frac{b}{d}$.

The following are illustrative examples.

Ex. 1. Solve $\frac{x^2-8x+15}{x^2-6x+6}=\frac{x-7}{x-5}$.

Alternately, we have $\dfrac{x^2-8x+15}{x-7}=\dfrac{x^2-6x+6}{x-5}$.

By division, $x-1+\dfrac{8}{x-7}=x-1+\dfrac{1}{x-5}$;

$\therefore\ \dfrac{8}{x-7}=\dfrac{1}{x-5}$; $\therefore\ x-7=8(x-5)=8x-40$;

$$\therefore\ -7x=-33 \text{ and } \therefore\ x=\tfrac{33}{7}=4\tfrac{5}{7}.$$

Ex. 2. Solve $\left(\dfrac{2x-15}{2x-13}\right)^2=\dfrac{x-15}{x-13}$.

Alternately, we have $\dfrac{(2x-15)^2}{x-15}=\dfrac{(2x-13)^2}{x-13}$;

$$\text{or } \frac{4x^2-60x+225}{x-15}=\frac{4x^2-52x+169}{x-13}.$$

By division, $4x+\dfrac{225}{x-15}=4x+\dfrac{169}{x-13}$;

$\therefore\ \dfrac{225}{x-15}=\dfrac{169}{x-13}$, or $225(x-13)=169(x-15)$;

$\therefore\ 225x-2925=169x-2535$: $\therefore\ 56x=390$;

$$\therefore\ x=\tfrac{390}{56}=6\tfrac{27}{28}.$$

364. An Important Formula. *If* $\dfrac{a}{b}=\dfrac{c}{d}$, *then will each*

$$\textit{fraction}=\frac{ma+nc}{mb+nd} \text{ or } =\frac{ma-nc}{mb-nd},$$

where m and n may be any quantities whatever, integral or fractional, positive or negative.

Ex. 1. Solve $\left(\dfrac{x+12}{x+8}\right)^2=\dfrac{(x+11)(x+13)}{(x+7)(x+9)}$.

Multiplying out, $\dfrac{x^2+24x+144}{x^2+16x+64}=\dfrac{x^2+24x+143}{x^2+16x+63}$.

$$\therefore\ \text{each}=\frac{\text{diff. of numerators}}{\text{diff. of denominators}}=\frac{144-143}{64-63}=1;$$

$$\therefore\ x^2+24x+144=x^2+16x+64;$$

$$\therefore\ 8x=-80 \text{ and } \therefore\ x=-10.$$

Ex. 2. Solve $\left(\dfrac{2x+9}{2x+7}\right)^3 = \dfrac{2x+11}{2x+5}$.

Multiplying by $\dfrac{2x+7}{2x+9}$, $\left(\dfrac{2x+9}{2x+7}\right)^2 = \dfrac{(2x+11)(2x+7)}{(2x+5)(2x+9)}$;

Multiplying out, $\dfrac{4x^2+36x+81}{4x^2+28x+49} = \dfrac{4x^2+36x+77}{4x^2+28x+45}$;

$\therefore$ each $= \dfrac{\text{diff. of numerators}}{\text{diff. of denominators}} = \dfrac{81-77}{49-45} = \dfrac{4}{4} = 1$.

$\therefore$ $4x^2+36x+81 = 4x^2+28x+49$;

$\therefore$ $8x = -32$ and $\therefore$ $x = -4$.

Exercise CXXXII.

Solve the following equations :—

1. $\dfrac{x+\frac{1}{2}}{7} - \dfrac{x}{2} = \dfrac{4}{35x} - \dfrac{5x}{14}$.

2. $\dfrac{2x}{3} - \dfrac{1-\frac{1}{2}x}{4x} = \dfrac{x-1}{2} + \dfrac{x}{6}$.

3. $\dfrac{2x+3}{4} + \dfrac{4x}{3} = \dfrac{1}{x} + \dfrac{6x+2}{3} - \dfrac{x+1}{6}$.

4. $\dfrac{3}{4x-3} = \dfrac{2}{3x-5}$.

5. $1 - \dfrac{x}{2}\left(1 - \dfrac{3}{4x}\right) = \dfrac{2}{3}\left(3 - \dfrac{5x}{2}\right) + 5\dfrac{13}{40}$.

6. $\dfrac{2x+5}{x+7} = \dfrac{6x-3}{3x-2}$.

$\dfrac{2x}{3} - \dfrac{1-\frac{1}{2}x}{4x} = \dfrac{2x+7}{12} - \dfrac{1-x}{2}$. (B. M. 1885).

8. $\dfrac{5x-7}{x+1} = \dfrac{15x-11}{3x+1}$.

9. $10\left(x+\dfrac{1}{2}\right) - 23 = 6x\left(\dfrac{1}{x} - \dfrac{1}{3}\right)$. (C. E. 1859).

10. $\dfrac{x+3}{2x-3} = \dfrac{2x}{4x-9}$.

11. $\dfrac{4x-17}{9} - \dfrac{3\frac{2}{3}-22x}{33} = x - \dfrac{6}{x}\left(1 - \dfrac{x^2}{54}\right)$.

12. $\dfrac{x-\frac{1}{2}}{x-1} - \dfrac{3}{5}\left(\dfrac{1}{x-1} - \dfrac{1}{3}\right) = \dfrac{23}{10(x-1)}$. (A. E. 1891).

13. $\dfrac{2x-3}{5x-2} = \dfrac{2x+6}{5x+37}$.

14. $\dfrac{4x+3}{9} + \dfrac{29-7x}{12-5x} = \dfrac{8x+19}{18}$. (C. E. 1868).

15. $\dfrac{2(4x+3)}{x+3} + \dfrac{3}{x+1} = 8$.

16. $\dfrac{8x+5}{14} + \dfrac{7x-3}{6x+2} = \dfrac{4x+6}{7}$.

17. $\dfrac{x-3}{x+2} = \dfrac{1}{2} + \dfrac{x-3}{2x-1}$.

18. $\dfrac{3-4x}{3(3-x)} + \dfrac{1}{2(1-x)} = 1\frac{1}{3}$.

19. $\frac{x(2x+3)}{2x+1}+\frac{1}{3x}=x+1.$

20. $\frac{x+4}{3x+5}+1\frac{1}{6}=\frac{3x+8}{2x+3}.$

21. $\frac{10x+47}{18}-\frac{12x+38}{13x+23}=\frac{5x+11}{9}.$ (M. M. 1871).

22. $\frac{6x+13}{15}-\frac{3x+5}{5x-25}=\frac{2x}{5}.$ (B. M. 1871).

23. $\frac{x-7}{x+7}=\frac{2x-15}{2x-6}-\frac{1}{2(x+7)}.$

24. $\frac{132x+91}{3x+1}+\frac{8x+5}{x-1}=52.$

25. $\frac{7x+1}{x-1}=\frac{35}{9}\left(\frac{x+4}{x+2}\right)+3\frac{1}{9}.$ (A. E. 1896).

26. $\frac{1}{x-1}+\frac{1}{x-4}=\frac{2}{x-3}.$ (C. E. 1860).

27. $3x-5 \quad x-5 \quad 2x-5.$ (C. E. 1871).

28. $6x+17 \quad 3x-10 \quad 1-2x$

29. $\frac{11}{12x+11}+\frac{5}{6x+5}=\frac{7}{4x+7}$

30. $15 \quad 14(x-1) \quad 21 \quad 6 \quad + \quad 105$

31. $\frac{25-\frac{1}{3}x}{x+1}+\frac{16x+4\frac{1}{5}}{3x+2}=5+\frac{23}{x+1}$

32. $\frac{1}{2}x-\frac{\frac{1}{3}(2x-3)-\frac{1}{4}(3x-1)}{\frac{1}{2}(x-1)}=\frac{3}{2}\left(\frac{x^2-\frac{1}{3}x+2}{3x-2}\right).$

33. $\frac{2x+3}{x+1}=\frac{4x+5}{4x+4}+\frac{3x+3}{3x+1}.$ (B. M. 1889).

34. $\frac{12}{x+2}=6-2\left(\frac{3x+2}{x+1}\right).$ (C. E. 1873).

35. $\frac{1}{3}-\frac{7x-1}{6\frac{1}{2}-3x}=\frac{8}{3}\cdot\frac{x-\frac{1}{2}}{x-2}.$ (C. E. 1888).

36. $\frac{1}{x-1}-\frac{1}{x}-\frac{1}{x+3}+\frac{1}{x+4}=0.$ (M. M. 1887)

37. $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-5}{x-6}-\frac{x-6}{x-7}.$ (C. E. 1865).

38. $\dfrac{x-8}{x-10}-\dfrac{x-5}{x-7}=\dfrac{x-7}{x-9}-\dfrac{x-4}{x-6}$. (B. M. 1887).

39. $\dfrac{x}{x-2}+\dfrac{9-x}{7-x}=\dfrac{x+1}{x-1}+\dfrac{8-x}{6-x}$. (M. M. 1885).

40. $\dfrac{x-2}{x-3}+\dfrac{x-3}{x-4}=\dfrac{x-1}{x-2}+\dfrac{x-4}{x-5}$. (C. E. 1887 & M. M. 1890).

41. $\dfrac{2x+11}{x+5}-\dfrac{9x-9}{3x-4}=\dfrac{4x+13}{x+3}-\dfrac{15x-47}{3x-10}$. (C. E. 1860 & B. M. 1895).

42. $\dfrac{6-5x}{5}-\dfrac{\text{[illegible]}}{14}\times\dfrac{7-2x^2}{x-1}-1\dfrac{1}{10}=\dfrac{1+3x}{7}-x+\dfrac{\text{[illegible]}}{35}$. (M. M. 1867).

43. $\dfrac{16x-27\frac{4}{5}}{3x-4}+\dfrac{77-x}{3(x-1)}=5+\dfrac{23}{x+1}$. (M. M. 1882).

44. $\dfrac{4x-17}{x-4}+\dfrac{10x-13}{2x-3}=\dfrac{8x-30}{2x-7}+\dfrac{5x-4}{\text{[illegible]}}$

45. $\dfrac{\text{[illegible]}}{x-5}+\dfrac{\text{[illegible]}}{x-9}+\dfrac{\text{[illegible]}}{x-1}=3$.

46. $\dfrac{x-4}{x-1}+\dfrac{x-7}{x-3}+\dfrac{x-2}{x-9}=3$. (M. M. 1875).

47. $\dfrac{5x-34}{x-7}+\dfrac{3x-26}{x-9}=\dfrac{5x-24}{x-5}+\dfrac{3x-32}{x-11}$.

48. $\dfrac{x^2-x+1}{x-1}+\dfrac{x^2+x+1}{x+1}=2x$.

49. $\dfrac{1+x}{1-x}-\dfrac{2+3x}{2-3x}=1+\dfrac{1+3x}{1-3x}$.

50. $\dfrac{x-4}{(x-1)(x-3)}+\dfrac{x-7}{(x-1)(x-6)}+\dfrac{x-9}{(x-3)(x-6)}=\dfrac{3}{x}$. (M. M. 1874).

51. $\dfrac{x^2+2x+2}{x+1}+\dfrac{x^2+8x+17}{x+4}=\dfrac{x^2+4x+5}{x+2}+\dfrac{x^2+6x+10}{x+3}$

52. $\dfrac{1}{5}\cdot\dfrac{x-4}{x-9}+\dfrac{1}{9}\cdot\dfrac{x-16}{x-25}-\dfrac{2}{13}\cdot\dfrac{x-36}{x-49}=\dfrac{92}{585}$.

53. $\dfrac{1}{x-9}+\dfrac{1}{x-25}\ \text{[illegible]}\ \dfrac{\text{[illegible]}}{x-49}$.

54. $\dfrac{9}{3x-4}+\dfrac{20}{4x+1}\ \text{[illegible]}\ \dfrac{\text{[illegible]}}{x+7}$.

55. $31\left\{\dfrac{24-5x}{x+1}+\dfrac{5-6x}{x+4}\right\}+370=29\left\{\dfrac{17-7x}{x+2}+\dfrac{8x+55}{x+3}\right\}$

56. $\left(\dfrac{8x-3}{4x-1}\right)^2 = \dfrac{4x-5}{x-1}$.

57. $\dfrac{x-11}{x-9} = \left(\dfrac{2x-19}{2x-17}\right)^2$.

58. $\dfrac{(x+4)(x+5)}{(x+6)(x+9)} = \dfrac{x+6}{x+12}$.

59. $\dfrac{2x-1}{x-1} + \dfrac{x-2}{x-3} = \dfrac{3x+2}{x-2}$.

60. $\dfrac{(x+1)(x+9)}{(x+2)(x+4)} = \dfrac{(x+6)(x+10)}{(x+5)(x+7)}$. (M. M. 1889).

61. $\dfrac{4 \cdot 05}{9x} - \dfrac{\cdot 3}{8-2x} = \dfrac{1 \cdot 8}{x} - \dfrac{3 \cdot 6}{2 \cdot 4 - 6x}$. (C. E. 1881).

62. $\dfrac{(x+9)(x+7)}{(x+6)(x+10)} = \dfrac{(x+2)(x+4)}{(x+1)(x+5)}$.

63. $\left(\dfrac{x+7}{x+9}\right)^3 = \dfrac{x+5}{x+11}$.

64. $\dfrac{x+3}{x+5} = \dfrac{x^2+3x+7}{x^2+5x+9}$.

65. $\left(\dfrac{x-7}{x-6}\right)^2 = \dfrac{(x-5)(x-9)}{(x-4)(x-8)}$.

66. $(x+1)(x+7)(x^2-20) = (x^2-3x+2)(x^2+11x+18)$.

67. $5 + \dfrac{2}{3 - \dfrac{1}{4-x}} = \dfrac{29}{5}$. (A. E. 1893).

68. $\dfrac{6}{7 - \dfrac{6}{7 - \dfrac{6}{7-x}}} = 1$. (B. M. 1891).

69. $\dfrac{3}{x-3} - \dfrac{4}{x+9} - \dfrac{5}{x-27} + \dfrac{6}{x-15} = 0$. (M. M. 1873).

70. $1 - \dfrac{1 - \dfrac{3}{3-x}}{3-x} = \dfrac{x}{x-3}$. (P. E. 1890).

71. $\dfrac{(x+1)^5 + (x-1)^5}{(x+1)^3 + (x-1)^3} = 10$. (M. M. 1877).

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365. Literal Equations. The following are typical examples of *Literal Equations.*

Ex. 1. Solve $\dfrac{a}{bx} + \dfrac{b}{ax} = a^2 + b^2$.

Multiplying by abx, the L. C. M. of the denrs., we have

$$a^2 + b^2 = ab(a^2+b^2)x,$$

$$\therefore\ x = \frac{a^2+b^2}{ab(a^2+b^2)} = \frac{1}{ab}.$$

Ex. 2. Solve $\dfrac{x+a}{x+b}=\dfrac{x+3a}{x+a+b}$. (C. E. 1892).

Multiplying crosswise, we obtain

$$(x+a)(x+a+b)=(x+3a)(x+b),$$

or $x^2+(2a+b)x+a(a+b)=x^2+(3a+b)x+3ab.$

Cancelling x^2 from both sides, we have

$$(2a+b)x+a(a+b)=(3a+b)x+3ab.$$

By transposition, $(2a+b)x-(3a+b)x=3ab-a(a+b)$,

or $\{(2a+b)-(3a+b)\}x=a\{3b-(a+b)\}$,

$\therefore\ -ax=a(2b-a)$, and $\therefore\ x=a-2b$, (dividing by a).

Ex. 3. Solve $\dfrac{m(x+a)}{x+b}+\dfrac{n(x+b)}{x+a}=m+n.$

By division, $m\left(1+\dfrac{a-b}{x+b}\right)+n\left(1-\dfrac{a-b}{x+a}\right)=m+n.$

$\therefore\ \dfrac{m(a-b)}{x+b}=\dfrac{n(a-b)}{x+a}$, or $\dfrac{m}{x+b}=\dfrac{n}{x+a}$, (dividing by $a-b$)

$\therefore\ m(x+a)=n(x+b)$, or $mx+ma=nx+nb$.

By transposition, $(m-n)x=nb-ma$, and $\therefore\ x=\dfrac{nb-ma}{m-n}$.

Ex. 4 Solve $\left(\dfrac{2x+a+c}{2x+b+c}\right)^2=\dfrac{x+a}{x+b}$.

Alternately, we have $\dfrac{(2x+a+c)^2}{x+a}=\dfrac{(2x+b+c)^2}{x+b}$;

Multiplying out, $\dfrac{4x^2+4(a+c)x+(a+c)^2}{x+a}=\dfrac{4x^2+4(b+c)x+(b+c)^2}{x+b}$,

By division, $4(x+c)+\dfrac{(a-c)^2}{x+a}=4(x+c)+\dfrac{(b-c)^2}{x+b}$;

$\therefore\ \dfrac{(a-c)^2}{x+a}=\dfrac{(b-c)^2}{x+b}$; $\quad\therefore\ (a-c)^2(x+b)=(b-c)^2(x+a)$

Multiplying out and transposing, we have

$$\{(a-c)^2-(b-c)^2\}x=a(b-c)^2-b(a-c)^2;$$

$$\therefore\ (a-b)(a+b-2c)x=a(b^2-2bc+c^2)-b(a^2-2ac+c^2)$$
$$=(a-b)c^2-ab(a-b)=(a-b)(c^2-ab);$$

$\therefore\ x=\dfrac{c^2-ab}{a+b-2c}$, (dividing by $a-b$).

Ex. 5. Solve $\dfrac{ax+b}{px+q}=\dfrac{ax^2+bx+c}{px^2+qx+r}$.

Here, $\dfrac{ax+b}{px+q}=\dfrac{x(ax+b)}{x(px+q)}=\dfrac{ax^2+bx+c}{px^2+qx+r}$;

$\therefore$ each $=\dfrac{\text{diff. of numerators}}{\text{diff. of denominators}}=\dfrac{c}{r}$.

$\therefore r(ax+b)=c(px+q)$, or $arx+br=pcx+cq$;

$\therefore (ar-pc)x=cq-br$, and $\therefore x=\dfrac{cq-br}{ar-pc}$.

Ex. 6. Solve $\dfrac{x-a}{3b+5c}+\dfrac{x-3b}{5c+a}+\dfrac{x-5c}{a+3b}=3$. (C. E. 1896).

Here, $\left(\dfrac{x-a}{3b+5c}-1\right)+\left(\dfrac{x-3b}{5c+a}-1\right)+\left(\dfrac{x-5c}{a+3b}-1\right)=0$,

$\therefore \dfrac{x-a-3b-5c}{3b+5c}+\dfrac{x-a-3b-5c}{5c+a}+\dfrac{x-a-3b-5c}{a+3b}=0$.

Hence $x-a-3b-5c=0$, Art. 357.

$\therefore x=a+3b+5c$.

Ex. 7. Solve $\dfrac{1}{x+6a}+\dfrac{2}{x-3a}+\dfrac{3}{x+2a}=\dfrac{6}{x+a}$.

By transposition, $\dfrac{1}{x+6a}+\dfrac{2}{x-3a}=\dfrac{6}{x+a}-\dfrac{3}{x+2a}$.

$\therefore \dfrac{x-3a+2(x+6a)}{(x+6a)(x-3a)}=\dfrac{6(x+2a)-3(x+a)}{(x+a)(x+2a)}$;

$\therefore \dfrac{3x+9a}{(x+6a)(x-3a)}=\dfrac{3x+9a}{(x+a)(x+2a)}$.

$\therefore 3x+9a=0$, Art. 357;

$\therefore 3x=-9a$ and $\therefore x=-3a$.

The other equation, $(x+6a)(x-3a)=(x+a)(x+2a)$, is inadmissible, for it will be found not to contain x at all.

Ex. 8. Solve $\left(\dfrac{x-a}{x-b}\right)^3=\dfrac{x-2a+b}{x+a-2b}$.

Adding 1 to both sides, we have

$$\frac{(x-a)^3+(x-b)^3}{(x-b)^3}=\frac{x-2a+b+x+a-2b}{x+a-2b},$$

or $$\frac{\{(x-a)+(x-b)\}\{(x-a)^2-(x-a)(x-b)+(x-b)^2\}}{(x-b)^3}=\frac{2x-a-b}{x+a-2b}.$$

or $$\frac{(2x-a-b)\{(x-a)^2-(x-a)(x-b)+(x-b)^2\}}{(x-b)^3}=\frac{2x-a-b}{x+a-2b}.$$

$\therefore\ 2x-a-b=0$, Art. 357.

$\therefore\ 2x=a+b$ and $\therefore\ x=\frac{1}{2}(a+b)$.

The other equation, namely,

$$\{(x-a)^2-(x-a)(x-b)+(x-b)^2\}(x+a-2b)=(x-b)^3,$$

when simplified, will be found not to contain x at all.

Exercise CXXXIII.

Solve the following equations :—

1. $a-\frac{bx}{a}=\frac{ax-b^2}{c}$. **2.** $\frac{a}{x}+\frac{b}{c}-\frac{d}{e}=0$. **3.** $\frac{x}{a}+\frac{x}{b}=c$.

4. $\frac{ax+b}{c}-\frac{a}{b}=\frac{cx+d}{e}$. **5.** $\frac{a(d^2+x^2)}{dx}=ac+\frac{ax}{d}$.

6. $\frac{a}{bx}-\frac{b}{ax}=a^2-b^2$. **7.** $\frac{ax}{b}+\frac{cx}{f}=gx+\frac{1}{f}(fh-cx)$.

8. $\frac{ax}{b(x+c)}+\frac{bx}{a(x+c)}=1$. **9.** $\frac{1}{ab-ax}+\frac{1}{bc-bx}=\frac{1}{ac-ax}$. (B.M. 1890).

10. $\frac{6x+a}{4x+b}=\frac{3x-b}{2x-a}$. **11.** $\frac{2x+a}{3(x-a)}+\frac{3x-a}{2(x+a)}=2\frac{1}{6}$.

12. $\frac{x}{a+x}=\frac{a+x}{x}-\frac{2a-b}{2x}$. **13.** $\frac{p-q}{qx+r}=\frac{p+q}{px-r}$. (C. E. 1866).

14. $(a+x)(b+x)-a(b+c)=\frac{a^2c}{b}+x^2$. (B. M. 1868).

15. $\frac{x-a}{b-a}=\frac{x-b}{a-b}$. (C. E. 1866). **16.** $\frac{x}{2x-a}+\frac{x}{2x-b}=1$. (C. E. 1899).

17. $\frac{a-b}{x-c}=\frac{a+b}{x+2c}$. (P. E. 1862). **18.** $\frac{a}{bx-p}=\frac{b}{ax-p}$. (P. E. 1893

19. $\frac{a}{x-a}-\frac{b}{x-b}=\frac{a-b}{x-c}$. (C. E. 1859). **20.** $\frac{a}{x+a}+\frac{b}{x+b}=\frac{a+b}{x+c}$

21. $\frac{x-a}{b-a}+\frac{x-c}{b-c}=2$. (A. E. 1893). **22.** $\frac{1}{x+a}+\frac{1}{x+b}=\frac{1}{x+a+b}$

23. $\frac{a-x^2}{bx}-\frac{b-x}{c}=\frac{c-x}{b}-\frac{b-x^2}{cx}$. (C. E. 1886).

24. $\frac{x-a}{b}+\frac{x-b}{c}+\frac{x-c}{a}=\frac{x-(a+b+c)}{abc}$. (C. E. 1865 and M. M. 1863).

25. $\frac{1}{x+a}+\frac{1}{x+b}=\frac{1}{x+a+b}+\frac{1}{x}$. (A. E. 1894).

26. $\frac{1}{x-a}-\frac{1}{x-a+c}=\frac{1}{x-b-c}-\frac{1}{x-b}$. (C. E. 1890).

27. $\frac{x-a}{x-a-1}-\frac{x-a-1}{x-a-2}=\frac{x-b}{x-b-1}-\frac{x-b-1}{x-b-2}$.

28. $\frac{3abc}{a+b}+\frac{a^2b^2}{(a+b)^3}+\frac{(2a+b)b^2x}{a(a+b)^2}=3cx+\frac{bx}{a}$.

29. $\frac{1}{(a-b)(x-a)}-\frac{1}{(c-d)(x-c)}=\frac{1}{(a-b)(x-b)}-\frac{1}{(c-d)(x-d)}$.

(C. E. 1891).

30. $\frac{(x+a)(x+b)}{x+a+b}=\frac{(x+c)(x+d)}{x+c+d}$. (A. E. 1892).

31. $\left(\frac{2x-a}{2x-b}\right)^2=\frac{x-a}{x-b}$. **32.** $\left(\frac{x+a}{x-b}\right)^2=\frac{x+2a}{x-2b}$.

33. $\frac{(x-a)(x+b)}{x-a+b}=\frac{x(x-c)-b(x+c)}{x-b-c}$. (M. M. 1886).

34. $\frac{1}{(x+a)^2-b^2}+\frac{1}{(x+b)^2-a^2}=\frac{1}{x^2-(a+b)^2}+\frac{1}{x^2-(a-b)^2}$.

35. $\frac{a+c}{x-2b}-\frac{b+c}{x-2a}=\frac{a-c}{x+2b}-\frac{b-c}{x+2a}$. (M. M. 1888).

36. $\frac{x-a}{3b+5c}+\frac{x-3b}{5c+a}+\frac{x-5c}{a+3b}=3$.

37. $\dfrac{a+x}{a^2+ax+x^2}+\dfrac{a-x}{a^2-ax+x^2}=\dfrac{2a^4}{x(a^4+a^2x^2+x^4)}$. (B. M. 1889).

38. $\dfrac{x+n}{x+m}=\left(\dfrac{2x+n+r}{2x+m+r}\right)^2$. **39.** $16\left(\dfrac{a-x}{a+x}\right)^3=\dfrac{a+x}{a-x}$. (C. E. 1886).

40. $\dfrac{a}{x+a}+\dfrac{b}{x+b}=\dfrac{a-c}{x+a-c}+\dfrac{b+c}{x+b+c}$. (B. M. 1882).

$\dfrac{x+4a+b}{x+a+b}+\dfrac{4x+a+2b}{x+a-b}=5$. **42.** $\dfrac{mx-a-b}{nx-c-d}=\dfrac{mx-a-c}{nx-b-d}$.

III. EQUATIONS INVOLVING SURDS.

366. The following general method is observed in solving **Irrational Equations** :—

Rule. *Transpose to one side of the equation a single radical term by itself, and then square or cube (as the case may be) both sides to get rid of this radical. Repeat this process until any remaining radicals in turn are removed.*

Ex. 1. Solve $\sqrt{(12+x)}=2+\sqrt{x}$.

Squaring both sides of the equation, we have

$$12+x=4+4\sqrt{x}+x\ ;\ \therefore\ 4\sqrt{x}=8, \text{ (by transposition)}.$$

Dividing by 4, $\sqrt{x}=2$, and $\therefore x=4$ (by squaring).

Ex. 2. Solve $3+x-\sqrt{(x^2+9)}=2$.

Transposing, $\sqrt{(x^2+9)}=1+x$;

Squaring both sides, $x^2+9=1+2x+x^2$;

Transposing and taking away x^2, we have

$$2x=8 \text{ and } \therefore x=4.$$

Ex. 3. Solve $\sqrt[m]{(a+x)}=\sqrt[2m]{(x^2+8ax+b^2)}$. (B. M. 1865).

Raising both sides to the $2m$th power, we have

$$\{(a+x)^{\frac{1}{m}}\}^{2m}=\{(x^2+8ax+b^2)^{\frac{1}{2m}}\}^{2m},$$

or $(a+x)^2=x^2+8ax+b^2$; $\therefore\ a^2+2ax+x^2=x^2+8ax+b^2$;

Cancelling x^2 and transposing, we obtain

$$6ax=a^2-b^2, \text{ and } \therefore\ x=\frac{a^2-b^2}{6a}.$$

Ex. 4. Solve $\sqrt{(2x+3)}=\dfrac{5}{\sqrt{(2x+3)}}+\sqrt{(2x+7)}$.

Clearing of fractions, *i.e.* multiplying by $\sqrt{(2x+3)}$,

$$2x+3=5+\sqrt{(2x+7)(2x+3)}\ ;\ \therefore\ \sqrt{(2x+7)(2x+3)}=2x-2\ ;$$

Squaring, $4x^2+20x+21=4x^2-8x+4$;

Taking away $4x^2$ and transposing, we obtain,

$$28x=-17, \text{ and } \therefore x=-\tfrac{17}{28}.$$

Ex. 5. Solve $\sqrt{(4x+5)}-\sqrt{x}=\sqrt{(x+3)}$.

Transposing, $\sqrt{(4x+5)}=\sqrt{x}+\sqrt{(x+3)}$.

Squaring, $4x+5=x+2\sqrt{(x^2+3x)}+x+3$

Transposing, $2\sqrt{(x^2+3x)}=2(x+1)$.

Dividing by 2 and squaring again, we have

$$x^2+3x=x^2+2x+1\ ;\ \therefore\ x=1, \text{ (by transposition).}$$

Exercise CXXXIV.

Solve the following equations :—

1. $15+\sqrt{(x+7)}=19$. (C.E. 1880).

2. $5+7\sqrt{(\frac{2}{3}x-6)}=19$. (C.E. 1859).

3. $\sqrt{(3x)}-4=\sqrt{(3x+4)}$. (C. E. 1863).

4. $\sqrt{(5x-1)}=1+\sqrt{(5x-2)}$. (C. E. 1875).

5. $\sqrt{5(x+2)}=\sqrt{5x}+2$.

6. $\frac{1}{3}\sqrt[3]{(17x-26)}+\frac{3}{8}=1\frac{1}{24}$.

7. $\sqrt{(x+9)}=1+\sqrt{x}$. (C. E. 1861-62).

8. $\sqrt{(2x+10)}+2\sqrt{(x+6)}=2$.

9. $\dfrac{8x+4}{\sqrt{(x+5)}}=4\sqrt{(x+5)}$. (C. E. 1868).

10. $\sqrt{x}-\sqrt{(4+x)}=\dfrac{2}{\sqrt{x}}$. (C. E. 1873).

11. $\sqrt{x}-\sqrt{(a+x)}=\sqrt{\dfrac{a}{x}}$.

12. $\sqrt{x}+\sqrt{(a+x)}=\dfrac{na}{\sqrt{(a+x)}}$.

13. $\sqrt{(bx)}+\sqrt{b(a+x)}=\sqrt{x}$.

14. $\sqrt{(bx+x^2)}=1+x$.

15. $\sqrt{(x-a)}=\sqrt{x+\sqrt{(b+x)}}$.

16. $a+x-\sqrt{(a^2+x^2)}=b$.

17. $a+x-\sqrt{(2ax+x^2)}=b$.

18. $a+x+\sqrt{(a^2+bx+x^2)}=b$.

19. $-\sqrt{x}+\sqrt{x+2\sqrt{(ax+a^2)}}=\sqrt{a}$.

20. $x+5=\sqrt{(x+5)}+6$.

21. $\sqrt{(9x+1)}+\sqrt{(4x+1)}=\sqrt{(x+1)}$.

22. $\sqrt{(x^2+x-2)}=3-x$.

23. $\sqrt{(5x+9)}-\sqrt{(2x-12)}=\sqrt{(3x+1)}$.

24. $x+\sqrt{(x^2+9)}=9$. (M. M. 1889).

25. $x-k+\sqrt{(k^2+x^2)}=m$. (C. E. 1879). **26.** $x-\sqrt{(2ax+x^2)}=a$.

27. $\sqrt{\{(x-a)^2+2ab+b^2\}}=x-a+b$. (B. M. 1886).

28. $x+\sqrt{(a^2+x^2)}=\dfrac{5a^2}{\sqrt{(a^2+x^2)}}$. **29.** $\sqrt{x}+\sqrt{(x+9)}=\dfrac{45}{\sqrt{(x+9)}}$.

30. $2(x+2)=1+\sqrt{(4x^2+9x+14)}$. (C. E. 1877).

31. $\sqrt{(x^2+11x+20)}-\sqrt{(x^2+5x-1)}=3$. (C. E. 1881).

32. $\sqrt{\{4x^2+20x+17-\sqrt{(16x^2+11x+10)}\}}=2(x+2)$. (C. E. 1878).

33. $\sqrt{(a^2+x^2)}+\sqrt{(a^2-x^2)}=d$. (P. E. 1892).

34. $\sqrt{(x+5)}+\sqrt{(x+6)}=\sqrt{3(x+11)}$.

35. $\dfrac{1}{\sqrt{x}}+\dfrac{1}{\sqrt{a}}=\sqrt{\left\{\dfrac{1}{a}+\sqrt{\left(\dfrac{4}{ax}+\dfrac{9}{x^2}\right)}\right\}}$.

367. Method of Identity. The following are typical examples.

Ex. 1. Solve $\sqrt{(4x+5)}+\sqrt{(4x-11)}=2$. . . . .(1)

We have $(4x+5)-(4x-11)=16$, (identically)

i.e. $\{\sqrt{(4x+5)}\}^2-\{\sqrt{(4x-11)}\}^2=16$.

Dividing this by the original equation, we get

$\sqrt{(4x+5)}-\sqrt{(4x-11)}=8$. . . . .(2)

Adding (1) and (2), $2\sqrt{(4x+5)}=10$;

Dividing by 2 and squaring, we have

$4x+5=25$; $\therefore$ $4x=20$ and $\therefore$ $x=5$.

Ex. 2. Solve $\sqrt{(2x+1)}-\sqrt{(3x+2)}=\sqrt{(4x+3)}-\sqrt{(5x+4)}$....(1)

Since $(2x+1)-(3x+2)=-(x+1)=(4x+3)-(5x+4)$ (identically)

$\therefore$ Dividing the above by the given equation, we have

$\sqrt{(2x+1)}+\sqrt{(3x+2)}=\sqrt{(4x+3)}+\sqrt{(5x+4)}$....(2)

Adding (1) and (2), we obtain $2\sqrt{(2x+1)}=2\sqrt{(4x+3)}$;

Dividing by 2 and squaring, $2x+1=4x+3$;

$\therefore$ $2x=-2$ and $\therefore$ $x=-1$.

368. Rationalisation of Denominator. The following is an illustrative example.

Ex. 1. Solve $\dfrac{\sqrt{(1+x)}-1}{\sqrt{(1-x)}+1}+\dfrac{\sqrt{(1-x)}+1}{\sqrt{(1+x)}-1}=4.$

Rationalising the denominator of each, we obtain

$$\frac{\{\sqrt{(1+x)}-1\}\{\sqrt{(1-x)}-1\}}{(1-x)-1}+\frac{\{\sqrt{(1-x)}+1\}\{\sqrt{(1+x)}+1\}}{(1+x)-1}=4,$$

or $$\frac{\sqrt{(1-x^2)}-\sqrt{(1-x)}-\sqrt{(1+x)}+1}{-x}+$$

$$\frac{\sqrt{(1-x^2)}+\sqrt{(1+x)}+\sqrt{(1-x)}+1}{x}=4.$$

Multiplying by x, and adding up, we have

$$2\sqrt{(1+x)}+2\sqrt{(1-x)}=4x, \text{ or } \sqrt{(1+x)}+\sqrt{(1-x)}=2x.$$

Squaring, $2+2\sqrt{(1-x^2)}=4x^2$, or $2\sqrt{(1-x^2)}=4x^2-2.$

Dividing by 2 and squaring again, we get

$$1-x^2=4x^4-4x^2+1\ ;\ \therefore\ 4x^4=3x^2\ ;\ x^2=0 \text{ or } 4x^2=3.$$

$$\therefore\ x=0 \text{ or } x=\pm\tfrac{1}{2}\sqrt{3}.$$

369. Componendo and Dividendo. *If* $\dfrac{a}{b}=\dfrac{c}{d}$, *then will*

$$\frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

The following is a typical example.

Ex. 1. Solve $\dfrac{a+x+\sqrt{(2ax+x^2)}}{a+x-\sqrt{(2ax+x^2)}}=b^2.$

By Componendo and Dividendo,

$$\frac{2(a+x)}{2\sqrt{(2ax+x^2)}}=\frac{b^2+1}{b^2-1}, \text{ or } \frac{a+x}{\sqrt{(2ax+x^2)}}=\frac{b^2+1}{b^2-1}.$$

Squaring, $\dfrac{a^2+2ax+x^2}{2ax+x^2}=\left(\dfrac{b^2+1}{b^2-1}\right)^2$; $\therefore\ \dfrac{a^2}{2ax+x^2}+1=\left(\dfrac{b^2+1}{b^2-1}\right)^2$;

$$\therefore\ \frac{a^2}{2ax+x^2}=\left(\frac{b^2+1}{b^2-1}\right)^2-1=\frac{4b^2}{(b^2-1)^2}.$$

Inverting the terms, $\dfrac{2ax+x^2}{a^2}=\dfrac{(b^2-1)^2}{4b^2}.$

Adding 1 to both sides, we obtain

$$\frac{a^2+2ax+x^2}{a^2}=\frac{(b^2-1)^2}{4b^2}+1, \text{ or } \left(\frac{a+x}{a}\right)^2=\frac{(b^2+1)^2}{4b^2}.$$

Ext. the sq. root, $\frac{a+x}{a}=\frac{b^2+1}{2b}$, or $1+\frac{x}{a}=\frac{b^2+1}{2b}$.

Transposing, $\frac{x}{a}=\frac{b^2+1}{2b}-1=\frac{(b-1)^2}{2b}$, and $\therefore\ x=\frac{a}{2b}(b-1)^2$

Exercise CXXXV.

Solve the following equations :—

1. $\sqrt{(5x-14)}-\sqrt{(5x-21)}=1.$ 2. $\sqrt{x}-\sqrt{(x-16)}=2$

3. $\sqrt{(x-2)}+\sqrt{(3x-15)}=\sqrt{(3x-10)}-\sqrt{(x+7)}$

4. $\sqrt{(x^2+9)}+\sqrt{(x^2-9)}=4+\sqrt{(34)}$

5. $\left(\frac{a^2}{x}+b\right)^{\frac{1}{2}}-\left(\frac{a^2}{x}-b\right)^{\frac{1}{2}}=c^{\frac{1}{2}}.$ (C. E 1865).

6. $\frac{1}{\sqrt{(1-x)}+1}+\frac{1}{\sqrt{(1+x)}-1}=\frac{1}{x}.$

7. $\frac{\sqrt{(x+1)}-\sqrt{(x-1)}}{\sqrt{(x+1)}+\sqrt{(x-1)}}=\frac{1}{2}.$ (B. M. 1863). 8. $\frac{\sqrt{x}+\sqrt{(3x-1)}}{\sqrt{x}-\sqrt{(3x-1)}}=1$

9. $\frac{\sqrt{(2x+1)}+\sqrt{(3x-2)}}{\sqrt{(2x+1)}-\sqrt{(3x-2)}}=\frac{1}{7}.$ 10. $\frac{a-\sqrt{(2ax-x^2)}}{a+\sqrt{(2ax-x^2)}}=b.$

11. $\frac{\sqrt{a}-\sqrt{\{a-\sqrt{(a^2-ax)}\}}}{\sqrt{a}+\sqrt{\{a-\sqrt{(a^2-ax)}\}}}=b.$

12. $\frac{1+x}{1+x+\sqrt{(1+x^2)}}+\frac{1-x}{1-x+\sqrt{(1+x^2)})}=a.$

13. $\frac{1}{\sqrt{x}-\sqrt{(x-2)}}+\frac{1}{\sqrt{x}+\sqrt{(x+2)}}=1.$ (C. F. A. 1882).

14. $\sqrt{(3x^2+16)}-\sqrt{(3x^2-16)}=8-4\sqrt{2}.$

15. $\sqrt{\{a+\sqrt{(a^2-x^2)}\}}+\sqrt{\{a-\sqrt{(a^2-x^2)}\}}=n\sqrt{\left\{\frac{a+x}{a+\sqrt{(a^2-x^2)}}\right\}}.$

370. Special Artifices. Some equations require special artifices for their solution.

Ex. 1. Solve $\dfrac{5x-9}{3+\sqrt{(5x)}}=1+\dfrac{\sqrt{(5x)}-3}{2}$.

We have $\dfrac{5x-9}{3+\sqrt{(5x)}}=\dfrac{\{\sqrt{(5x)}+3\}\{\sqrt{(5x)}-3\}}{3+\sqrt{(5x)}}=\sqrt{(5x)}-3$.

Hence the equation reduces to

$$\sqrt{(5x)}-3=1+\frac{\sqrt{(5x)}-3}{2}; \;\therefore\; \frac{\sqrt{(5x)}-3}{2}=1.$$

$\therefore\; \sqrt{(5x)}-3=2$, or $\sqrt{(5x)}=5$.

Squaring $5x=25$ and $\therefore\; x=5$.

Ex. 2. Solve $\sqrt[3]{(a+\sqrt{x})}+\sqrt[3]{(a-\sqrt{x})}=\sqrt[3]{b}$.

Cubing both sides, we get

$$a+\sqrt{x}+a-\sqrt{x}+3\sqrt[3]{\{(a+\sqrt{x})(a-\sqrt{x})\}}\times\{\sqrt[3]{(a+\sqrt{x})}+\sqrt[3]{(a-\sqrt{x})}\}=b.$$

Substituting $\sqrt[3]{b}$ for $\sqrt[3]{(a+\sqrt{x})}+\sqrt[3]{(a-\sqrt{x})}$,

$$2a+3\sqrt[3]{(a^2-x)b}=b, \text{ or } 3\sqrt[3]{(a^2-x)b}=b-2a.$$

Cubing, $27(a^2-x)b=(b-2a)^3$; or $a^2-x=\dfrac{(b-2a)^3}{27b}$.

$$\therefore\; x=a^2-\frac{(b-2a)^3}{27b}=\frac{8a^3+15a^2b+6ab^2-b^3}{27b}.$$

Ex. 3. Solve $\dfrac{a+\sqrt{(a^2-x^2)}}{a-\sqrt{(a^2-x^2)}}=64\cdot\dfrac{\sqrt{(a+x)}-\sqrt{(a-x)}}{\sqrt{(a+x)}+\sqrt{(a-x)}}$.

Since $\dfrac{a+\sqrt{(a^2-x^2)}}{a-\sqrt{(a^2-x^2)}}=\dfrac{2a+2\sqrt{(a^2-x^2)}}{2a-2\sqrt{(a^2-x^2)}}=\left\{\dfrac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}-\sqrt{(a-x)}}\right\}^2$,

therefore multiplying both sides by $\dfrac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}-\sqrt{(a-x)}}$,

we get $\left\{\dfrac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}-\sqrt{(a-x)}}\right\}^3=64$.

Extracting the cube root, $\dfrac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}-\sqrt{(a-x)}}=4$.

$$\frac{\sqrt{(a+x)}}{\sqrt{(a-x)}}=\frac{5}{3}; \text{ squaring } \frac{a+x}{a-x}=\frac{25}{9};$$

By Art. 369, $\dfrac{a}{x}=\dfrac{34}{16}$, and $\therefore\; x=\dfrac{16}{34}a$.

Ex. 4. Solve $\dfrac{9x+3}{\sqrt{(9x+7)}+2}+\dfrac{9x-9}{\sqrt{(9x+16)}-5}=6.$

Since $9x+3=(9x+7)-4=\{\sqrt{(9x+7)}+2\}\{\sqrt{(9x+7)}-2\}$;

and $9x-9=(9x+16)-25=\{\sqrt{(9x+16)}+5\}\{\sqrt{(9x+16)}-5\}$;

Therefore the equation reduces to

$$\sqrt{(9x+7)}-2+\sqrt{(9x+16)}+5=6,$$

$$\text{or } \sqrt{(9x+16)}+\sqrt{(9x+7)}=3 \quad \ldots\ldots(1)$$

Now, since $(9x+16)-(9x+7)=9$, (identically).

Therefore dividing the above by the given equation,

$$\sqrt{(9x+16)}-\sqrt{(9x+7)}=3 \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Hence, adding (1) and (2), we obtain,

$$2\sqrt{(9x+16)}=6, \text{ or } \sqrt{(9x+16)}=3.$$

Squaring, $9x+16=9$; $\therefore$ $9x=-7$, and $\therefore$ $x=-\frac{7}{9}$.

Ex. 5. Solve $\sqrt{(x^2+39x+374)}-\sqrt{(x^2+20x+51)}$

$$=\tfrac{19}{2}\sqrt{\left(\frac{x+22}{x+17}\right)}.$$

Transposing, $\sqrt{(x^2+39x+374)}-\tfrac{19}{2}\sqrt{\left(\dfrac{x+22}{x+17}\right)}$

$$=\sqrt{(x^2+20x+51)}.$$

Squaring and observing that $x^2+39x+374=(x+22)(x+17)$.

$$x^2+39x+374-19(x+22)+\tfrac{361}{4}\left(\frac{x+22}{x+17}\right)=x^2+20x+51.$$

Transposing and simplifying, we have

$$\tfrac{361}{4}\left(\frac{x+22}{x+17}\right)=95, \text{ or } \frac{x+22}{x+17}=95\times\tfrac{4}{361}=\tfrac{20}{19}.$$

$$\therefore\ 19x+418=20x+340 \text{ and } \therefore\ x=78.$$

Ex. 6. Solve $\sqrt{\left(\dfrac{5}{3+x}\right)}+\sqrt{\left(\dfrac{3}{3-x}\right)}=2\sqrt[4]{\left(\dfrac{15}{9-x^2}\right)}.$

Transposing, $\sqrt{\left(\dfrac{5}{3+x}\right)}+\sqrt{\left(\dfrac{3}{3-x}\right)}-2\sqrt[4]{\left(\dfrac{15}{9-x^2}\right)}=0$

Extracting the square root, $\sqrt[4]{\left(\dfrac{5}{3+x}\right)}-\sqrt[4]{\left(\dfrac{3}{3-x}\right)}=0.$

Transposing and raising both sides to the 4th power,

$$\frac{5}{3+x}=\frac{3}{3-x}\,;\ \therefore\ \frac{3+x}{3-x}=\frac{5}{3}.$$

By Art. 369, $\frac{3}{x}=\frac{8}{2}=4$; and $\therefore\ x=\frac{3}{4}$.

Ex. 7. Solve $\frac{1}{a}\sqrt{(a+x)}+\frac{1}{x}\sqrt{(a+x)}=\frac{1}{b}\sqrt{x}$.

Here, $\left(\frac{1}{a}+\frac{1}{x}\right)\sqrt{(a+x)}$ or $\frac{a+x}{ax}\sqrt{(a+x)}=\frac{1}{b}\sqrt{x}$.

$$\therefore\ \frac{(a+x)^{\frac{3}{2}}}{ax}=\frac{x^{\frac{1}{2}}}{b},\ \text{or}\ (a+x)^{\frac{3}{2}}=\frac{a}{b}x^{\frac{3}{2}}.$$

Squaring and taking the cube root, we have

$$a+x=\left(\frac{a}{b}\right)^{\frac{2}{3}}.x\,;\ \therefore\ a=\left\{\left(\frac{a}{b}\right)^{\frac{2}{3}}-1\right\}x\,;\ \therefore\ x=\frac{ab^{\frac{2}{3}}}{a^{\frac{2}{3}}-b^{\frac{2}{3}}}.$$

Ex. 8. Solve $6\sqrt{(5x)}+\frac{200+120\sqrt{(5x)}}{9x-5}=9x+5$.

Transposing and resolving into factors, we have

$$\frac{40\sqrt{5}(\sqrt{5}+3\sqrt{x})}{(3\sqrt{x}+\sqrt{5})(3\sqrt{x}-\sqrt{5})}=9x-6\sqrt{(5x)}+5,$$
$$=(3\sqrt{x}-\sqrt{5})^2.$$

Cancelling the common factor and clearing of fractions,

$$40\sqrt{5}=(3\sqrt{x}-\sqrt{5})^3\ \text{or}\ (2\sqrt{5})^3=(3\sqrt{x}-\sqrt{5})^3.$$

Taking the cube root of both sides, $2\sqrt{5}=3\sqrt{x}-\sqrt{5}$;

Transposing, $3\sqrt{x}=3\sqrt{5}$; $\therefore\ \sqrt{x}=\sqrt{5}$ and $\therefore\ x=5$.

Exercise CXXXVI.

Solve the following equations :—

1. $\frac{x-ax}{\sqrt{x}}=\frac{\sqrt{x}}{x}$. (C. E. 1885).

2. $\frac{ax-1}{\sqrt{(ax)}+1}=4+\frac{\sqrt{(ax)}-1}{2}$. (C. E. 1885).

3. $\dfrac{ax-b^2}{\sqrt{(ax)}+b}=\dfrac{\sqrt{(ax)}-b}{n}-c.$ 4. $\sqrt{\left(\dfrac{x+a}{x+b}\right)}=1+\dfrac{a-b}{2(x+c)}.$

5. $\dfrac{\sqrt{x}+a}{(\sqrt{x}-b)(\sqrt{x}-c)}+\dfrac{\sqrt{x}+b}{(\sqrt{x}-a)(\sqrt{x}-c)}+\dfrac{\sqrt{x}+c}{(\sqrt{x}-a)(\sqrt{x}-b)}=0.$ (C. E. 1867).

6. $\sqrt[3]{(1+x)}+\sqrt[3]{(1-x)}=\sqrt[3]{2}.$ (C. E. 1885).

7. $\sqrt[3]{(a+x)}+\sqrt[3]{(a-x)}=b.$

8. $\sqrt{\{(2a+x)^2+b^2\}}+\sqrt{\{(2a-x)^2+b^2\}}=2a.$

9. $\dfrac{a^2-x^2}{a+\sqrt{(a^4+x^4)}}=b-\sqrt{(a^2+x^2)}.$

10. $\sqrt{\{a^2x^2+b\sqrt{(4a^2x^2+abx+b\sqrt{9a^2x^2+2abx})}\}}=ax+b.$

11. $\dfrac{x-\sqrt{(x^2-a^2)}}{\sqrt{(x+a)}+\sqrt{(x-a)}}=2\sqrt{a}.$ 12. $\dfrac{1+\sqrt{x}+x}{1-\sqrt{x}+x}=\dfrac{62}{63}\cdot\dfrac{1+\sqrt{x}}{1-\sqrt{x}}.$

13. $\sqrt{(x^2+33x+260)}-\sqrt{(x^2+18x+35)}=\frac{15}{2}\sqrt{\left(\dfrac{x+20}{x+13}\right)}.$

14. $\dfrac{1}{a}\sqrt[4]{(a+x)}+\dfrac{1}{x}\sqrt[4]{(a+x)}=\dfrac{1}{b}\sqrt[4]{x}.$

15. $\sqrt{(1+a)}\left(\dfrac{1+x}{1-x}\right)^{\frac{1}{4}}+\sqrt{(1-a)}\left(\dfrac{1-x}{1+x}\right)^{\frac{1}{4}}=2\sqrt[4]{(1-a^2)}.$

16. $\sqrt{(1+x+x^2)}+\sqrt{(1-x+x^2)}=2k.$

17. $\left(\dfrac{x}{a}+\dfrac{a}{b}\right)^{\frac{1}{2}}+9\left(\dfrac{x}{a}-\dfrac{a}{b}\right)^{\frac{1}{2}}=6\left(\dfrac{x^2}{a^2}-\dfrac{a^2}{b^2}\right)^{\frac{1}{4}}.$

18. $\dfrac{1-5x}{1+5x}\sqrt{\left(\dfrac{1+9x}{1-9x}\right)}=1.$ 19. $\dfrac{2a\sqrt{(1+x^2)}}{x+\sqrt{(1+x^2)}}=a+b.$

20. $\dfrac{1+x+\sqrt{(2x+x^2)}}{1+x-\sqrt{(2x+x^2)}}=8.\dfrac{\sqrt{(2+x)}-\sqrt{x}}{\sqrt{(2+x)}+\sqrt{x}}.$

21. $\dfrac{x-1}{\sqrt{x}+1}=1+\dfrac{\sqrt{x}-1}{2}.$ (M. M. 1866).

22. $\dfrac{3x-1}{\sqrt{(3x)}-1}=1+\dfrac{\sqrt{(3x)}-2}{2}.$ (B. M. 1875).

23. $\dfrac{x^2-a}{x-\sqrt{a}}+\dfrac{x^2-a}{x+\sqrt{a}}=x-\dfrac{x-\sqrt{a}}{2}.$ (M. M. 1864).

24. $a\sqrt{\left(\frac{1+x}{1-x}\right)}+(a+2)\sqrt{\left(\frac{1-x}{1+x}\right)}=2\sqrt{\{a(a+2)\}}$. (M. M. 1869).

25. $\sqrt{(15-9x)}+\sqrt{(10-4x)}=\sqrt{(5-x)}$. (M. M. 1874).

26. $\sqrt{\left(\frac{x-a}{x-b}\right)}+\frac{a}{x}=\sqrt{\left(\frac{x-b}{x-a}\right)}+\frac{b}{x}$. (M. M. 1875).

27. $\sqrt{\frac{x}{a}}+\sqrt{\left\{\frac{(b-c)(ac-bx)+ac^2}{abc}\right\}}=1$.

IV. EXPONENTIAL EQUATIONS.

371. In Exponential equations, the unknown quantity or quantities enter as an index or exponent of a quantity.

Thus, $a^x=b$, is an *Exponential equation.*

372. The simplest cases of Exponential equations that we shall here consider present the following obvious facts :—

(1) *If* $a^x=a^c$, *then* $x=c$.

(2) *If* $a^x=1$, *then* $x=0$, *for* $a^0=1$. (Art. 106).

Ex. 1. Solve $3^{x-4}=27$.

Since $27=3^3$, $\therefore\ 3^{x-4}=3^3$ and $\therefore\ x-4=3$.

$\therefore\ x=4+3=7$.

Ex. 2. Solve $2^{x+2}.3^{x+3}=648$.

Since $648=8\times 81=2^3\times 3^4$, $\therefore\ 2^{x+2}.3^{x+3}=2^3.3^4$.

$\therefore\ \frac{2^{x+2}}{2^3}\times\frac{3^{x+3}}{3^4}=1$, or $2^{x-1}\times 3^{x-1}=1$.

$\therefore\ (2.3)^{x-1}=1$; $\therefore\ x-1=0$ and $\therefore\ x=1$.

Ex. 3. Solve $\left.\begin{array}{l}a^x.a^{y+1}=a^7 \ldots\ldots (1)\\ a^{2y}.a^{3x+5}=a^{20} \ldots\ldots (2)\end{array}\right\}$. (C. E. 1879).

$$\left.\begin{array}{ll}\text{From (1), } a^{x+y+1}=a^7, & \therefore\ x+y+1=7 \text{ or } x+y=6 \ldots\ldots\ldots(3)\\ \text{From (2) } a^{2y+3x+5}=a^{20}, & \therefore\ 2y+3x+5=20, \text{ or } 3x+2y=15\ldots(4)\end{array}\right\}$$

From (4) subtract twice (3), thus $x=3$.

Hence, from (3) $y=6-x=3$. Thus $x=y=3$.

Ex. 4. Solve $\left.\begin{array}{l} 2^{x+1}+3^y=97 \;\ldots\ldots\;(1) \\ 2^x+3^{y+2}=737 \ldots\ldots\ldots(2) \end{array}\right\}$

From (1) $2.2^x+3^y=97$, and from (2) $2^x+9\,3^y=737$.

Now writing X for 2^x and Y for 3^y, we have

$$\left.\begin{array}{r} 2X+Y=97 \\ \text{and } X+9Y=737 \end{array}\right\} \text{ whence } X=8,\ Y=81.$$

Therefore $2^x=8=2^3$; so that $x=3$.

Similarly, $3^y=81=3^4$; so that $y=4$.

Exercise CXXXVII.

Solve the following equations :—

1. $2^{x+2}=32.$ 2. $3.2^{x+3}=48$ 3. $a^{mx-n}=1.$

4. $a^{mx-p}=b^{mx-p}.$ 5. $3^{x-2}=9a^{x-4}.$ 6. $2^{x+1}-2^x-16=0$

7. $4^{x+2}.5^{x+3}=40000.$ 8 $3^x=9^{y-1},\ 16^{y-x}=8^{y-2}.$

9. $\left.\begin{array}{r} a^{x+1}.a^{2y+3}=a^{17} \\ a^{2x}.a^{y+1}=a^{15} \end{array}\right\}$ 10 $\left.\begin{array}{r} 4^{2y-1}=16^{x+y} \\ 3^{x+3y}=9^{2x+3} \end{array}\right\}$ 11. $\left.\begin{array}{r} 2^x.4^y=1024 \\ 3^x\div 9^y=\frac{1}{9} \end{array}\right\}$

12. $\left.\begin{array}{r} 2^x=8^{y+1} \\ 9^y=3^{x-9} \end{array}\right\}$ 13. $\left.\begin{array}{r} 2^{2x}=16\times 4^{x+y} \\ x=6y \end{array}\right\}$ 14. $\begin{array}{r} 4^x.2^y=2^{10} \\ 2^{7x+2}.16^{2y}=2^{55} \end{array}$

15. $a^{-x}(a^x+b^{-x})=\dfrac{a^2b^2+1}{a^2b^2}.$ 16. $7^{\left(\frac{x}{2}-\frac{y}{3}\right)}=2401,\ 6^{\left(\frac{x}{4}+\frac{y}{2}\right)}=1296.$

17. $\left.\begin{array}{r} a^{2x}.a^{1-y}=a^{10} \\ a^{2x}.a^{5-7y}=a^{24} \end{array}\right\}$ 18. $\left.\begin{array}{r} 3^{x+1}+2^y=85 \\ 3^x-2^{y+2}=11 \end{array}\right\}$ 19. $\left.\begin{array}{r} x^{2x+3}=4y^x \\ x^{y-4}=1 \end{array}\right\}$

20. $z^x=y^{2x},\ 2^z=2\times 4^x,\ x+y+z=16.$

V. HARDER PROBLEMS.

373. We shall add here a few Problems of a harder type than those considered in Art. 162 with their solutions.

Ex. 1. A farmer bought equal numbers of two kinds of sheep, one kind at *Rs.*6 each, the other at *Rs* 8 each ; if he had expended his money equally in the two kinds, he would have had three sheep more than he did. How many of each kind did he buy ? (C. E. 1898)

Let x be the number of each kind of sheep bought.

Then $2x$ is the total number of sheep bought.

The sum expended on one kind $=Rs.6x$ and that on the other kind $=Rs.8x$, so that the whole sum expended $=Rs.(6x+8x)=Rs.14x$

If he had expended the sum equally in the two kinds, each kind would have cost him *Rs.*$7x$, and then he could have bought $\frac{7x}{6}$ sheep of the first kind and $\frac{7x}{8}$ sheep of the second kind; thus the total number bought would have been $\left(\frac{7x}{6}+\frac{7x}{8}\right)$ sheep.

Hence, by the question, we have

$$\frac{7x}{6}+\frac{7x}{8}=2x+3, \text{ whence } x=72.$$

Thus the number of sheep bought of each kind$=72$.

Ex. 2. A bankrupt paid only 17*s.* 6*d.* in the pound to his creditors, and then gave $\frac{4}{5}$ of what he still owed to the lawyers. This left him £20 for his current expenses What was the amount of his debt? (C. E. 1886).

Let x be the amount of his debts in pounds.

Since, in £1 he paid 17*s.* 6*d.* or $17\frac{1}{2}$*s.*

$\therefore$ in £x,........£$\frac{17\frac{1}{2}}{20}x$ or £$\frac{7}{8}x$.

Thus, he still owed £$(x-\frac{7}{8}x)$ or £$\frac{1}{8}x$; but of this amount he gave away $\frac{4}{5}$ths,

$\therefore$ $\frac{1}{5}$ of £$\frac{1}{8}x$ or £$\frac{1}{40}x$ remained.

$\therefore$ By the question, we have

$\frac{1}{40}x=20$; whence $x=800$.

Hence the amount of debts$=$£800.

Ex. 3. Divide the number 127 into four such parts that the first increased by 18, the second diminished by 5, the third multiplied by 6, and the fourth divided by $2\frac{1}{2}$, shall be equal. (B. M. 1883).

Let x be the common result in each case.

Then, since the first part$+18=x$, $\therefore$ the first part$=x-18$.

The second part$-5=x$, $\therefore$ the second part$=x+5$.

The third part$\times 6=x$. $\therefore$ the third part$=\frac{1}{6}x$.

The fourth part$\div 2\frac{1}{2}=x$, $\therefore$ the fourth part$=2\frac{1}{2}\times x$

$\therefore$ By the question, we have

$$(x-18)+(x+5)+\tfrac{1}{6}x+2\tfrac{1}{2}\times x=127;$$

whence $x=30$, and the parts are 12, 35, 5, 75.

374. Work and Cistern. If an agent can do a work in m days, then the average daily work of the agent is $1/m$, taking *unity* to represent the work. Similarly, if an agent does $\frac{m}{n}$ths of any work in one day, he will do $\frac{1}{n}$th of it in $\frac{1}{m}$th of a day, and therefore the *whole* work in $\frac{n}{m}$ days.

Ex. 1. A and B together can do a piece of work in 15 days; A can do it alone in 24 days; how long would B take to do it alone? (C. E. 1877).

Let x be the no. of days B alone would take to do the work.

In one day, B does $\frac{1}{x}$th of the work and A does $\frac{1}{24}$th of the work. Therefore they together can do $\left(\frac{1}{x}+\frac{1}{24}\right)$ of the work in one day.

But, by the question, A and B together can do $\frac{1}{15}$ of the work in a day,

$$\therefore \frac{1}{x}+\frac{1}{24}=\frac{1}{15}\text{ ; whence } x=40.$$

Hence B can do the work alone in 40 days.

Ex. 2. A can do a piece of work in 10 days; but after he has been upon it 4 days, B is sent to help him, and they finish it together in 2 days. In what time would B have done the whole?

Let x be the no. of days B would have taken.

In one day A does $\frac{1}{10}$th of the work and B does $\frac{1}{x}$th of the work. Therefore in 4 days, A does $\frac{4}{10}$ or $\frac{2}{5}$th of the work and in 2 days, A and B together do $\frac{2}{10}+\frac{2}{x}$ or $\left(\frac{1}{5}+\frac{2}{x}\right)$th of the work.

$$\therefore \text{ By the question, } \frac{2}{5}+\left(\frac{1}{5}+\frac{2}{x}\right)=1\text{ ; whence } x=5.$$

Hence B can do the whole work alone in 5 days.

Ex. 3. A cistern can be filled in half-an-hour by a pipe A, and emptied in 20 min. by another pipe B. After A had been opened 20 min., B is also opened for 12 min., when A is closed, and B remains open for 5 min. more, and now there are 13 gallons in the cistern; how much would it contain when full?

Let x be the no. of gallons that would fill the cistern.

In one min. **A** brings in $\frac{1}{30}x$ gals., and **B** carries out $\frac{1}{20}x$ gals. Now **A** is opened altogether for 32 min. during which time it brings in $\frac{32}{30}x$ or $\frac{16}{15}x$ gals., and **B** for 17 min., during which time it carries out $\frac{17}{20}x$ gals.

∴ By the question, $\frac{16}{15}x - \frac{17}{20}x = 13$; whence $x = 60$.

Hence, the cistern can hold 60 gals. when full.

375. Percentage. Problems relating to *percentages* require a knowledge of *Profit and Loss.*

(i) If an article be sold at a *profit* of a per cent., then

$$\textit{the selling price} = \left(1 + \frac{a}{100}\right) \times \textit{cost.}$$

(ii) If an article be sold at a *loss* of a per cent., then

$$\textit{the selling price} = \left(1 - \frac{a}{100}\right) \times \textit{cost.}$$

Ex. 1. How much are eggs a score, when a rise of 20 per cent. in the price would make a difference of 4 scores in the number which could be bought for *Rs.*10.

Let x *as.* be the price of a score of eggs.

The number of scores of eggs which can be bought for *Rs.*10 $= \frac{10 \times 16}{x}$ or $\frac{160}{x}$.

On a rise of 20 per cent. in the price, the price of a score of eggs will be $x\left(1 + \frac{20}{100}\right)$ or $\frac{6}{5}x$ *as.*, and the number of scores of eggs which can be bought for *Rs* 10 $= \frac{10 \times 16}{\frac{6}{5}x}$ or $\frac{400}{3x}$.

∴ By the question, $\frac{160}{x} - \frac{400}{3x} = 4$; whence $x = 6\frac{2}{3}$.

Hence the price of a score of eggs is 6*a.* 8*p.*

Ex. 2. A person bought an article and sold it at a profit of 6 per cent. Had he bought it at 4 per cent. less, and sold at *Re.*1. 3*a.* more, his profit would have been 12 per cent. For how much did he buy it? (P. E. 1891).

Let x be the cost price in rupees.

The actual selling price at 6 per cent. profit

$$= Rs.\,x\left(1 + \frac{6}{100}\right) = Rs.\,\frac{53}{50}x.$$

If he had bought the article at 4 per cent. less, the cost price would have been $Rs. x(1-\frac{4}{100}) = Rs. \frac{24}{25}x$, and the selling price on this cost at 12 per cent. profit

$$= Rs. \frac{24}{25}x(1+\frac{12}{100}) = Rs. \frac{672}{625}x.$$

$\therefore$ By the question, $\frac{672}{625}x = \frac{53}{50}x + 1\frac{3}{16}$; whence $x = 78\frac{1}{8}$.

Hence the required cost is $Rs. 78.\ 2a.$

Ex. 3. How many bundles of hay at *Rs.*5. per thousand must a *ghaswala* mix with 5600 bundles at *Rs* 6 per thousand in order that he may gain 20 per cent. by selling the whole at 11 *as.* per hundred. (C. E. 1875).

Let x be the number of bundles required.

Price of x bundles at *Rs.*5 per 1000 $= Rs. \frac{5}{1000}x = Rs. \frac{1}{200}x$.

. ....5600........*Rs.*6........... $Rs. \frac{56}{10} \times 6 = Rs. \frac{168}{5}$.

$\therefore$ the total cost of $(x+5600)$ bundles $= Rs. (\frac{1}{200}x + \frac{168}{5})$.

The selling price at 20 per cent. profit on the above

$$= Rs. (\frac{1}{200}x + \frac{168}{5})(1+\frac{20}{100}) = Rs. \frac{6}{5}(\frac{1}{200}x + \frac{168}{5}).$$

But the selling price of $(x+5600)$ bundles at 11*as.* per hundred $= \frac{11}{100}(x+5600)$ *as.* $= Rs. \frac{11}{1600}(x+5600)$.

$\therefore$ By the question, $\frac{6}{5}(\frac{1}{200}x + \frac{168}{5}) = \frac{11}{1600}(x+5600)$;
whence $x = 2080$.

Hence the no. of bundles required $= 2080$.

376. Mixture. The following are illustrative Examples.

Ex. 1. There are two bars of metal, the first containing 14 oz. of silver and 6 of tin, the second containing 8 oz. of silver and 12 of tin; how much must be taken from each to form a bar of 20 oz. containing equal weights of silver and tin?

Let x be the no. of oz. to be taken from the first bar,

then $20-x$ is the no.................................second......

Now, since $14+6=20$, $\therefore \frac{14}{20}$ or $\frac{7}{10}$ of the first bar, and therefore of *every* oz. of it, is silver;

and since $8+12=20$, $\therefore \frac{8}{20}$ or $\frac{2}{5}$ of every oz. of the second bar is silver.

Thus, in x oz. of first bar, there are $\frac{7}{10}x$ silver and in $(20-x)$ oz. of second, there are $\frac{2}{5}(20-x)$ silver.

But there are to be altogether 10 oz. of silver in the compound,

$\therefore$ by the question, $\frac{7}{10}x+\frac{2}{5}(20-x)=10$;

whence $x=6\frac{2}{3}$, and $20-x=13\frac{1}{3}$.

Hence $6\frac{2}{3}$ oz. to be taken from first and $13\frac{1}{3}$ oz. from second.

Ex. 2. A vessel contains a mixture of wine and water, so that for every 5 gallons of wine there are 3 gallons of water; another vessel contains a mixture of 9 gallons of wine and 7 gallons of water. What quantity should be taken of each mixture so as to produce a new mixture of 30 gallons containing $1\frac{1}{2}$ times as much wine as water?

Let x be the no. of gallons to be taken from the first vessel.

then $30-x$ is the.... . ...second.......

Since $5+3=8$, in every 8 gals. from the first vessel, we take 5 gals. of wine and 3 gals. of water.

$\therefore$ in x gals. from the 1st, we take $\frac{5}{8}x$ gals. of wine and $\frac{3}{8}x$ gals. of water..

Again, since $9+7=16$, in every 16 gals. from the second vessel, we take 9 gals. of wine and 7 gals. of water.

$\therefore$ in $(30-x)$ gals. from the 2nd, we take $\frac{9}{16}(30-x)$ gals. of wine and $\frac{7}{16}(30-x)$ gals. of water.

Thus, wine in the new mixture $=\{\frac{5}{8}x+\frac{9}{16}(30-x)\}$ gals.

and water....... .... $=\{\frac{3}{8}x+\frac{7}{16}(30-x)\}$ gals.

$\therefore$ By the question, $\frac{5}{8}x+\frac{9}{16}(30-x)=1\frac{1}{2}\{\frac{3}{8}x+\frac{7}{16}(30-x)\}$;

whence $x=18$ and $30-x=12$.

Hence 18 gals. to be taken from 1st and 12 gals. from 2nd.

377. Provisions. In questions involving provisions, bear in mind that *the total quantity of food consumed = number of men × time × rate of allowance per man.*

Ex. 1. A garrison had sufficient provisions for 30 months, but at the end of 4 months the number of troops was doubled, and 3 months after, it was re-inforced with 400 men more, on which accounts the provisions lasted only 15 months altogether. Required the number of men in the garrison before the augmentation took place. (B. E. 1871).

Let x be the required no. of men,

and a the monthly allowance per man.

Then the total quantity of provisions $=30ax$.

Again, x men in the first 4 months consumed $4ax$.

$2x$ ....next 3 months consumed $6ax$.

and $2x+400$ ...... last 8 months consumed $8(2x+400)a$.

$\therefore$ By the question, $4ax+6ax+8(2x+400)a=30ax$;

Dividing both sides by a, $4x+6x+8(2x+400)=30x$;

whence $x=800$.

Hence, the no. of men in the garrison at first was 800.

Ex. 2. A ship left Bombay on a voyage of 3 weeks, with provisions for that time at the rate of 1 seer a day for each man. At the end of a week a storm arose which washed 4 men overboard and so damaged the vessel that the speed was reduced by half and each man could be allowed only $\frac{3}{4}$ of a seer *per diem*. What was the original number of the crew? (B. E. 1863).

Let x be the original number of the crew.

The total quantity of provisions in the ship $=3\times7\times1\times x$ seers $=21x$ seers.

Again, x men in the first week consumed $1\times7\times1\times x$ or $7x$ seers

and $x-4$ men in the remaining four weeks, which was increased from two to four, on account of the vessel's speed being diminished by half, consumed $4\times7\times\frac{3}{4}(x-4)$ or $21(x-4)$ seers.

$\therefore$ By the question, $21x=7x+21(x-4)$;

whence $x=12$.

Hence, the original number of the crew was 12.

378. Motion. In questions involving distance, time and rate, keep in mind that

time = distance/rate; distance = time × rate.

Ex. 1. A person has just a hours at his disposal. How far may he ride in a coach which travels b miles an hour, so as to return home in time, walking back at the rate of c miles an hour?

Let x be the required distance in miles.

Then $\frac{x}{b}=$ time in hours taken by the coach.

and $\frac{x}{c}=$ .... in walking back.

$\therefore$ By the question, $\frac{x}{b}+\frac{x}{c}=a$; whence $x=\frac{abc}{b+c}$.

Hence the required distance is $\frac{abc}{b+c}$ miles.

Ex. 2. A person walks from A to B, a distance of $7\frac{1}{2}$ miles, in 2 hours $17\frac{1}{2}$ minutes and returns in 2 hours 20 minutes, his rate of walking up-hill, down-hill and on a level road being 3, $3\frac{1}{2}$ and $3\frac{1}{4}$ miles per hour respectively. Find the length of the level road between A and B. (B. E. (1884).

Let x be the length of the level road between A and B in miles.

Then the remainder of the distance between A and B is $7\frac{1}{2}-x$ miles, which is partly up-hill and partly down-hill.

The whole time taken is 2 hrs. $17\frac{1}{2}$ min. + 2 hrs. 20 min. = 4 hrs. $37\frac{1}{2}$ min. = $4\frac{5}{8}$ hours, during which the person goes up-hill $7\frac{1}{2}-x$ miles, down-hill $7\frac{1}{2}-x$ miles and on the level road $2x$ miles.

The time to go $7\frac{1}{2}-x$ miles up-hill $=\dfrac{7\frac{1}{2}-x}{3}$ hours.

" " $7\frac{1}{2}-x$ miles down-hill $=\dfrac{7\frac{1}{2}-x}{3\frac{1}{2}}$ hours

" " $2x$ miles on the level $=\dfrac{2x}{3\frac{1}{4}}$ hours.

$\therefore$ By the question, $\dfrac{7\frac{1}{2}-x}{3}+\dfrac{7\frac{1}{2}-x}{3\frac{1}{2}}+\dfrac{2x}{3\frac{1}{4}}=4\frac{5}{8}$;

whence $x=4\frac{7}{8}$.

Hence the length of the level road is $4\frac{7}{8}$ miles.

Ex. 3. A hare is 50 of her own leaps before a greyhound; she takes 4 leaps for every 3 that he takes, but he covers as much ground in 2 leaps as she does in 3. How many leaps must the greyhound take to catch the hare?

Let $3x$ be the no. of leaps taken by the greyhound,

then $4x$ is the no. of leaps taken by the hare in the same time.

and $4x+50$ is the total no. of hare's leaps.

Again, let a denote the no. of inches in one leap of the hare,

then $3a$ is the no. of inches in 2 leaps of the greyhound,

$\therefore$ $\frac{3}{2}a$ is the no. of inches in one leap of the greyhound.

Thus $3x$ leaps of the greyhound $=\frac{3}{2}a\times 3x$ or $\frac{9}{2}ax$ inches.

and $4x+50$ leaps of the hare $=a(4x+50)$ inches.

$\therefore$ By the question, $\frac{9}{2}ax=a(4x+50)$;

Dividing by a, we have $\frac{9}{2}x=4x+50$; whence $x=100$.

Hence the greyhound must take 300 leaps.

Ex. 4. A train, 195 ft. long, runs at the rate of 20 miles an hour; another 245 ft. long, runs on a parallel rail (i) in the opposite direction, (ii) in the same direction, at the rate of 30 miles an hour: how long will they take to pass each other?

Let x be the time in hours in which they pass each other.

When the trains pass each other, they travel over a distance = sum of the lengths of the two trains

$$= 195 \text{ ft.} + 245 \text{ ft.} = 440 \text{ ft.} = \tfrac{1}{12} \text{ mile.}$$

(i) When travelling in the opposite direction we have

$$20x + 30x = \tfrac{1}{12}\text{; whence } x = \tfrac{1}{600}.$$

Hence the reqd. time $= \frac{1}{600}$ hour. = 6 sec.

(ii) When travelling in the same direction, we have

$$30x - 20x = \tfrac{1}{12}\text{; whence } x = \tfrac{1}{120}.$$

Hence the reqd. time $= \frac{1}{120}$ hour. = 30 sec.

Ex. 5. The termini of a railway 126 miles long are at **A** and **C**, and the station **B**, at which a certain train stops 15 minutes, is 70 miles from **A**. The whole journey from **A** to **C** takes 15 minutes less than twice as long as the journey from **A** to **B**. Determine the average rate of the train, including all stoppages except that at **B** (P. I. E. 1887).

Let x be the rate of the train in miles per hour.

The train takes $\frac{126}{x}$ hours to travel from **A** to **C**.

Also the train takes $\frac{70}{x}$ hours to travel from **A** to **B**.

$\therefore$ By the question, $\frac{126}{x} + \frac{1}{4} = \frac{2 \times 70}{x} - \frac{1}{4}$; (for 15 min. $= \frac{1}{4}$ hr.)

whence $x = 28$.

Hence the rate of the train is 28 miles per hour.

379. Motion up and down a stream. In such questions remember that the

(i) *rate of rowing down a stream = rate on still water + rate of current.*

and (ii) *rate of rowing up a stream = rate on still water − rate of current.*

Ex. 1. A crew which can pull at the rate of 9 miles an hour, finds that it takes twice as long to come up a river as to go down; find the rate at which the river flows.

Let x be the rate of the stream in miles per hour.

Then the crew can row $9+x$ miles per hour *down* the stream and $9-x$ miles per hour *up* the stream.

Let l denote the distance rowed in miles.

Then $\dfrac{l}{9-x}$ = time in hours taken to row *up* the stream,

and $\dfrac{l}{9+x}$ = ... *down* .. ..

$\therefore$ By the question, $\dfrac{l}{9-x}=2\times\dfrac{l}{9+x}$;

Dividing by l, $\dfrac{1}{9-x}=\dfrac{2}{9+x}$; whence $x=3$.

Hence the rate of the stream is 3 miles per hour.

Ex. 2. A person rowed down a river, a distance of 15 miles, in $1\frac{1}{4}$ hours with the stream, and rowed back again in $3\frac{3}{4}$ hours. Find the rate of the stream per hour.

Let x be the rate of the stream per hour in miles.

Since the person rowed 15 miles in $1\frac{1}{4}$ hours, therefore his rate of rowing down the stream is $(15\div 1\frac{1}{4})$ or 12 miles per hour.

Hence his rate of rowing in still water is $12-x$ miles per hour, and therefore his rate of rowing back was $12-x-x$ or $12-2x$ miles per hour.

$\therefore$ By the question, $\dfrac{15}{12-2x}=3\frac{3}{4}$; whence $x=4$.

Hence the rate of the stream is 4 miles per hour.

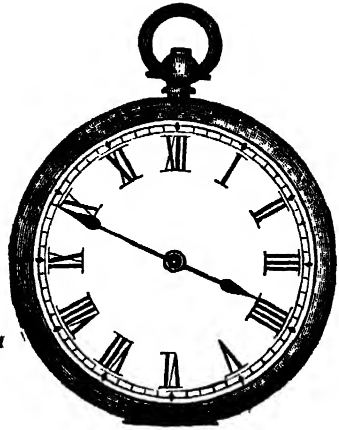
380. Clocks. In questions relating to clocks and watches, bear in mind that the *minute-hand travels twelve times as fast as the hour-hand.*

(i) *When the hands are coincident,* i.e., *in the same direction, there are no spaces between them.*

(ii) *When the hands are in exactly opposite directions, they are separated by 30 minute-spaces.*

(iii) *When the hands are at right angles to each other* (which happens **twice** in an hour), *they are separated by 15 or 45 minute spaces.*

Ex. 1. Find (a) the instant of time between 3 and 4 o'clock at which the hour-hand and the minute-hand are exactly in *the same direction*, (b) that at which they are exactly *opposite* each other. (E. F. 1862).



(*a*) Let x be the reqd. no. of minutes after 3 o'clock,

then $\frac{1}{12}x$ = the no. of minute-spaces the hour-hand will travel over in x minutes.

At 3 o'clock the hour-hand was ahead of the minute-hand by 15 minute-spaces,

$\therefore$ the minute-hand has to travel $(15+\frac{1}{12}x)$ minute-spaces to overtake the hour hand.

$\therefore$ By the question, $x=15+\frac{x}{12}$;

whence $x=16\frac{4}{11}$.

coincident

Hence the hands are in opposite direction at $16\frac{4}{11}$ min. past 3.

(*b*) Let x be the reqd. no, of minutes past 3 o'clock.

then $\frac{1}{12}x$ = the no. of minute spaces the hour-hand will travel over in x minutes.

At 3 o'clock the hour-hand was ahead of the minute-hand by 15 minute-spaces, and the hour-hand must be ahead 30 minute spaces when the two hands are opposite to each other.

$\therefore$ By the question, $x=15+\frac{x}{12}+30$; whence $x=49\frac{1}{11}$.

Hence the hands are opposite each other at $49\frac{1}{11}$ min. past 3.

Ex. 2. At what times are the hands of a watch at right angles to each other (*a*) between 2 and 3 o'clock, (*b*) between 5 and 6 o'clock.

(*a*) Let x be the reqd. no. of minutes after 2 o'clock.

then $\frac{1}{12}x$ = the no. of minute spaces the hour-hand will move over in x minutes.

At 2 o'clock the hour-hand was ahead of the minute-hand by 10 minute-spaces, and the minute-hand must be ahead of the hour-hand 15 or 45 minute-spaces when the two hands are at right angles to each other.

$\therefore$ By the question, $x-\left(10+\frac{x}{12}\right)=15$ or 45, since the minute-hand is to be in advance of the hour-hand.

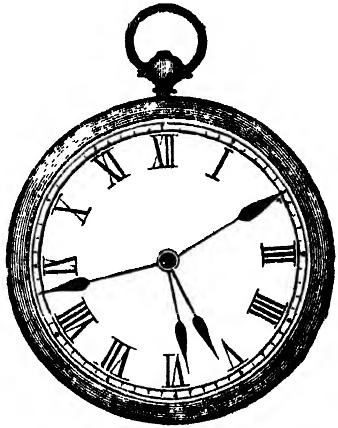
whence $x=27\frac{3}{11}$ or 60.

Hence the hands are at right angles *once* at $27\frac{3}{11}$ min past 2 and *again* at 3 o'clock.

(*b*) Let x be the reqd. no. of minutes after 5 o'clock.

then $\frac{1}{12}x$ = the no. of minutes-spaces the hour-hand will move over in x minutes.

At 5 o'clock the hour-hand was ahead of the minute-hand by 25 minute-spaces,



$\therefore$ at x minutes past 5 the hour-hand is separated from the mark 12 by $(25+\frac{1}{12}x)$ minute-spaces.

$\therefore$ By the question, when the hands are at right angles,

$$x-\left(25+\frac{x}{12}\right)=15 \quad \text{or} \quad -15,$$

according as the minute-hand is the more or less advanced of the two,

whence $x=43\frac{7}{11}$ or $10\frac{10}{11}$.

Hence the hands are at right angles *once* at $10\frac{10}{11}$ min. past 5 and *again* at $43\frac{7}{11}$ min. past 5.

Ex. 3. Find the time after h o'clock at which the hour and minute-hands of a watch are distant d of the minute divisions from each other.

Let x be the reqd. no. of minutes after h o'clock

then $\frac{1}{12}x$ = the no. of minutes-spaces the hour-hand will travel over in x minutes.

At h o'clock the hour-hand was ahead of the minute-hand by $5h$ minute-spaces,

$\therefore$ at x minutes past h the hour-hand is separated from the 12 o'clock mark by $(5h+\frac{1}{12}x)$ minute-spaces.

$\therefore$ By the question, $x-\left(5h+\dfrac{x}{12}\right)=+d$ or $-d$, according as the minute-hand is the more or less advanced of the two,

whence $x=\frac{12}{11}(5h\pm d)$.

Ex. 4. A man who went out between 5 and 6 and returned between 6 and 7, found that the hands of his watch has exactly changed places. When did he go out? (P. E. 1894).

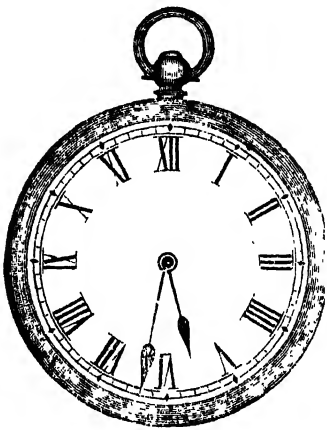
Let x be the reqd. no. of minutes after 5 o'clock,

then $\frac{1}{12}x$ = the no. of minute-spaces the hour-hand will travel over in x minutes.

At 5 o'clock the two hands are separated from each other by 25 minute-spaces,

$\therefore$ at x minutes past 5 the hour-hand is separated from the 12 o'clock mark by $(25+\frac{1}{12})x$) minute-spaces.

When the two hands exchange places, the minute-hand will be $(25+\frac{1}{12}x)$ minute-spaces and the hour-hand x minute-spaces respectively from the mark 12,



$\therefore$ the hour-hand will be $\frac{1}{12}\times(25+\frac{1}{12}x)$ minute-spaces after the mark 6, *i. e.* $\{30+\frac{1}{12}(25+\frac{1}{12}x)\}$ minute-spaces after the mark 12.

$\therefore$ By the question, $x=30+\frac{1}{12}\times\left(25+\frac{x}{12}\right)$; whence $x=32\frac{4}{13}$.

Hence the man's hour of departure is $32\frac{4}{13}$ min. past 5.

381. Formation of Squares. In questions relating to formation of squares remember that the

(i) *no. forming a* **solid** *square* = (*no. in front*)^2.

(ii) *no. forming a* **hollow** *square* = (*no. in front*)^2 − (*no. in front* − *twice depth*)^2.

or = *4 times the depth* × (*no. in front* − *depth*).

Ex. 1. A number of troops being formed into a solid square, it was found there were 60 over; but it would require 41 more to increase the side of the square by one. Find the number.

Let x be the no. of men in a side of the front square.

Then the total no. of men is x^2+60.

Again, to increase the side of the square by one man, the total no. of men in the square will be $(x+1)^2$.

$\therefore$ By the question, $x^2+60=(x+1)^2-41$; whence $x=50$.

Hence the no. of men required $=50^2+60=2560$.

Ex. 2. An officer can form his men into a hollow square 5 deep and also into a hollow square 6 deep, but the front in the latter formation contains 4 men fewer than in the former ; find the number of men. (C. E. 1887).

Let x be the no. of men in the front of the hollow square, 5 deep, then $x-4$ is the no. of men in the front of the hollow square, 6 deep.

The total no. of men in the first case $=x^2-(x-10)^2=20x-100$.

and the total no. of men in the second case $=(x-4)^2-\{(x-4)-12\}^2$

$$=(x-4)^2-(x-16)^2$$
$$=24x-240.$$

$\therefore$ By the question, $20x-100=24x-240$; whence $x=35$.

Hence, the reqd. of men $=20\times35-100=600$.

Exercise CXXXVIII.

1. Find a number such that if $\frac{1}{5}$ of it be subtracted from 20, and $\frac{1}{11}$ of the remainder from $\frac{1}{4}$ of the original number, 12 times the second remainder shall be half the original number.

2. A gamester at one sitting lost $\frac{1}{4}$ of his money, and then won *Rs.*10 ; at a second he lost $\frac{1}{3}$ of the remainder, and then won *Rs.*3 ; and now he has *Rs.*63 left. How much money had he at first ?

3. A fish was caught whose tail weighed 9lbs. ; his head weighed as much as his tail and half his body, and his body weighed as much as his head and tail. What did the fish weigh ?

4. A father's age is four times that of his eldest son and five times that of his younger son : when the elder son has lived to three times his present age, the father's age will exceed twice that of his younger son by 3 years. Find their present ages. (B.M. 1882).

5. A farmer bought equal numbers of two kinds of sheep, one at £3 each, and the other at £4 each. Had he expended his money equally in the two kinds, he would have had two more sheep than he had. How many did he buy ? (A.E. 1891)

6. A person dies worth *Rs* 130000 ; some of this he leaves to Charity, and twelve times as much to his eldest son, whose share is half as much again as that of each of his two brothers, and two-thirds as much again as that of each of his five sisters ; find the amount of the bequest to the Charity.

7. A composition of copper and tin containing 140 cubic inches weighs 42lbs. 3 oz. How many ounces of each are there, if a cubic inch of copper weighs $5\frac{1}{4}$oz. and a cubic inch of tin $4\frac{1}{4}$ oz. (M.M. 1891).

8. What quantity of corn at *Rs.*5 per maund must a tradesman mix with 560 maunds at *Rs.*6 per maund, in order to gain 20 per cent. by selling the whole at 2*a.* 6*p.* per seer? (P. E. 1888).

9. Divide the number 834 into two parts such that 30 per cent. of one part exceeds 40 per cent. of the other part by 6. (C.E. 1893).

10. A can do a piece of work in 9 days, B in twice that time, C can only do $\frac{3}{4}$ as much as A in a day; how long would A, B and C, working together require to do the same piece of work? (C.E. 1876).

11. A and B can reap a field together in 7 days, which A alone could reap in 10 days; in what time could B alone reap it?

12. A cistern can be filled in 15 min. by two pipes, A and B, running together; after A has been running by itself for 5 min., B is also turned on, and the cistern is filled in 13 min. more; in what time would it be filled by each pipe separately?

13. A does $\frac{5}{9}$ of a piece of work in 10 days, when B comes to help him, and they take 3 days more to finish it; in what time would they have done the whole, each separately, or both together?

14. A man and his wife could drink a cask of beer in 20 days, the man drinking half as much again as his wife; but $\frac{18}{25}$ of a gallon having leaked away, they found that it only lasted them together for 18 days, and the wife herself for 2 days longer. How much did it contain when full?

15. A boy buys a certain number of oranges at 3 for 2*d.*, and one-third of that number at 2 for 1*d.*; at what price must he sell them to get 20 per cent. profit? If his profit be 5*s.* 4*d.*, find the number bought. (C.E. 1885).

16. A cask A contains 12 gallons of wine and 18 gallons of water; and another cask B contains 9 gallons of wine and 3 gallons of water; find how many gallons must be drawn from each cask so as to produce by their mixture 7 gallons of wine and 7 gallons of water.

17. A and B can reap a field together in 12 hours, A and C in 16 hours, and A by himself in 20 hours; in what time could (i) B and C together, (ii) A, B and C together, reap it?

18. A and B can perform a certain task in 30 days, working together. After 11 days, however, B is called away, and A finished it by himself 28 days after. How long would it take A to do the whole of the work by himself?

19. Two vessels contain mixtures of wine and water; in one there is twice as much wine as water, and in the other, three times as much water as wine. Find how much must be drawn off from

each, to fill a third vessel which holds 15 gallons, in order that its contents may be half wine and half water. (B. M. 1890).

20. I bought a horse and a carriage for £90; I sold the horse at a gain of 12 per cent. and the carriage at a loss of 4 per cent. and gained on the whole 6 per cent. Find the prime cost of the carriage. (B. M. 1885).

21. A ship sails with a supply of biscuit for 60 days, at a daily allowance of a pound a head; after being at sea 20 days, she encounters a storm in which 5 men are washed overboard and damage sustained that would cause a delay of 24 days, and it is found that each man's daily allowance must be reduced to $\frac{5}{7}$ths of a pound. Find the original number of the crew.

22. A person walked out a certain distance at the rate of $3\frac{1}{2}$ miles an hour, and then ran part of the way back at the rate of 7 miles an hour, walking the remaining distance in 7 minutes. He was out 35 minutes. How far did he run? (A. E. 1890).

23. Two persons started at the same time from **A**. One rode on horse back at the rate of $7\frac{1}{2}$ miles an hour and arrived at **B**, 30 minutes later than the other, who travelled the same distance by train at the rate of 30 miles an hour. Find the distance between **A** and **B**. (C. E. 1873).

24. Of the candidates in a certain examination, 45 per cent. passed. If there had been 30 more candidates of whom 19 failed, the number of successful candidates would have been 44·8 per cent How many candidates were there? (C. E. 1890).

25. A man rides one-third of the distance from **A** to **B** at the rate of a miles an hour, and the remainder at the rate of $2b$ miles an hour. If he had travelled at a uniform rate of $3c$ miles an hour, he could have ridden from **A** to **B** and back again in the same time.

Prove that $\frac{2}{c}=\frac{1}{a}+\frac{1}{b}$. (C. E. 1889).

26. **A** travels at the rate of 3 miles an hour; **B** leaves the same place two hours after **A** and travels at the rate of 5 miles an hour; when and where will **B** overtake **A**? (C. F. A. 1869).

27. At what time are the hands of a watch together between 5 and 6 o'clock? (C. E. 1886).

28. **AB** is a railway 220 miles long, and three trains (**P**, **Q**, **R**) travel upon it at the rate of 25, 20 and 30 miles per hour respectively; **P** and **Q** leave **A** at 7 A. M., and 8-15 A. M., respectively and **R** leaves **B** at 10-30 A. M. When and where will **P** be equidistant from **Q** and **R**? (C. E. 1870).

29. The express leaves Bristol at 3 P. M., and reaches London at 6 P. M.; the ordinary train leaves London at 1-30 P. M., and arrives at Bristol at 6 P. M. If both trains travel uniformly, find the time when they will meet. (C. E. 1892).

30. **A** and **B** start together from the same point on a walking match round a circular course. After half an hour, **A** has walked three complete circuits, and **B** four and a half. Assuming that each walks with uniform speed, find when **B** next overtakes **A**. (P. E. 1892).

31. **A** alone can do a piece of work in a hours, **A** and **C** together can do it in b hours, and **C**'s work is $1/n$th of **B**'s The work has to be completed in c hours. Find (i) how long after **A** has commenced, **B** and **C** should *relieve* him, so as to finish the work in time; (ii) how long after **A** has commenced, **B** and **C** should *join* him so that the three working together might just complete the work in time? (M. M. 1867).

32. A person sets out to walk to a certain town. But when he has accomplished a quarter of his journey, he finds that if he continues at the same pace, he will have gone only $\frac{7}{9}$ths of the whole distance when he ought to be at his destination. He therefore increases his speed by a mile per hour, and arrives just in time. Find his rates of walking. (M. M. 1875).

33. A person has a number of rupees which he tries to arrange in the form of a square. On the first attempt he has 116 over. When he increases the side of the square by 3 rupees, he wants 25 rupees to complete the square. How many rupees has he? (B. M. 1875).

34. A number of troops being formed into a solid square, it was found there were 60 over; but, when formed into a column with 5 men more in front than before and 3 less in depth, there was just one man wanting to complete it. Find the number.

35. Divide 90 into four such parts that if the first be increased by 5, the second diminished by 4, the third multiplied by 3, and the fourth divided by 2, the results shall all be equal.

36. Two casks, **A** and **B**, contain mixtures of wine and water; in **A** the quantity of wine is to the quantity of water as 4 to 3; in **B** the like proportion is that of 2 to 3. If **A** contain 84 gallons, what must **B** contain, so that when the two are put together, the new mixture may be half wine and half water?

37. A privateer running at the rate of 10 miles an hour discovers a ship 18 miles off, running at the rate of 8 miles an hour; how many miles can the ship run before it is overtaken? (B. M. 1865).

38. An officer can form the men of his regiment into a hollow square 12 deep. The number of men in the regiment is 1296. Find the the number of men in the front of the hollow square.

39. An officer can form his men into a hollow square 4 deep, and also into a hollow square 8 deep; the front in the latter formation contains 16 men fewer than in the former formation; find the number of men.

40. A regiment was drawn up in a solid square; when some time after it was again drawn up in a solid square it was found that there were 5 men fewer in a side; in the interval 295 men had been removed from the field; what was the original number of men in the regiment?

41. At what time are the hands of a watch opposite to each other (*a*) between 1 and 2 o'clock; (*b*) between 7 and 8 o'clock.

42. At what times between 7 and 8 o'clock are the hour and minute-hands of a watch at right angles to one another?

43. It is between 11 and 12 o'clock, and it is observed that the number of minute spaces between the hands is two-thirds of what it was 10 minutes previously; find the time.

44. The distance from a place **P** to another place **Q** is $3\frac{3}{4}$ miles; two persons **A** and **B** start together from **P** to go to **Q**, the former by carriage which travels at the rate of 6 miles an hour, the latter walking at the rate of 3 miles an hour. If **A** remains at **Q** for 15 minutes, and then returns by the carriage to **P**, find where he will meet **B**. (C. E. 1882).

45. A smuggler had a quantity of brandy, which he expected would produce £9. 18*s*: after he had sold 10 gallons, a revenue officer seized one-third of the remainder, in consequence of which the smuggler makes only £8. 2*s*.; required the number of gallons he had and the price per gallon.

46. A hare is 80 of her own leaps before a greyhound; she takes 3 leaps for every 2 that he takes, but he covers as much ground in one leap as she does in 2. How many leaps will the hare have taken before she is caught?

47. If 19 lbs. of gold weigh 18 lbs. in water, and 10 lbs. of silver weigh 9 lbs. in water, find the quantity of gold and silver in a mass of gold and silver weighing 106 lbs. in air and 99 lbs. in water. (B. M. 1888).

48. The denominator of a fraction exceeds the numerator by 4, and if 5 be taken from each, the sum of the reciprocal of the new fraction and four times the original fraction is 5. Find the original fraction. (B. M. 1892).

49. Two towns **L** and **M** are 30 miles apart. **A** sets off from **L** to **M**, and **B** from **M** to **L** at the same moment. **A** reaches **M** 16

hours, and B reaches L 36 hours after they have met on the road. Find the time taken by each to perform the journey. (P. E. 1887).

50. A crew which can pull at the rate of 8 miles an hour down the stream, finds that it takes twice as long to come up a river as to come down. At what rate does the stream flow?

51. At what time between 4 and 5 o'clock are the hands of a watch coincident?

52. A person buys a piece of land at *Rs.*300 a cottah, and by selling it in allotments finds the value increased three-fold, so that he clears *Rs.*1500, and retains 25 cottahs for himself; how many cottahs were there?

53. Divide the number 88 into four parts such that the first increased by 2, the second diminished by 3, the third multiplied by 4, and the fourth divided by 5, may all be equal.

54. A merchant goes to three bazars in succession. At the first he gains 15 per cent. on his capital; and at the second 20 per cent. upon his increased capital; and at the third 25 per cent. on what he then possessed: on his return home he finds that he has gained *Rs.*2639. What was his original capital? (M. M. 1863).

55. The charge for the first class tickets of admission to an exhibition was *Rs.*4 each and the charge for second class tickets was *Rs.*2. 8*a.* The whole number of tickets sold was 259, and the total amount received for them was *Rs.*731. 8*a.* How many first class tickets were sold, and how many second class tickets? (B. M. 1869).

56. A receives a fixed sum as pocket money at the beginning of every week, and in each week he spends half of all that he had at its beginning. He had no money before the first pocket money was given him and at the end of the third week he has 1*s.* 2*d.* What was his weekly allowance? (B. M. 1876).

57. Find a number such that whether divided into two equal parts, or into three equal parts, the product of the parts shall be the same. (B. M. 1864).

58. A man walks from the University towards Malabar Hill at the rate of 3 miles an hour, runs part of the way back at the rate of $8\frac{1}{3}$ miles an hour and then walks the remainder in 1 hour. 5 min. He was out 2 hours 44 min.; find how far he had gone? (B. M. 1886).

CHAPTER XVII.

HARDER SIMULTANEOUS EQUATIONS.

I. EQUATIONS INVOLVING FRACTIONS.

382. The following are illustrative Examples.

Ex. 1. Solve
$$\left.\begin{aligned} \frac{x}{2}+\frac{18x-9y}{8x-7}&=5+\frac{x-6}{2}\ldots\ldots\ldots(1) \\ \frac{x}{2}+\frac{y+1}{3}&=x\ldots\ldots\ldots\ldots\ \ldots\ldots(2) \end{aligned}\right\}$$

From (1), $\frac{x}{2}+\frac{18x-9y}{8x-7}=5+\frac{x}{2}-3$, or $\frac{18x-9y}{8x-7}=2$.

Multiplying across, $18x-9y=16x-14$, or $2x-9y=-14$....(3)

From (2) $\frac{y+1}{3}=\frac{x}{2}$, or $2y+2=3x$, or $3x-2y=2$...............(4)

Multiply (3) by 3 and (4) by 2 ; thus

$$\left.\begin{aligned} 6x-27y&=-42 \\ 6x-\ \ 4y&=\ \ \ 4 \end{aligned}\right\} \quad \begin{aligned} &\text{By subtraction,} \\ &-23y=-46, \quad \therefore\ y=2. \end{aligned}$$

From (4) $3x=2y+2=4+2=6$; $\therefore\ x=2$.

Ex. 2. Solve
$$\left.\begin{aligned} \frac{3}{x+y}+\frac{2}{x-y}&=1\tfrac{3}{4}\ldots\ldots\ldots\ldots\ (1) \\ \frac{4}{x+y}+\frac{3}{x-y}&=2\tfrac{1}{2}\ldots\ldots\ \ \ldots\ldots(2) \end{aligned}\right\}$$

Multiply (1) by 3 and (2) by 2 ; thus

$$\left.\begin{aligned} \frac{9}{x+y}+\frac{6}{x-y}&=\tfrac{21}{4} \\ \frac{8}{x+y}+\frac{6}{x-y}&=5 \end{aligned}\right\} \quad \begin{aligned} &\text{By subtraction,} \\ &\frac{1}{x+y}=\tfrac{1}{4}, \quad \therefore\ x+y=4\ \ldots(3) \end{aligned}$$

From (1) $\frac{2}{x-y}=1\frac{3}{4}-\frac{3}{x+y}=1\frac{3}{4}-\frac{3}{4}$, substituting (3)

$$=1, \quad \therefore\ x-y=2\ldots\ldots\ \ \ldots\ldots\ldots(4)$$

Hence, from (3) and (4) by addition and subtraction,

$$2x=6, \text{ or } x=3 \text{ and } 2y=2, \text{ or } y=1.$$

Exercise CXXXIX.

Solve the following equations :—

1. $\left.\begin{aligned} x-\frac{2y-x}{23-x}&=20-\frac{59-2x}{2} \\ y+\frac{y-3}{x-18}&=30-\frac{73-3y}{3} \end{aligned}\right\}$ **2.** $\left.\begin{aligned} 4x-\frac{11(x+1)-\frac{1}{3}y}{17-3x}&=20-\frac{103-8x}{2} \\ 8y+\frac{3(y-1)}{5x-10}&=50-\frac{147-24y}{3} \end{aligned}\right\}$

3. $\left.\begin{aligned} \frac{6x+6y+11}{2x+2y-7}&=3+\frac{64}{6x+7y-28} \\ 4x+3y-7&=\frac{32x^2-18y^2-35}{8x-6y+7} \end{aligned}\right\}$ **4.** $\left.\begin{aligned} \frac{3}{x+y}+\frac{8}{x-y}&=2\tfrac{3}{4} \\ \frac{6}{x-y}-\frac{5}{x+y}&=\tfrac{1}{4} \end{aligned}\right\}$

5. $\left.\begin{aligned} 3y+11&=\frac{4x^2-y(x+3y)}{x-y+4}+31-4x \\ (x+7)(y-2)+3&=2xy-(x+1)(y-1) \end{aligned}\right\}$ **6.** $\left.\begin{aligned} \frac{30}{x-y}+\frac{44}{x+y}&=10 \\ \frac{40}{x-y}+\frac{55}{x+y}&=13 \end{aligned}\right\}$

7. $\left.\begin{aligned} \frac{4x^2+2xy+288-6y^2}{2x+13-2y}&=2x+3y-131 \\ 5x-4y&=22 \end{aligned}\right\}$ (B. M. 1874).

8. $\left.\begin{aligned} x+\frac{5x-2y}{y-7}&=13\tfrac{2}{3}-\frac{44-3x}{3} \\ 4x-13\tfrac{1}{3}&=\frac{7x-3y}{x-1}-\frac{55-12x}{3} \end{aligned}\right\}$ **9.** $\left.\begin{aligned} \frac{4}{x}-\frac{5}{y}&=\frac{x+y}{xy}+1\tfrac{22}{35} \\ xy&=\tfrac{35}{34}(y-x) \end{aligned}\right\}$

(B. M. 1877)

10. $2{\cdot}4x+{\cdot}32y-\frac{{\cdot}36x-{\cdot}05}{{\cdot}5}={\cdot}8x+\frac{2{\cdot}6+{\cdot}005y}{{\cdot}25}, \quad \frac{{\cdot}04y+{\cdot}1}{{\cdot}3}=\frac{{\cdot}07x-{\cdot}1}{{\cdot}6}.$

II. LITERAL EQUATIONS.

383. The following are typical solutions deserving of notice.

Ex. 1. Solve $\left.\begin{aligned} axy&=c(bx+ay)\ldots\ldots\ldots(1) \\ bxy&=c(ax-by)\ldots\ldots\ldots(2) \end{aligned}\right\}$

Dividing both (1) and (2) by cxy, we have

$\frac{a}{c}=\frac{b}{y}+\frac{a}{x}\ldots(3)$ Multiply (3) by a, $\frac{a^2}{c}=\frac{ab}{y}+\frac{a^2}{x}$

$\frac{b}{c}=\frac{a}{y}-\frac{b}{x}\ldots(4)$ (4) by b, $\frac{b^2}{c}=\frac{ab}{y}-\frac{b^2}{x}$

By subtraction, $\dfrac{a^2-b^2}{c}=\dfrac{a^2+b^2}{x}$; $\therefore\ x=\dfrac{a^2+b^2}{a^2-b^2}\cdot c$.

From (3) $\dfrac{b}{y}=\dfrac{a}{c}-\dfrac{a}{x}=\dfrac{a}{c}-\dfrac{a^2-b^2}{a^2+b^2}\cdot\dfrac{a}{c}=\dfrac{a}{c}\cdot\dfrac{2b^2}{a^2+b^2}=\dfrac{2ab^2}{c(a^2+b^2)}$

$\therefore\ \dfrac{1}{y}=\dfrac{2ab}{c(a^2+b^2)}$ and $\therefore\ y=\dfrac{a^2+b^2}{2ab}\cdot c$.

Ex. 2. Solve
$$\left.\begin{aligned} 2x-a-\frac{y-b}{a+b}&=a \quad\ldots\ldots\ldots\ (1)\\ 2y-b+\frac{x-a}{b}&=2b-y\ \ldots\ldots\ldots(2)\end{aligned}\right\}\ \text{(M. M. 1864)}$$

From (1) $2(x-a)=\dfrac{y-b}{a+b}$; from (2) $\dfrac{x-a}{b}=-3(y-b)$.

Writing X for $x-a$ and Y for $y-b$, we obtain

$$2X=\frac{Y}{a+b}\ldots\ldots(3) \text{ and } \frac{X}{b}=-3Y\ldots\ldots(4)$$

From (3) $X=\dfrac{Y}{2(a+b)}$ and from (4) $X=-3bY$.

$\therefore\ \dfrac{Y}{2(a+b)}=-3bY$; whence $Y=0$ and $\therefore\ X=0$.

$\therefore\ x-a=0=y-b$, and $x=a,\ y=b$

Ex. 3 Solve $xyz=a(yz-zx-xy)=b(zx-xy-yz)$
$=c(xy-yz-zx)\ldots(1),(2)\ \&\ (3)$.

Dividing (1) by $axyz$, (2) by $bxyz$ and (3) by $cxyz$, we get

$$\left.\begin{aligned}\frac{1}{a}&=\frac{1}{x}-\frac{1}{y}-\frac{1}{z}\ldots(4)\\ \frac{1}{b}&=\frac{1}{y}-\frac{1}{z}-\frac{1}{x}\ldots(5)\\ \frac{1}{c}&=\frac{1}{z}-\frac{1}{x}-\frac{1}{y}\ldots(6)\end{aligned}\right\}$$

Adding (4) and (5), we have

$$-\frac{2}{z}=\frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab}.$$

$$\therefore\ z=-\frac{2ab}{a+b}.$$

Adding (4) and (6), we have

$$-\frac{2}{y}=\frac{1}{a}+\frac{1}{c}=\frac{a+c}{ac},$$

$$\therefore\ y=-\frac{2ac}{a+c}.$$

Adding (5) and (6), we have

$$-\frac{2}{x}=\frac{1}{b}+\frac{1}{c}=\frac{b+c}{bc},$$

$$\therefore\ x=-\frac{2bc}{b+c}.$$

Ex. 4. Solve $\left.\begin{array}{l}\dfrac{a-b}{x}+\dfrac{a+b}{y}=\dfrac{2(a^2+b^2)}{a^2-b^2}\ldots(1)\\[2ex]\dfrac{a+b}{x}+\dfrac{a-b}{y}=2\ldots\ldots\ldots\ldots(2)\end{array}\right\}$ (B. M. I

Adding (1) and (2) and dividing by $2a$, we have

$$\frac{1}{x}+\frac{1}{y}=\frac{2a}{a^2-b^2}\ldots\ldots\ldots\ldots(3)$$

Subtracting (1) from (2) and dividing $2b$, we have

$$\frac{1}{x}-\frac{1}{y}=-\frac{2b}{a^2-b^2}\ldots\ldots(4)$$

Adding (3) and (4) and dividing by 2, we get

$$\frac{1}{x}=\frac{1}{a+b}, \text{ and } \therefore\ x=a+b.$$

Subtracting (4) from (3) and dividing by 2, we ge

$$\frac{1}{y}=\frac{1}{a-b}, \text{ and } \therefore\ y=a-b.$$

Exercise CXL.

Solve the following equations :—

1. $ax-by=a-b, \dfrac{x}{2a}+\dfrac{y}{2b}=\dfrac{1}{a+b}$. (M. M. 1868).

2. $\left.\begin{array}{l}(a-b)x+(b-c)y+cz=1.\\ 2ax+by+\frac{3}{2}cz=2.\\ (a+b-c)x+(a-2b+2c)y+2bz=3.\end{array}\right\}$ (M. M. 1867).

3. $\left.\begin{array}{l}ax+by+cz=a+b+c.\\ \dfrac{ax}{b+c}+\dfrac{by}{a+c}=1,\ \dfrac{2x}{b+c}+\dfrac{2y}{a+c}=\dfrac{1}{a}+\dfrac{1}{b}\end{array}\right\}$ (M. M. 1865).

4. $\dfrac{a}{x}+\dfrac{b}{y}=\dfrac{a}{y}+\dfrac{b}{z}=\dfrac{a}{z}+\dfrac{b}{x}=c$. (M. M. 1873).

5. $(a^2+b^2)(x-1)=ab(2x-y),\ 4x=y+2$. (M. M. 1875).

6. $\left.\begin{array}{l}2ab(x-y)=xy(a-b)\\ 2ab(x+y)+xy(a+b+2ab)=0\end{array}\right\}$ (B. M. 1897).

III. RULE OF CROSS MULTIPLICATION.

384. Theorem. If $\left.\begin{array}{l} ax+by+cz=0 \;\ldots\ldots\; (1) \\ \text{and } a'x+b'y+c'z=0 \;\ldots\ldots\; (2) \end{array}\right\}$

then will $\dfrac{x}{bc'-b'c}=\dfrac{y}{ca'-c'a}=\dfrac{z}{ab'-a'b}$.

Proof. Multiply (1) by c', and (2) by c; thus we have

$$\left.\begin{array}{l} ac'x+bc'y+cc'z=0 \;\ldots\ldots\ldots (3) \\ a'cx+b'cy+cc'z=0 \;\ldots\ldots\; (4) \end{array}\right\}$$

Subtract (4) from (3); thus $(ac'-a'c)x+(bc'-b'c)y=0$;

$\therefore\; (bc'-b'c)y=-(ac'-a'c)x=(ca'-c'a)x$;

Dividing each by $(bc'-b'c)(ca'-c'a)$, we have

$$\frac{y}{ca'-c'a}=\frac{x}{bc'-b'c}. \quad \text{Similarly, } \frac{y}{ca'-c'a}=\frac{z}{ab'-a'b}.$$

Hence $\dfrac{x}{bc'-b'c}=\dfrac{y}{ca'-c'a}=\dfrac{z}{ab'-a'b}$.

Thus, it appears that when we have two equations of the type represented by (1) and (2), we may always by the above formula write down the ratios $x:y:z$ in terms of the coefficients of the unknown quantities by the following Rule :—

Write down the coefficients of y, z and x in the two given equations as shewn below, commencing with those of y.

$$\begin{array}{ccccccc} & x & & y & & z & \\ b & & c & & a & & b \\ b' & & c' & & a' & & b' \end{array}$$

and multiply b by c', and b' by c for the denominator of x, giving the + (*plus*) sign to the first product and the − (*minus*) sign to the second; similarly treat c, a' and c', a for that of y; and a, b' and a', b for that of z. Thus we obtain the required result.

Ex. 1. Solve $\left.\begin{array}{l} x-2y+z\;=0 \;\ldots\ldots\ldots (1) \\ 9x-8y+3z=0 \;\ldots\ldots\ldots (2) \\ 2x+3y+5z=36 \;\ldots\ldots\ldots (3) \end{array}\right\}$ (C. E. 1887).

From (1) and (2), by the *Rule of Cross Multiplication*,

$$\frac{x}{-2\times3-(-8)\times1}=\frac{y}{1\times9-1\times3}=\frac{z}{1\times(-8)-9\times(-2)}$$

or $\dfrac{x}{-6+8}=\dfrac{y}{9-3}=\dfrac{z}{-8+18}$, or $\dfrac{x}{2}=\dfrac{y}{6}=\dfrac{z}{10}=k$ (suppose),

then $x=2k$, $y=6k$ and $z=10k$ …………… (4)

Substituting these values of x, y and z in (3), we have

$$4k+18k+50k=36, \text{ or } 72k=36; \quad \therefore k=\tfrac{1}{2}.$$

Hence from (4), $x=1, y=3$ and $z=5$.

Note.—The above relations may be reduced to the form $x/1=y/3=z/5$ and thus we may take each $=k$.

Ex. 2. Solve

$$\left.\begin{aligned} x+y+z&=a+b+c \ldots\ldots\ldots\ldots(1) \\ bx+cy+az=cx+ay+bz&=a^2+b^2+c^2 \ldots(2) \text{ \& } (3) \end{aligned}\right\}$$

$$\left.\begin{aligned} &\text{From (1)} \quad (x-b)+(y-c)+(z-a)=0. \\ &\text{From (2)} \quad b(x-b)+c(y-c)+a(z-a)=0. \end{aligned}\right\}$$

Then, by the *Rule of Cross Multiplication*, we get

$$\frac{x-b}{c-a}=\frac{y-c}{a-b}=\frac{z-a}{b-c}=k, \text{ suppose,}$$

then $x-b=(c-a)k$, $y-c=(a-b)k$ and $z-a=(b-c)k$,

or $x=b+(c-a)k$, $y=c+(a-b)k$ and $z=a+(b-c)k$.

Substituting these values in (3), we have

$$bc+c(c-a)k+ac+a(a-b)k+ab+b(b-c)k=a^2+b^2+c^2;$$

$$\therefore (a^2+b^2+c^2-bc-ca-ab)k=a^2+b^2+c^2-bc-ca-ab;$$

$$\therefore k=1.$$

Hence $x=b+c-a$, $y=c+a-b$ and $z=a+b-c$.

385. The above formula may advantageously be applied in solving Simultaneous Equations involving only two unknown quantities x and y.

Ex. 1. Solve

$$\left.\begin{aligned} ax+by+c&=0 \ldots\ldots\ldots\ldots (1) \\ a'x+b'y+c'&=0 \ldots\ldots\ldots\ldots (2) \end{aligned}\right\}$$

The equations may be written thus :—

$$\left.\begin{aligned} ax+by+c.1&=0 \\ a'x+b'y+c'.1&=0 \end{aligned}\right\}$$

Hence by the formula, we have

$$\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{1}{ab'-a'b}.$$

Hence $x=\dfrac{bc'-b'c}{ab'-a'b}$ and $y=\dfrac{ca'-c'a}{ab'-a'b}$.

Exercise CXLI.

Solve the following equations :—

1. $$\left.\begin{array}{r} x-2(3y-2z)=0 \\ 2y+3(x-z)=0 \\ 5x+7y+9z=67 \end{array}\right\} \text{(C. E. 1900).}$$

2. $$\begin{array}{r} 2x-3y+4z=0 \\ 7x+2y-6z=0 \\ 9x+5y-10z=8 \end{array}$$

3. $$\left.\begin{array}{l} x+y+z=0,\ bcx+cay+abz=0 \\ ax+by+cz+(b-c)(c-a)(a-b)=0 \end{array}\right\} \text{(C. E. 1896).}$$

4. $$\left.\begin{array}{c} 4ax+(a+1)y=(3a-1)z \\ (3a-1)(x-y-1)+z=0 \\ x+y=z \end{array}\right\} \text{(M. M. 1874).}$$

5. $$\left.\begin{array}{r} x+6y-5z=0 \\ 7x-6y+z=0 \\ 3x-4y+2z=4 \end{array}\right\}$$

6. $$\left.\begin{array}{c} x+y+z=(a-b)(b-c)(a-c) \\ ax+by+cz=0 \\ a^2x+b^2y+c^2z=0 \end{array}\right\} \text{(M. M. 1869).}$$

7. $$\left.\begin{array}{l} x+y+z=ax+by+cz=0 \\ \dfrac{x}{b-c}+\dfrac{y}{a-c}+\dfrac{z}{a-b}=1 \end{array}\right\} \text{(M. M. 1878).}$$

8. $$\left.\begin{array}{l} ax+by+cz=a+b \\ bx+cy+az=b+c \\ cx+ay+bz=c+a \end{array}\right\} \text{(M. M. 1879).}$$

9. $$\left.\begin{array}{r} 2(4x+9y)=7(2y+z) \\ 7(x+2y)=8(y+z) \\ 5x+2y-3z=4 \end{array}\right\}$$

10. $$\left.\begin{array}{l} x+y+z=a+b+c \\ ax+by+cz=bc+ca+ab \\ (b-c)x+(c-a)y+(a-b)z=0 \end{array}\right\}$$

11. $$\left.\begin{array}{l} 4x-13y+8z=0 \\ 7x+6y-9z=0 \\ \dfrac{4}{x}+\dfrac{7}{y}-\dfrac{10}{z}=1\dfrac{1}{12} \end{array}\right\}$$

12. $$\left.\begin{array}{l} x+y+z=0 \\ (b+c)x+(c+a)y+(a+b)z=0 \\ bcx+cay+abz=1 \end{array}\right\} \text{(C. E. 1906).}$$

13. $$\left.\begin{array}{r} 2ax-by-cz=0 \\ ax-2by+cz=0 \\ ax+by-cz=1 \end{array}\right\}$$

14. $$\left.\begin{array}{l} x+y+z=a+b+c \\ bx+cy+az=cx+ay+bz=bc+ca+ab \end{array}\right\}$$

15. $$\left.\begin{array}{l} x+y+z=0,\ ax+by+cz=0 \\ bcx+cay+abz+(b-c)(c-a)(a-b)=0 \end{array}\right\}$$

16. $$\frac{6y-4x}{3z-7x}=\frac{5z-x}{2y-3z}=\frac{y-2z}{3y-2x}=1.$$

IV. METHOD OF UNDETERMINED MULTIPLIERS

386. Consider the following three equations containing three unknown quantities :—

$$\left.\begin{aligned} ax+by+cz&=d \ldots\ldots\ldots\ldots\ldots (1)\\ a'x+b'y+c'z&=d' \ldots\ldots\ldots\ldots\ldots (2)\\ a''x+b''y+c''z&=d'' \ldots\ldots\ldots\ldots\ldots (3)\end{aligned}\right\}$$

Let l, m, n be three quantities whose values are at present undetermined.

Multiply (1) by l, (2) by m, (3) by n, and add, then

$$(al+a'm+a''n)x+(bl+b'm+b''n)y+(cl+c'm+c''n)z$$
$$=ld+md'+nd'' \ldots\ldots (4).$$

Let such values be given to l, m and n as will make the coefficients of y and z each equal to zero, then

$$\left.\begin{aligned} bl+b'm+b''n&=0 \ldots\ldots\ldots\ldots (5)\\ cl+c'm+c''n&=0 \ldots\ldots\ldots\ldots (6)\end{aligned}\right\}$$

and $(al+a'm+a''n)x=ld+md'+nd''$.

$$\therefore\ x=\frac{ld+md'+nd''}{al+a'm+a''n} \ldots\ldots\ldots (7)$$

From (5) and (6), by the *Rule of Cross Multiplication*, we have

$$\frac{l}{b'c''-b''c'}=\frac{m}{b''c-bc''}=\frac{n}{bc'-b'c}=k \text{ (suppose)}$$

where k may be any quantity whatever.

$$\therefore\ l=k(b'c''-b''c'),\ m=k(b''c-bc''),\ n=k(bc'-b'c).$$

Hence, $k(b'c''-b''c')$, $k(b''c-bc'')$, $k(bc'-b'c)$ are the multipliers which will eliminate y and z and give the value of x. The value of x is found by substituting the values of l, m, n in (7); thus, we get

$$x=\frac{d(b'c''-b''c')+d'(b''c-bc'')+d''(bc'-b'c)}{a(b'c''-b''c')+a'(b''c-bc'')+a''(bc'-b'c)}.$$

As in the value of x, k becomes cancelled, we can evidently take $b'c''-b''c'$, $b''c-bc''$ and $bc'-b'c$ for the multipliers which eliminate y and z and give the value of x. Hence, the following Rule to find x :—

Multiply (1) by $b'c''-b''c'$, (2) by $b''c-bc''$ and (3) by $bc'-b'c$, and add; then y and z will vanish and x can be easily found.

Similarly the values of y and z may be obtained. It will be noticed that

(i) the value of y can be found from that of x by changing a, a', a'' into b, b', b'' respectively and *vice versâ* ;

(ii) the value of z can be found from that of x by changing a, a', a'' into c, c', c'' respectively and *vice versâ* ;

(iii) the values of x, y and z have the same denominators.

Ex. 1. Solve
$$\left.\begin{aligned} x+5y-4z&=5 \;\ldots\ldots\ldots(1)\\ 3x-2y+2z&=14\;\ldots\ldots\ldots(2)\\ -10x+8y+\;z&=6\;\ldots\ldots\ldots(3)\end{aligned}\right\}\text{(C. E. 1867).}$$

Here, $b'c''-b''c'=(-2)\times 1-8\times 2=-2-16=-18.$

$b''c-bc''=8\times(-4)-5\times 1=-32-5=-37.$

$bc'-b'c=5\times 2-(-2)\times(-4)=10-8=2.$

Hence, multiply (1) by -18, (2) by -37, and (3) by 2.

$$\therefore\left.\begin{aligned} -18x-90y+72z&=-90\\ -111x+74y-74z&=-518\\ -20x+16y+2z&=12\end{aligned}\right\}$$

By addition, $-149x=-596$; $\therefore\; x=4.$

Similarly, $y=5$ and $z=6$.

Exercise CXLII.

Solve the following equations :—

1. $\left.\begin{aligned} x+y+z&=6\\ 3x-y+2z&=7\\ 4x+3y-z&=7\end{aligned}\right\}$ (B. M. 1880).

2. $\left.\begin{aligned} 5x+2y+z&=30\\ \tfrac{1}{2}x+\tfrac{4}{5}y-\tfrac{1}{10}z&=4\\ 2x+5y+10z&=129\end{aligned}\right\}$ (M. M. 1865).

3. $\left.\begin{aligned} x+2y+3z&=20\\ 2x+3y-5z&=-7\\ 4x-5y+7z&=21\end{aligned}\right\}$ (C. E. 1898).

4. $\left.\begin{aligned} 2x+3y+4z&=38\\ 3x-2y+5z&=26\\ 4x+6y-3z&=21\end{aligned}\right\}$ (C. E. 1901).

5. $\left.\begin{aligned} x+2y+3z&=\tfrac{10}{6}\\ 2x+3y+z&=2\\ 3x-4y-7z&=\tfrac{1}{6}\end{aligned}\right\}$ (M. M. 1899).

6. $\left.\begin{aligned} x+y+z&=1\\ ax+by+cz&=d\\ a^2x+b^2y+c^2z&=d^2\end{aligned}\right\}$

7. $\left.\begin{aligned} x+ay+bcz&=a^2\\ x+by+caz&=b^2\\ x+cy+abz&=c^2\end{aligned}\right\}$

8. $\left.\begin{aligned} ax+by+cz&=d\\ a^2x+b^2y+c^2z&=d^2\\ a^3x+b^3y+c^3z&=d^3\end{aligned}\right\}$

V. EASY HIGHER SIMULTANEOUS EQUATIONS.

387. The following are typical examples with their solutions.

Ex. 1. Solve $x+y=8$..................(1)
$x^2+y^2=34$..................(2)

We have $(x-y)^2=2(x^2+y^2)-(x+y)^2=2\times 34-8^2$, from (1) & (2)
$=68-64=4$.

Taking the sq. root, we have $x-y=\pm 2$
From (1) $x+y=8$

By addition and subtraction, we obtain
$2x=10$ or 6 and $2y=6$ or 10; $\therefore x=5$ or 3 and $y=3$ or 5.

Ex. 2. Solve $x^2+y^2=a^2$..................(1)
$xy=b^2$..................(2) (C. E. 1883).

We have $(x+y)^2=x^2+y^2+2xy=a^2+2b^2$, from (1) and (2)
and $(x-y)^2=x^2+y^2-2xy=a^2-2b^2$, from (1) and (2).

Taking square roots, we have

$x+y=\pm\sqrt{(a^2+2b^2)}$
and $x-y=\pm\sqrt{(a^2-2b^2)}$
Hence $x=\frac{1}{2}\{\pm\sqrt{(a^2+2b^2)}\pm\sqrt{(a^2-2b^2)}\}$.
and $y=\frac{1}{2}\{\pm\sqrt{(a^2+2b^2)}\mp\sqrt{(a^2-2b^2)}\}$.

Ex. 3. Solve $xy=10$......(1), $yz=30$......(2), $zx=12$......(3).

Multiplying (1), (2) and (3) together, we get
$x^2y^2z^2=10\times 30\times 12=10^2\times 36=10^2\times 6^2$.

Taking the square root, $xyz=\pm 10\times 6=\pm 60$..................(4).

Divide (4) by (1), (2) and (3) separately;
thus, $z=\pm 6$, $x=\pm 2$ and $y=\pm 5$.

$=\frac{1}{9}(3a+b)(x+y)^2$.........(1)
$(a-b)x-(a+b)y=\frac{1}{9}(a-3b)(x+y)^2$.........(2)

Dividing (1) by (2), we have

$$\frac{(a+b)x+(a-b)y}{(a-b)x-(a+b)y}=\frac{3a+b}{a-3b}.$$

By Comp. and Divd., $\dfrac{ax-by}{bx+ay}=\dfrac{2a-b}{a+2b}$.

Now, multiplying across, we obtain

$$a^2x - aby + 2abx - 2b^2y = 2abx + 2a^2y - b^2x - aby\ ;$$

$$\therefore\ (a^2+b^2)x = 2(a^2+b^2)y\ ;\quad \therefore\ x = 2y, \text{ dividing by } a^2+b^2.$$

Substitute this value of x in (1), and we have

$$2(a+b)y + (a-b)y = \tfrac{1}{9}(3a+b)(3y)^2, \text{ or } (3a+b)y = (3a+b)\,y^2\ ;$$

$$\therefore\ y = 0 \text{ or } 1, \text{ and }\ \therefore\ x = 0 \text{ or } 2.$$

Exercise CXLIII.

Solve the following equations :—

1. $\left.\begin{aligned} xy+3y&=20 \\ 5y-4&=2xy \end{aligned}\right\}$ (P. E. 1890)

2. $\left.\begin{aligned} x+y&=7 \\ x^2+y^2&=25 \end{aligned}\right\}$

3. $\left.\begin{aligned} xy=2,\ yz&=6 \\ zx&=3 \end{aligned}\right\}$

4. $\left.\begin{aligned} (a+b)x-(a-b)y&=\tfrac{1}{21}(3a+7b)(x^2-y^2) \\ (a-b)x+(a+b)y&=\tfrac{1}{21}(7a-3b)(x^2-y^2) \end{aligned}\right\}$ (M. M. 1871).

5. $x(y+z)=22,\ y(z+x)=40,\ z(x+y)=42.$ (M. M. 1857).

6. $\left.\begin{aligned} (6a+b)x+(a+6b)y&=\tfrac{1}{45}(a+b)xy \\ (9a-2b)x-(2a-9b)y&=\tfrac{1}{49}(a+b)xy \end{aligned}\right\}$

7. $\left.\begin{aligned} 5x+4y+3z&=48xyz \\ 3x+6y+5z&=46xyz \\ x+2y+3z&=18xyz \end{aligned}\right\}$

8. $\left.\begin{aligned} xy+2(x+y)&=8 \\ xz+2(x+z)&=11 \\ yz+2(y+z)&=16 \end{aligned}\right\}$

9. $\left.\begin{aligned} x(x+y+z)&=18 \\ y(x+y+z)&=27 \\ z(x+y+z)&=36 \end{aligned}\right\}$

10. $\left.\begin{aligned} 3x-4y+7z&=0 \\ 2x-y-2z&=0 \\ 5x^2-3y^2+4z^2&=1 \end{aligned}\right\}$

11. $xy+\dfrac{x}{y}=10,\ xy^2-x=6y.$

12. $x^2+y^2=74,\ xy=35.$

VI. SIMULTANEOUS EQUATIONS INVOLVING SURDS.

388. The following are illustrative Examples.

Ex. 1. Solve $\left.\begin{aligned} \sqrt{x}+\sqrt{y}&=2\ldots(1) \\ x+y&=3\ldots(2) \end{aligned}\right\}$ (M. M. 1860).

$$\left.\begin{aligned} &\text{Squaring (1)}\ x+2\sqrt{(xy)}+y=4 \\ &\text{From (2)}\ x\qquad\qquad\ \ +y=3 \end{aligned}\right\}\ \begin{aligned} &\text{By subtraction,} \\ &2\sqrt{(xy)}=1, \text{ and } \therefore\ 4xy=1\ldots(3) \end{aligned}$$

Now, since $(x-y)^2=(x+y)^2-4xy=9-1$, from (2) and (3)

$$\left.\begin{aligned}&=8;\ \therefore\ x-y=\pm 2\sqrt{2}\\ &\quad\text{and } x+y=3\end{aligned}\right\}$$

$\therefore$ By addition and subtraction, we have

$x=\frac{1}{2}(3\pm 2\sqrt{2})$ and $y=\frac{1}{2}(3\mp 2\sqrt{2})$.

Ex. 2. Solve
$$\left.\begin{aligned}\sqrt{(x+y)}+\sqrt{(x-y)}&=\tfrac{1}{2}\{xy-y\sqrt{(x^2-y^2)}\}\ldots(1)\\ \sqrt[4]{(x+y)}+\sqrt[4]{(x-y)}&=\sqrt{2}\ldots\ldots\ldots\ldots\ldots\ldots(2)\end{aligned}\right\}$$

From (1) $\sqrt{(x+y)}+\sqrt{(x-y)}=\frac{1}{4}y\{2x-2\sqrt{(x^2-y^2)}\}$

$$=\tfrac{1}{4}y\{\sqrt{(x+y)}-\sqrt{(x-y)}\}^2.$$

$\therefore\ \frac{1}{4}y\{\sqrt{(x+y)}-\sqrt{(x-y)}\}^3=\{\sqrt{(x+y)}+\sqrt{(x-y)}\}\times$
$\{\sqrt{(x+y)}-\sqrt{(x-y)}\}=(x+y)-(x-y)=2y.$

$\therefore\ \{\sqrt{(x+y)}-\sqrt{(x-y)}\}^3=8$, and $\therefore\ \sqrt{(x+y)}-\sqrt{(x-y)}=2\ldots(3)$

or $\{\sqrt[4]{(x+y)}+\sqrt[4]{(x-y)}\}\{\sqrt[4]{(x+y)}-\sqrt[4]{(x-y)}\}=2$, Art. 324.

$\therefore\ \sqrt[4]{(x+y)}-\sqrt[4]{(x-y)}=2/\sqrt{2}$, from (2)

$$=\sqrt{2}.$$

and $\sqrt[4]{(x+y)}+\sqrt[4]{(x-y)}=\sqrt{2}$, from (2)

$\therefore\ \sqrt[4]{(x+y)}=\sqrt{2}$ and $\sqrt[4]{(x-y)}=0$ (by addition and subtraction)

Hence $x+y=4$ and $x-y=0$, (raising to the 4th power).

$\therefore\ x=2$ and $y=2$.

Exercise CXLIV.

Solve the following equations :—

1. $\sqrt{y}-\sqrt{(y-2x)}=\sqrt{(48-2x)}$, $y(x-15)=36$. (M. M. 1868).

2. $$\left.\begin{aligned}\sqrt{(x+\sqrt{y})}+\sqrt{(x-\sqrt{y})}&=2\sqrt{(a+b)}\\ x-2\sqrt{(x^2-y)}&=a-2b\end{aligned}\right\}$$

3. $\frac{1}{2}(x+y)=x-y=\sqrt{(x+2y-1)}$.

4. $$\left.\begin{aligned}\sqrt{x}-\sqrt{(15+x)}&=\sqrt{(x+y)}\\ 3\sqrt{(15+x)}+2\sqrt{(x+y)}&=9\sqrt{(15+x)}\end{aligned}\right\}$$

5. $\dfrac{5\sqrt{(x+y)}}{x}+\dfrac{5\sqrt{(x+y)}}{y}=10\dfrac{2}{3}$, $\dfrac{3\sqrt{(x-y)}}{y}-\dfrac{3\sqrt{(x-y)}}{x}=\dfrac{4}{5}$.

Miscellaneous Simultaneous Equations.

Solve the following equations :—

1. $\frac{1}{x}+\frac{1}{y}=3,\ \frac{1}{y}+\frac{1}{z}=4,\ \frac{1}{z}+\frac{1}{x}=5.$ (M. M. 1863).

2. $xyz=(xy+xz-yz)=4(yz+xy-xz)$
$=6(xz+yz-xy).$ (M. M. 1864).

3. $x+y=2(z+1),\ y+z=x+1,\ z+x=3y+1.$ (M. M. 1866).

4. $\frac{x+2y+1}{2}=\frac{y+3z+2}{3}=\frac{z+4x+3}{4}=2.$ (M. M. 1871).

5. $\left.\begin{aligned}4x-5y+6z&=3\\8x+7y-3z&=2\\7x+8y+9z&=1\end{aligned}\right\}$ (B. M. 1862).

6. $\left.\begin{aligned}ax-by&=\tfrac{3}{2}(b-a)\\by+2cz&=\tfrac{3}{2}(a-c)\\ax+by+cz&=0\end{aligned}\right\}$
(M. M. 1872).

7. $\left.\begin{aligned}u+2x+y+2z&=-3\\2u+x+2y+z&=3\\u+2x+12y+z&=21\\u+x+6y+z&=10\end{aligned}\right\}$ (B. M. 1872).

8. $\left.\begin{aligned}2x-3y+z+1&=0\\5x-3z&=6\\3x+2y&=4\end{aligned}\right\}$
(M. M. 1888.)

9. $ax-by=\frac{1}{2}(b-a),\ ax+by=c(1+z),\ by-cz=\frac{1}{2}(c-b).$ (M. M. 1870).

10. $\left.\begin{aligned}a(x+y)+b(x-y)&=a^2-ab+b^2\\a(x+y)-b(x-y)&=a^2+ab+b^2\end{aligned}\right\}$ (M. M. 1876).

11. $\frac{1}{1-x+y}-\frac{1}{x+y-1}=\frac{2}{3},\ \frac{1}{1-x+y}-\frac{1}{1-x-y}=\frac{4}{3}.$ (B. M. 1889).

12. $\frac{5}{x-3}+\frac{3}{y-2}=5\frac{1}{2},\ \frac{9}{x-3}+\frac{2}{y-2}=6\frac{1}{2}.$

13. $\frac{1}{x}+\frac{3}{y}-\frac{2}{z}=6,\ \frac{3}{x}+\frac{1}{z}=5,\ \frac{2}{y}+\frac{5}{z}=16.$ (M. M. 1896).

14. $\left.\begin{aligned}x^2+y^2&=13\\xy&=\ 6\end{aligned}\right\}$

15. $\left.\begin{aligned}a(x+y)-b(x-y)&=2a^2\\(a^2-b^2)(x-y)&=4a^2b\end{aligned}\right\}$

16. $\left.\begin{aligned}&ax+by+cz=3,\ a^2x+b^2y+c^2z=a+b+c,\\&(x+y)ab+(y+z)bc+(z+x)ca=2(a+b+c)\end{aligned}\right\}$

17. $\frac{4}{x}+2y-3\sqrt{z}=2,\ \frac{6}{x}+4\sqrt{z}=11,\ 3y-4\sqrt{z}=1.$

18 $\frac{b}{x}+\frac{a+c}{y}=m,\ \frac{a-c}{x}+\frac{b}{y}=n.$ (M. M. 1887).

VII. HARDER PROBLEMS.

389. We shall consider here a few harder Problems involving Simultaneous Equations with their Solutions.

Ex. 1. A pound of tea and three pounds of sugar cost six shillings, but if sugar were to rise 50 per cent. and tea 10 per cent., they would cost seven shillings. Find the price of tea and sugar. (B. M. 1866).

Let x be the price of a ℔. of tea in shillings.

and y .... sugar ....

Then $x+3y=6$ ....(1)

Again, the price of tea rising 10 per cent. and of sugar 50 per cent. the price of 1 lb. of tea $=x(1+\frac{10}{100})=\frac{11}{10}x$ shillings and of 1 lb. of sugar $=y(1+\frac{50}{100})=\frac{3}{2}y$, shillings.

$\therefore \frac{11}{10}x+3\times\frac{3}{2}y=7$ ....(2)

Multiply (1) by 3; thus $3x+9y=18$

,, (2) by 2; thus $\frac{11}{5}x+9y=14$

By subtraction, $\frac{4}{5}x=4$ and $\therefore x=5$;

Again, from (1), substituting x, we have

$5+3y=6$, and $\therefore y=\frac{1}{3}$.

Hence the price of tea per ℔. is 5 shillings and of sugar $\frac{1}{3}s$. or $4d.$ per ℔.

Ex. 2. A and B went out to shoot. A shot 3 pheasants for every 5 partridges, and B 5 pheasants for every 9 partridges. A shot 4 birds to B's 5; how many pheasants, and how many partridges had they brought down when they had shot 126 birds? (M. M. 1866).

Let $3x$ be the no. of pheasants shot by A,

and $5y$ .... B.

Since A shoots 3 pheasants for every 5 partridges,

and B...... 5 .... 9 ....

$\therefore$ A shoots $5x$ and B shoots $9y$ partridges.

Hence, **A** shoots $3x+5x$ or $8x$ birds and **B** shoots $5y+9y$ or $14y$ birds.

$\therefore$ for every 4 birds of **A**, **B** shoots $4\times\frac{14y}{8x}=\frac{7y}{x}$ birds.

$\therefore$ By the question,

$$\left.\begin{array}{l} 8x+14y=126 \quad \ldots.(1), \\ \text{and } \frac{7y}{x}=5 \ldots\ldots..(2) \end{array}\right\} \quad \begin{array}{l}\text{Solving, we have} \\ x=7, y=5.\end{array}$$

Hence, the total no. of pheasants shot $=3x+5y=3\times7+5\times5$ $=21+25=46$ and no. of partridges $=126-46=80$.

Ex. 3. A tradesman sells two articles together for *Rs* 46, making 10 per cent. on one, and 20 per cent. on the other. If he had sold each article at 15 per cent. profit, the result would have been the same. At what price does he sell each article? (C. E. 1891).

Let x be the prime cost of the first article in rupees
and y...........................second..... ....

The selling price of the first article at 10 per cent. profit $=x(1+\frac{10}{100})Rs.$ or $\frac{11}{10}xRs.$ and of the second at 20 per cent. profit $=y(1+\frac{20}{100})Rs.$ or $\frac{6}{5}yRs.$

Again, the selling price of the whole at 15 per cent. profit $=(x+y)(1+\frac{15}{100})Rs.$ or $\frac{23}{20}(x+y)Rs.$

$\therefore$ By the question, $\frac{11}{10}x+\frac{6}{5}y=\frac{23}{20}(x+y)=46.$

$\therefore$ $11x+12y=460$ and $x+y=40$.

Solving which $x=y=20$.

Hence the required prices are $Rs.\frac{11}{10}x$ or $Rs.22$ and $Rs.\frac{6}{5}y$ or $Rs.24$ respectively.

Ex. 4. Two men **A** and **B** are employed on a piece of work which has to be finished in 14 days. In 3 days they do $\frac{1}{5}$th of the work, and then **A**'s place is taken by **C**. **B** and **C** work for one day, and do $\frac{1}{20}$th of the whole work, and then **B**'s place is taken by **A**. **A** and **C** finish the work a day before the appointed time. Find the time in which the work could have been done (1) by each working separately, (2) by all working together. (M. M. 1886).

(1) Let x be the no. of days in which **A** can do the work,
y...... ....**B**...............,
and z.........................**C**........ ......

Then the daily work of A, B and C $=\frac{1}{x}, \frac{1}{y}$, and $\frac{1}{z}$ respectively.

Since A and B do $\frac{1}{5}$th of the work in 3 days,

$$\therefore\ 3\left(\frac{1}{x}+\frac{1}{y}\right)=\frac{1}{5}\ \ldots\qquad\ldots\ (1)$$

Again, since B and C do $\frac{1}{20}$th work in 1 day,

$$\therefore\ \frac{1}{y}+\frac{1}{z}=\frac{1}{20}\ \ldots\qquad\ldots\ (2)$$

Also the work is finished a day earlier, *i.e.* in 13 days.

$\therefore$ A and C do in $(13-3-1)$ or 9 days the remaining work which $=1-(\frac{1}{5}+\frac{1}{20})=\frac{3}{4}$.

$\therefore$ By the question, $9\left(\frac{1}{x}+\frac{1}{z}\right)=\frac{3}{4}\ \ldots\qquad\ldots\ (3)$

Solving equations (1), (2) and (3), we have

$x=20, y=60$ and $z=30$.

Hence A, B and C can do the work in 20, 60 and 30 days respectively.

(2) Again, from (1) $\frac{1}{x}+\frac{1}{y}=\frac{1}{15}$

" (2) $\frac{1}{y}+\frac{1}{z}=\frac{1}{20}$

" (3) $\frac{1}{x}+\frac{1}{z}=\frac{1}{12}$

Adding and dividing by 2,

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2}\times\frac{1}{5}=\frac{1}{10}.$$

Hence A, B and C can together do the whole work in 10 days.

Ex. 5. A train running from A to B meets with an accident 50 miles from A, after which it moves with $\frac{3}{5}$ths of its original velocity and arrives at B 3 hours late. Had the accident happened 50 miles further on, it would have been only 2 hours late. Find the distance from A to B, and the original speed of the train. (M. M. 1857).

Let x be the distance from A to B in miles,

and y the original velocity of the train per hour in miles.

Then the usual time $=\frac{x}{y}$ hours.

The accident happening 50 miles from A, the train travels the first 50 miles at the rate of y miles per hour, and the remaining distance $(x-50)$ miles at the rate of $\frac{3}{5}y$ miles per hour,

and $\therefore$ the whole time taken $=\left(\frac{50}{y}+\frac{x-50}{\frac{3}{5}y}\right)$ hours.

Similarly, supposing the accident to have happened 100 miles from **A**, the whole time taken would be

$$=\left(\frac{100}{y}+\frac{x-100}{\frac{3}{5}y}\right) \text{ hours.}$$

$\therefore$ By the question,

$$\frac{50}{y}+\frac{x-50}{\frac{3}{5}y}=\frac{x}{y}+3\ldots(1)\ ;\quad \frac{100}{y}+\frac{x-100}{\frac{3}{5}y}=\frac{x}{y}+2\ldots(2).$$

Subtracting, (2) from (1), we have

$$-\frac{50}{y}+\frac{50}{\frac{3}{5}y}=1\ ;\ \text{when } y=33\tfrac{1}{3}.$$

From (1), $50+\frac{5}{3}(x-50)=x+3y$; substituting y in this,

we have $50+\frac{5}{3}(x-50)=x+3\times33\frac{1}{3}=x+100$.

whence $x=200$.

Hence distance $=200$ miles and rate $=33\frac{1}{3}$ miles per hour.

Ex. 6. **A** challenged **B** to ride a bicycle race of 1040 yds. ; he first gave **B** a start of 120 yds. and lost by 5 seconds ; he then gave **B** 5 second's start and won by 120 ft. How long does each take to ride the distance ? (C.E. 1881).

Let x be **A**'s time in seconds to ride the whole distance, and y...**B**'s ....

In the first race **A** rode 1040 yds. and **B** $(1040-120)$ or 920 yds.

$\therefore$ By the question, $x=\frac{920}{1040}y+5=\frac{23}{26}y+5\ldots(1)$.

In the second race **A** rode 1040 yds. and **B** (1040 yds. $-$ 120 ft.) or 1000 yds.

$\therefore$ By the question, $x+5=\frac{1000}{1040}y=\frac{25}{26}y\ldots\ldots(2)$.

Subtracting (1) from (2), we have $5=\frac{1}{13}y-5$; $\therefore$ $y=130$.

From (1) $x=\frac{23}{26}\times130+5=115+5=120$.

Hence **A** takes 120 sec. or 2 min. and **B** 130 sec. or 2 min. 10 sec.

Ex. 7. A boat goes up-stream 30 miles and down-stream 44 miles in 10 hours ; it also goes up-stream 40 miles and down-stream 55 miles in 13 hours ; find the rate of the stream and of the boat : (C.E. 1880).

Let x be the rate of rowing in still water in miles per hour, and y the rate of the stream in miles per hour.

Then motion up-stream $=(x-y)$ miles per hour and motion down-stream $=(x+y)$ miles per hour.

$$\therefore \text{ By the question, } \left.\begin{aligned}\frac{30}{x-y}+\frac{44}{x+y}&=10\ \ldots\ldots(1)\\ \text{and}\quad \frac{40}{x-y}+\frac{55}{x+y}&=13\ldots\ldots\ldots(2)\end{aligned}\right\}$$

Subtract 3 times (2) from 4 times (1);

$$\therefore \frac{11}{x+y}=1 \text{ and } \therefore x+y=11 \quad \ldots \quad .. \; (3).$$

Subtract 5 times (1) from 4 times (2);

$$\therefore \frac{10}{x-y}=2 \text{ and } \therefore x-y=5 \quad \ldots \quad \ldots (4)$$

Adding (3) and (4), we have $2x=16$ and $\therefore x=8$.

Subtracting (4) from (3), we have $2y=6$, and $\therefore y=3$.

Hence the rate of the boat $=8$ miles per hour and of the stream $=3$ miles per hour.

Ex. 8. Two trains, 92 ft. and 84 ft. long respectively, are moving with uniform velocities on parallel rails in opposite directions, and are observed to pass each other in $1\frac{1}{2}$ sec.; but when they are moving in the same direction, their velocities being the same as before, the faster train is observed to pass the other in 6 seconds. Find the rates at which the trains are moving.

Let x be the speed of the faster train in miles per hour,

and y the................other............ ....

then $(x+y)$ miles per hour is the *relative* speed of the two trains moving in opposite directions and $(x-y)$ miles per hour is their *relative* speed moving in the same direction. Also the trains pass each other when they have travelled a distance equal to the sum of the lengths of the two trains, which $=(92+84)$ft. $=176$ ft.

$$\therefore \text{By the question,} \left.\begin{array}{r} \dfrac{176}{1760\times 3(x+y)}=\dfrac{1\frac{1}{2}}{60\times 60} \ldots \; . \; (1) \\ \text{and} \quad \dfrac{176}{1760\times 3(x-y)}=\dfrac{6}{60\times 60} \ldots\ldots\ldots (2) \end{array}\right\}$$

$$\left.\begin{array}{ll} \text{From (1), we have} & x+y=80 \\ \quad ,, \quad (2) & x-y=20 \end{array}\right\} \text{Hence } x=50 \text{ and } y=30.$$

Hence the rate of the faster train is 50 miles and of the other 30 miles per hour.

Ex. 9. A certain number consists of two digits whose sum is 8, another number is obtained by reversing the digits. If the product of these two is 1855, find the number. (B. M. 1877).

Let x be the digit in the tens' place,

and y the digit in the units' ......

then the number $=10x+y$.

Also the number formed by reversing the digits is $10y+x$.

$\therefore$ By the question, $x+y=8 \cdots\cdots(1)$

$$(10x+y)(10y+x)=1855 \ldots\ldots(2)$$

From (2), $10x^2+101xy+10y^2=1855 \ldots\ldots(3)$

Squaring (1) and multiplying by 10, we have

$$10x^2+20xy+10y^2=640 \ldots\ldots\ldots\ldots(4)$$

Subtracting (4) from (3), $81xy=1215$; $\therefore xy=15 \ldots\ldots(5)$

Now $(x-y)^2=(x+y)^2-4xy=8^2-4\times 15$, from (1) and (5)

$$=64-60=4; \quad \therefore x-y=\pm 2.$$

Hence, we find $x=5$ or 3 and $y=3$ or 5.

Hence the number reqd. $=53$ or 35.

Exercise CXLV.

1. A man has in his purse sovereigns and shillings. If he receive as many sovereigns as he has in his purse, and pay away his shillings and an equal number of sovereigns he will have 8 coins. But if he double the number of his shillings, retaining the original number of sovereigns, he will have 9 coins. How many sovereigns and how many shillings were in his purse at first? (B. M. 1861).

2. Water is admitted into a cistern by three cocks, two of which are exactly equal. When they are all open, $\frac{5}{12}$ths of the cistern is filled in 4 hours; and if one of the equal cocks is stopped, [illegible]ths of the cistern is filled in 10 hours and 40 minutes. In how many hours would each cock fill the cistern? (A. I. E. 1889).

3. The fore-wheel of a carriage makes six revolutions more than the hind-wheel in going 120 yards; if the circumference of the fore-wheel be increased by one-fourth of its present size, and the circumference of the hind-wheel by one-fifth of its present size, the six will be changed to four. Required the circumference of each wheel.

4 A railway train after travelling for one hour meets with an accident which delays it one hour, after which it proceeds at $\frac{3}{5}$ths of its former rate, and arrives at the terminus 3 hours behind time; had the accident happened 50 miles further on, the train could have arrived 1 hour 20 minutes sooner. Required the length of the line. (B. M. 1866).

5. A letter-carrier has to go daily from P to Q in a prescribed time. If he goes a mile an hour faster than his ordinary rate,

he arrives at Q half an hour before the time. But if he goes a mile an hour slower, he arrives three-quarters of an hour too late. Find his ordinary rate, and the distance from P to Q. (M. M. 1884).

6. A train travelled a certain distance at a uniform rate. Had the speed been 6 miles an hour more, the journey would have occupied 4 hours less: and had the speed been 6 miles an hour less, the journey would have occupied 6 hours more. Find the distance. (P. E. 1889).

7. A set of bearers on a journey perform one-third of the distance at a certain rate and then halt one hour to take their food. The remainder of the journey is accomplished at only two-thirds of the former rate, and the bearers reach their destination in 7 hours after first starting. Had they travelled at the former rate $4\frac{1}{2}$ miles further than they did before halting, they might have halted $22\frac{1}{2}$ minutes longer and yet reached the end of their journey in the same time. Find the length of the journey. (M. M. 1885).

8. In a quarter of a mile race, A gives B a start of 22 yards, and beats him by 2 seconds: and in a 300 yards race, he gives him a start of 2 seconds, and beats him by $10\frac{1}{3}$ yards. Find the rates of each. (M. M. 18[illegible]

9. A room of which the floor is rectangular is such that the addition of a foot to the height will increase the area of the walls as much as the addition of a foot to both the length and breadth, the increase in each case being 60 square feet; and if the floor be made square, the perimeter remaining the same as before, its area will be increased by 9 square feet. Find the length, breadth and height of the room. (M. M. 1868).

10. A horse-man travelling at a walking pace of 4 miles an hour meets a bandy going in the opposite direction at the rate of 2 miles an hour; after proceeding at the same pace for half an hour, he turns and canters back till he overtakes the bandy. If he had continued for another quarter of an hour before turning, the bandy would have been $\frac{3}{8}$ths of a mile further on before it was overtaken. Find the rate at which the horse-man cantered. (M. M. 1869).

11. A and B play four games of chance of which A wins the first and last, and B the other two. The amount which each stakes for the first game is half the whole sum of money possessed by both together, and for the other games half the money possessed by the loser of the preceding game. At the end of the fourth game, A finds that he has 18 shillings less than he would have had if he had won them all, and B finds that he has 9 shillings less than he had at starting. Find the amount of money possessed by each at first. (M. M. 1871).

12. A man rowing against a stream meets a log of wood which is being carried down by the current. He continues rowing in the same direction for a quarter of an hour longer and then turns and rows down the stream, overtaking the log $1\frac{1}{2}$ miles lower down than the point where he first met it. Find the rate at which the current flows. (M. M. 1874).

13. A boat's crew rowed $3\frac{1}{2}$ miles down a river and up again in 100 minutes. Had the stream been half as strong again, they would have taken $31\frac{1}{4}$ minutes longer. Find the rate of the stream. (B. M. 1860).

14 A merchant has a certain number of Back Bay and Mazagon Shares. The market rate for the two shares was *Rs.*2000, but Mazagon Shares rose 10 per cent. and Back Bay fell 20 per cent. The value of the two shares became in consequence $12\frac{1}{2}$ per cent. less than before. Find the original market value of each share. (B. M. 1867).

15. A criminal having escaped from prison, travelled 10 hours before his escape was known. He was pursued so as to be gained upon 3 miles an hour. After his pursuers had travelled 3 hours, they met an express going at the same rate as themselves, who met the criminal 2 hours 24 minutes before In what time after the commencement of the pursuit will they overtake him? (B. M. 1883).

16. A mail coach runs between two places **A** and **B**, and back again. A traveller who starts walking from **A** 5 hours before the mail coach is overtaken by it half way between **A** and **B**. He then doubles his rate of walking and meets the mail coach on its return journey 3 miles from **B**. The traveller then goes to **B** at the same rate and returns, and by the time he comes again midway between **A** and **B**, the mail coach reached **A** Find the distance between **A** and **B**, and the rate at which the mail coach runs. (M. M. 1878).

17. A gentleman went out for a walk; and after having been out 12 minutes, was overtaken by his servant who had run from the house at thrice his master's pace. The master then bade the servant run back at the same rate to the house and bring his cigars, while he walked on at his former pace. If the master was one mile from the house when overtaken the second time, at what rate did he walk? (M. M. 1873).

18. **A** and **B** start together on a certain journey. When they have walked a distance of a miles, **A** finds it necessary to return home, and goes at twice his former rate. He then starts again at m/n times his original pace, and just at the end of the journey overtakes **B**, who since **A** left him, had gone at n/m times the original pace. How long was the journey? (M. M. 1865).

19. A and B (one of whom could do the work alone in a less number of days than the other) agree to reap a field for *Rs*.20. If they had worked together every day, the field would have been reaped in 15 days; but at the end of 7 days A left off working for 4 days; and it consequently took $16\frac{1}{2}$ days to reap the field. In how many days could A alone, and in how many days could B alone, have reaped the field; and what part of the *Rs*.20 ought each to receive for the work he actually did? (B.M. 1869).

20. A person left Poona in the Sattara mail buggy at 2 P.M. and having proceeded a certain distance he went out of the buggy and returned to Poona on foot, walking at the rate of 3 miles an hour, and he reached Poona at 8 P.M. Had he gone 6 miles further in the buggy he would not have got back to Poona till 10 hours 40 min. P.M. How far did he go towards Sattara and what was the speed of the buggy? (B. M. 1870).

21. A person rows from Cambridge to Ely, a distance of 20 miles, and back again, in 10 hours, the stream flowing uniformly in the same direction all the time; and he finds that he can row 2 miles against the stream in the same time that he rows 3 miles with it. Find the time of his going and returning.

22. Some smugglers found a cave, which would just exactly hold the cargo of their boat, *viz* 13 bales of silk and 33 casks of rum. While unloading, a revenue cutter came in sight, and they were obliged to sail away, having landed only 9 casks and 5 bales, and filled one-third of the cave. How many bales separately, or how many casks, would it hold?

23. A number consists of three digits, the right-hand one being zero. If the left-hand and middle digits be interchanged, the number is diminished by 180; if the left-hand digit be halved, and the middle and right-hand digits be interchanged, the number is diminished by 336; find the number. (B.M. 1887).

24. In a half-mile race A gives B 22 yards' start and wins by 6 seconds. In a three quarter mile race he gives him 20 seconds' start, but is beaten by 29 yds. 1 ft. In what time can each of them run a mile? (M. M. 1892).

25. Two trains start at the same time from A and B for the junction C. The train from A should run at 24 miles an hour and reach the junction half an hour before that from B, which travels 18 miles an hour. But the former is so retarded as only to run at an average rate of 22 miles an hour. The two trains arrive at the junction at the same time. How far are A and B respectively from C, and how long were the trains upon the road? (M. M. 1862).

26. A and B start from opposite ends of a straight course, each walking uniformly, A, who is the faster walker, at the rate of 4 miles

an hour and meet at the end of 2 hours. If, when **A** reached the middle point of the course, they had interchanged their rates of walking, they would have met a quarter of a mile nearer the middle point. Find **B**'s rate of walking, and the length of the course. (M. M. 1870).

27. Two cyclists ride from **A** to **B**, a distance of 55 miles, and the first arrives 30 minutes before the second. They then ride from **B** to **A**, the first giving the second a start of 4 miles, and yet arriving 6 minutes before him. Find the rate of each cyclist in miles per hour. (B. M. 1901).

28. **A** and **B** start simultaneously from Poona to go to Kirkee. **A** would reach Kirkee half a hour before **B**, but missing his way, goes a mile and back again needlessly, during which he walks at twice his former pace, and he reaches Kirkee 6 min. before **B**; **C** starts 20 min. after **A** and **B**, and walking at the rate of $2\frac{1}{2}$ miles an hour arrives at Kirkee 10 min. after **B**. Find the rates of walking of **A** and **B** and the distance from Poona to Kirkee. (B. M. 1868).

CHAPTER XVIII.

RATIO, PROPORTION AND VARIATION.

I. RATIO.

390. The **Ratio** of one quantity to another is that relation which the former bears to the latter in respect of magnitude, when the comparison is made by considering, not *by how much* the one is greater or less than the other, but *what number of times* it contains it, or is contained in it, *i. e.* what *multiple*, *part*, or *parts*, or in other words, what *fraction* the first is of the second.

This is, in fact, the way in which we naturally, and, as it were, unconsciously, compare the magnitude of quantities. Thus the mere numerical *difference* between 999 and 1000 is the same as between 1 and 2; but no one would hesitate to say that 999 compared with 1000, is much *greater* than 1 compared with 2. The reason is, that the mind considers intuitively that 999 is a much greater fraction of 1000 than 1 is of 2; and this is what we should express by saying that the ratio of 999 to 1000 is greater than that of 1 to 2. On the other hand, we should say at once that 1001 compared with 1000, is much *less* than 2 compared with 1, the fraction in the former case being less than that in the latter.

391. The ratio, then, of one quantity to another is represented by the fraction obtained by dividing the former by the latter.

Thus, the ratio of 6 to 3 is $\frac{6}{3}$ or 2 ; that of 15 to 40 is $\frac{15}{40}$ or $\frac{3}{8}$; that of $4a$ to $6b$ is $\frac{4a}{6b}$ or $\frac{2a}{3b}$.

392. The two quantities compared (if they are not mere numbers, or algebraical quantities expressing numbers) must be of the same kind, or one could not be a fraction of the other.

Thus, the ratio of *Rs.*9 to *Rs.*12 is the same as that of 9 mds. to 12 mds., or of 9 to 12, or of 3 to 4, or of $\frac{3}{4}$ to 1 ; since, in each of these pairs of quantities, the first is $\frac{3}{4}$ of the second, and hence $\frac{3}{4}$ is the value of each of these ratios ; in saying which we may suppose, if we please, a tacit reference to 1, *i.e.* in saying that the ratio of *Rs.*9 to *Rs.*12 is $\frac{3}{4}$, we may either imply that *Rs.*9 is $\frac{3}{4}$ of *Rs.*12, or that the ratio of *Rs.*9 to *Rs.*12 is the same as that of $\frac{3}{4}$ to 1.

393. The ratio of one quantity to another is expressed by two points placed between them, as $a : b$, where a and b are the **terms** of the ratio ; the first term a is called the **antecedent**, and the second term b is called the **consequent** of the ratio.

394. A ratio is said to be a ratio of **greater inequality**, of **less inequality**, or of **equality**, according as the antecedent is *greater* than, *less* than, or *equal* to, the consequent.

Thus, the ratio $5 : 4$ is one of *greater inequality*, the ratio $4 : 5$ is one of *less inequality*, and the ratio $a : a$ is one of *equality*.

395. Problems upon ratios are solved by representing them by their corresponding fractions, which may now be treated by the ordinary rules.

Thus ratios are **compared** with one another, by reducing the corresponding fractions to common denominators, and comparing the numerators.

Ex. 1. Compare the ratios $5 : 7$ and $4 : 9$.

Here, $5 : 7 = \frac{5}{7}$ and $4 : 9 = \frac{4}{9}$.

Now, $\frac{5}{7}, \frac{4}{9} = \frac{45}{63}, \frac{28}{63}$; but $45 > 28$, $\therefore\ 5 : 7 > 4 : 9$.

Ex. 2. Find the ratio of $\frac{5}{7} : \frac{4}{9}$.

The reqd. ratio $= \frac{5}{7} \div \frac{4}{9} = \frac{5}{7} \times \frac{9}{4} = \frac{45}{28}$.

396. *A ratio of greater inequality is diminished, and a ratio of less inequality increased by adding the same positive quantity to both its terms.*

Let $\frac{a}{b}$ be the given ratio, and let x be added to each term, the resulting ratio being $\frac{a+x}{b+x}$, where x is a positive quantity.

Now, $\frac{a}{b}-\frac{a+x}{b+x}=\frac{ab+ax-ab-bx}{b(b+x)}=\frac{x(a-b)}{b(b+x)}$; and $a-b$ is positive or negative according as a is greater or less than b.

Hence, $\frac{a}{b}>\frac{a+x}{b+x}$, if a be $>b$, *i.e.* if $a:b$ be a ratio of *greater* inequality,

and $\frac{a}{b}<\frac{a+x}{b+x}$, if a be $<b$, *i.e.* if $a:b$ be a ratio of *less* inequality.

In like manner, it may be shewn that *a ratio of greater inequality is increased, and of less inequality diminished, by subtracting the same quantity from both its terms.*

397. Compound Ratio. If the fractions denoting two or more ratios be multiplied together, the resulting fraction is said to be the **ratio compounded of the ratios** represented by them.

Thus, if $a:b$, $c:d$, $e:f$, &c., be any ratios; their *compound ratio* will be ace &c. : bdf &c., or $=\frac{ace\ \&c.}{bdf\ \&c}$.

(i) The ratio $a^2:b^2$ is called the **duplicate** (*i. e.* squared) ratio of $a:b$.

(ii) The ratio $a^3:b^3$ is called the **triplicate** ratio of $a:b$.

(iii) The ratio $\sqrt{a}:\sqrt{b}$ is called the **subduplicate** ratio of $a:b$.

Ex. 1. What is the ratio compounded of $2:3$, $6:7$, $14:15$?

The reqd. ratio $=\frac{2}{3}\times\frac{6}{7}\times\frac{14}{15}=\frac{8}{15}$ or $8:15$.

Ex. 2. What is the duplicate ratio of $2:3$?

The reqd. ratio $=2^2:3^2=4:9$.

398. The ratio of any two quantities cannot always be expressed exactly by the ratio of two integers. For, if either, or both, of the terms of a ratio be a surd quantity, then no two integers can be found which will *exactly* measure their ratio.

Thus, the ratio of $\sqrt{7}:4$ cannot be exactly expressed by any two integers.

399. When the ratio of any two quantities can be expressed exactly as that between two integers, the quantities are said to be **commensurable**; otherwise, they are said to be **incommensurable.**

Although the ratio of two incommensurable quantities cannot be expressed *exactly* by the ratio of two integers, we can always find two integers whose ratio differs from that required by as small a quantity as we please.

Thus, $\frac{\sqrt{7}}{4} = \frac{2 \cdot 6457513\ldots}{4} = \cdot 6614378\ldots$

and therefore $\frac{\sqrt{7}}{4}$ is $> \frac{6614378}{10000000}$ and $< \frac{6614379}{10000000}$,
and it is obvious that any degree of approximation may be arrived at by calculating the value of $\sqrt{7}$ to more places of decimals.

Ex. 1. If $\frac{5x-2y}{3x+4y} = \frac{2}{9}$, find the ratio $x : y$.

Multiplying crosswise and transposing, we have

$39x = 26y$; $\therefore$ $3x = 2y$ and $\therefore$ $x/y = \frac{2}{3}$.

Ex. 2. If $2x^2 - 11xy + 12y^2 = 0$, find the ratio $x : y$.

Dividing by y^2, $2\left(\frac{x}{y}\right)^2 - 11\frac{x}{y} + 12 = 0$. Putting k for $\frac{x}{y}$,

$2k^2 - 11k + 12 = 0$, or $(2k-3)(k-4) = 0$.

$\left.\begin{array}{l}\therefore\ 2k-3=0, \text{ which gives } k=\frac{3}{2} \\ \text{and } k-4=0, \text{ which gives } k=4\end{array}\right\}$ Hence $\frac{x}{y} = \frac{3}{2}$ or 4.

Ex. 3. If $\frac{x}{y} = \frac{3}{4}$, find the value of $\frac{7x-4y}{3x+y}$.

$$\frac{7x-4y}{3x+y} = \frac{\frac{7x}{y} - 4}{\frac{3x}{y} + 1} = \frac{\frac{21}{4} - 4}{\frac{9}{4} + 1} = \frac{5}{4} \times \frac{4}{13} = \frac{5}{13}.$$

Ex. 4. If $2a : 3b$ be in the duplicate ratio of $2a - x : 3b - x$, prove that $x^2 = 6ab$.

We have, $\frac{2a}{3b} = \left(\frac{2a-x}{3b-x}\right)^2$.

Multiplying crosswise and squaring, we get

$2a(9b^2 - 6bx + x^2) = 3b(4a^2 - 4ax + x^2)$;

$\therefore$ $18ab^2 - 12abx + 2ax^2 = 12a^2b - 12abx + 3bx^2$;

$\therefore$ $x^2(2a - 3b) = 6ab(2a - 3b)$; $\therefore$ $x^2 = 6ab$,

since $2a - 3b$ is not zero, by supposition.

400. It appears from Art. 396 that by adding the same positive quantity to both the terms of a ratio, it is made more nearly equal to unity, and by taking x, the quantity added, large enough, it may be made to differ as little as possible from unity. The following is an illustrative example.

Ex. A is 32 years old, B is 5 years old. What is the least number of years after which the ratio of their ages will be less than 3 : 1?

After x years, A's age will be $32+x$ and B's $5+x$ years.

The value of the ratio $\frac{32+x}{5+x}$, when x is gradually increased, will become less and less than $\frac{32}{5}$, and nearer and nearer to unity.

When $\frac{32+x}{5+x}=\frac{3}{1}$, we have $x=8\frac{1}{2}$, and for this value of x the ratio of their ages will become 3 : 1, and for any *greater* value of x the ratio will be *less* than 3 : 1.

Hence, the least number of years required is 9.

Exercise CXLVI.

1. Which is the greater ratio :—

(1) 3 : 4 or 4 : 5? (2) 13 : 14 or 23 : 24?

(3) 3 : 7, 7 : 11 or 11 : 15? (4) $x+y : y$ or $4x : x+y$?

(5) $a+b : a-b$ or $a^2+b^2 : a^2-b^2$, supposing $a > b$?

(6) $x^2+y^2 : x+y$ or $x^3+y^3 : x^2+y^2$?

(7) $x^2+y^2 : 1$ or $x^6+y^6 : x^4-x^3y+x^2y^2-xy^3+y^4$?

2. Find the ratio compounded of

(1) 3 : 5, 10 : 21 and 14 : 15. (2) 7 : 9, 102 : 105 and 15 : 17.

(3) 169 : 200 and the duplicate ratio of $15a^2 : 26b^2$.

(4) $3a : 4b$ and the subduplicate ratio of $25b^4 : 49a^4$.

(5) $x^2-9x+20 : x^2-6x$ and $x^2-13x+42 : x^2-5x$.

(6) $a+b : a-b$, $a^2+b^2 : (a+b)^2$ and $(a^2-b^2)^2 : a^4-b^4$.

(7) $a^2+1 : a^2-1$, $a^4+1 : a^4-1$ and $(a^2-1)^2(a^4-1) : a^8+1$.

3. What is the ratio compounded of the duplicate ratio of $a+b : a-b$, and the difference of the duplicate ratios of $a : a$ and $a : b$, supposing $a > b$?

4. Shew that the ratio $a+b : a-b$ is greater or less than the ratio $a^2+b^2 : a^2-b^2$, according as the ratio $a : b$ is one of greater or less inequality.

5. If $a : b$ is a ratio of greater inequality, show that $a : b$ is greater than $a^2+b^2 : 2ab$. (B. P. E. 1888).

6. If $4a : 5b$ be in the duplicate ratio of $4a+x : 5b+x$, find the value of x.

7. What quantity must be (i) added to, (ii) taken from each of the terms of the ratio $a : b$, that it may become equal to $c : d$?

8. If $2x+5 : 3x+10$ be in the duplicate ratio of $3 : 4$, find x.

9. If $a : b$ be the subduplicate ratio of $a-x : b-x$, find x.

10. If $\frac{17x+3y}{5y-7x}=\frac{76}{3}$, find the ratio $x : y$.

11. If $7x-4y : 3x+y=5 : 13$, find the ratio $x : y$.

12. If $\frac{a}{b}=\frac{3}{4}$ and $\frac{c}{d}=\frac{5}{7}$, find the value of $\frac{3ac-bd}{4bd-7ac}$.

13. If $\frac{x}{y}=\frac{5}{7}$, find the value of $\frac{x+y}{y-x}$.

14. If $\frac{a}{b}=\frac{10}{3}$, find the value of $\frac{a-3b}{2a-5b}$.

15. If $\frac{2x^2-3y^2}{x^2+y^2}=\frac{2}{41}$, find the ratio $x : y$.

16. If $6y^2+35x^2=29xy$, find the ratio $x : y$.

17. If $a : b$ be in the duplicate ratio of $a+x : b+x$, find x. (P. E. 1896).

18. What number must be added to each term of the ratio $8 : 5$, to make it equal to the ratio $4 : 3$?

19. A certain ratio becomes $4 : 5$, if 2 be added to each of its terms; and becomes $3 : 4$, if 1 be subtracted from each of its terms; find the ratio.

20. If from each term of the ratio $a : b$, the quantity $\frac{(p-1)ab}{pb-a}$ be subtracted, shew that the resulting ratio will be $a : pb$.

21. Two armies number 11,000 and 7,000 men respectively; before they fight each is reinforced by 1,000 men: in favour of which army is the increase? (C. E. 1879).

22. If $x : y$ be the ratio $a : b$ in its lowest terms, prove that $\frac{x+1}{y+1}>\frac{a+1}{b+1}$, if $b>a$. (C. F. A. 1882).

23. Two persons are now of ages 36 and 31 years. After how many years will the ratio of their ages be less than the ratio of $17 : 15$?

24. What is the greatest integer which when subtracted from both the terms of $8 : 13$ will give a ratio greater than $1 : 3$?

401. Many properties of ratios can easily be proved by assuming a single letter k to represent a ratio, or to represent each of several equal ratios.

Important Theorem. *If* $a : b = c : d = e : f = \&c.,$

then each ratio $= \left(\dfrac{pa^n + qc^n + re^n + \ldots}{pb^n + qd^n + rf^n + \ldots}\right)^{\frac{1}{n}}$,

where p, q, r, &c., n are any quantities whatever.

Let $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = \&c. = k$,

Then $a = bk$, $c = dk$, $e = fk$, &c.

$\therefore$ $a^n = b^n k^n$, $c^n = d^n k^n$, $e^n = f^n k^n$, &c.

$\therefore$ $pa^n = pb^n k^n$, $qc^n = qd^n k^n$, $re^n = rf^n k^n$, &c.

By addition, $pa^n + qc^n + re^n + \ldots = (pb^n + qd^n + rf^n + \ldots)k^n$.

$$\therefore \frac{pa^n + qc^n + re^n + \ldots}{pb^n + qd^n + rf^n + \ldots} = k^n.$$

$$\therefore \left(\frac{pa^n + qc^n + re^n + \ldots}{pb^n + qd^n + rf^n + \ldots}\right)^{\frac{1}{n}} = k = \frac{a}{b} = \frac{c}{d} = \&c.$$

Hence the theorem.

Corollaries. (1) Putting $n = 1$,

each ratio $= \dfrac{pa + qc + re + \ldots}{pb + qd + rf + \ldots}$

(2) Putting $p = q = r = \ldots = n = 1$,

each ratio $= \dfrac{a + c + e + \ldots}{b + d + f + \ldots}$

Thus, we see that each ratio is equal to the *ratio of the sum of all the antecedents to the sum of all the consequents.*

(3) each ratio $= \dfrac{a - c}{b - d} = \dfrac{c - e}{d - f} = \dfrac{a - e}{b - f} = \ldots$

Thus, we see that each ratio is equal to the *ratio of the difference of any two antecedents to the difference of the two corresponding consequents.*

402. If $\dfrac{a}{b}$, $\dfrac{c}{d}$, $\dfrac{e}{f}$, &c., are unequal, the ratio $\dfrac{a + c + e + \ldots}{b + d + f + \ldots}$ lies in magnitude between the greatest and least of these ratios.

Suppose $\dfrac{a}{b} > \dfrac{c}{d} > \dfrac{e}{f} > \&c.$

Let $\frac{a}{b}=k$. Then $\frac{c}{d}<k$, $\frac{e}{f}<k$, &c.

$\therefore\ a=bk,\ c<dk,\ e<fk$, &c. ;

$\therefore\ a+c+e+\ldots<(b+d+f+\ldots)k$,

$\therefore\ \frac{a+c+e+\ldots}{b+d+f+\ldots}<k$, *i.e.* $<\frac{a}{b}$, the greatest of the ratios.

Again, let $\frac{p}{q}=k'$ the least. Then $\frac{a}{b}>k'$, $\frac{c}{d}>k'$, $\frac{e}{f}>k'$, &c.

$\therefore\ p=qk',\ a>bk',\ c>dk',\ e>fk'$, &c. ;

$\therefore\ a+c+e+\ldots>(b+d+f+\ldots)k'$;

$\therefore\ \frac{a+c+e+\ldots}{b+d+f+\ldots}>k'$ *i.e.* $>\frac{p}{q}$, the least of the ratios.

Hence the theorem.

Ex. 1. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that $\frac{a+c+e}{b+d+f}=\left(\frac{ace}{bdf}\right)^{\frac{1}{3}}$.

Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$.

Then $a=bk,\ c=dk,\ e=fk$. Multiplying out, we have

$$ace=bdfk^3,\ \text{and}\ \therefore\ \frac{ace}{bdf}=k^3\ \text{and}\ \therefore\ \left(\frac{ace}{bdf}\right)^{\frac{1}{3}}=k.$$

Again, adding the three equations, we have

$$a+c+e=(b+d+f)k,\ \text{and}\ \therefore\ \frac{a+c+e}{b+d+f}=k.$$

Hence $\frac{a+c+e}{b+d+f}=\left(\frac{ace}{bdf}\right)^{\frac{1}{3}}$.

Ex. 2. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that each $=\left(\frac{a^6-3a^3c^2+2c^2e^3}{b^6-3b^3d^2+2d^2f^3}\right)^{\frac{1}{6}}$.

Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$. Then $a=bk,\ c=dk,\ e=fk$.

$\therefore\ a^6=b^6k^6,\ 3a^3c^2=3b^3d^2k^5,\ 2c^2e^3=2d^2f^3k^5$.

$\therefore\ a^6-3a^3c^2+2c^2e^3=(b^6-3b^3d^2+2d^2f^3)k^6$.

$\therefore\ \frac{a^6-3a^3c^2+2c^2e^3}{b^6-3b^3d^2+2d^2f^3}=k^6$ and $\therefore\ \left(\frac{a^6-3a^3c^2+2c^2e^3}{b^6-3b^3d^2+2d^2f^3}\right)^{\frac{1}{6}}=k.$

Exercise CXLVII.

1. If $\frac{x}{b-c}=\frac{y}{c-a}=\frac{z}{a-b}$, shew that $x+y+z=0$.

2. If $\frac{a}{x+y}=\frac{b}{y+z}=\frac{c}{z-x}$, shew that $a-b+c=0$.

3. If $\frac{x+2y}{3x-z}=\frac{2x-z}{3y+x}=\frac{5y-3x+z}{4y-4x+z}$, prove that each $=1$.

4. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, prove that each $=\frac{ab+bc+cd}{b^2+c^2+d^2}$.

5. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that each $=\sqrt[3]{\left(\frac{3a^2c+5c^2e+4ce^2}{3b^2d+5d^2f+4f^2d}\right)}$.

6. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that each $=\sqrt[6]{\left(\frac{3ac^2e^2-4a^2c^3+5c^6}{3bd^2f^2-4b^2d^3+5f^6}\right)}$

7. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove $\frac{a^4+5c^2e^2+e^4}{b^4+5d^2f^2+f^4}=\frac{a^2c^2}{b^2d^2}$.

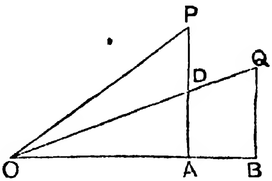
8. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove $\sqrt{\left(\frac{a^6b-2c^6e+3a^4c^2e^2}{b^7-2d^6f+3b^4cd^2e^2}\right)}=\frac{ace}{bdf}$.

II. GRAPHIC REPRESENTATION OF RATIO.

403. *To represent the ratio $\frac{a}{b}$ graphically.*

Take an abscissa OA to represent b, and an ordinate AP to represent a on the same scale.

Then $\frac{AP}{OA}$ represents the ratio $\frac{a}{b}$ graphically. The magnitude of the angle AOP enables us to determine whether this ratio is greater or less than another ratio represented in a similar way.



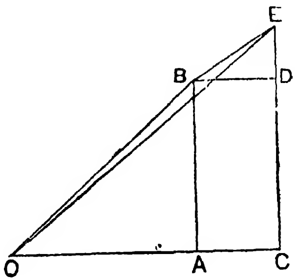
If $\frac{c}{d}$ be a second ratio, represent it by $\frac{BQ}{OB}$, and let OQ cut AP at D.

Then by similar $\triangle$s **BOQ, AOD,** $\frac{AD}{OA} = \frac{BQ}{OB} = \frac{c}{d}$.

Thus, we compare the ratios $\frac{a}{b}$ and $\frac{c}{d}$ by means of the lengths of **AP** and **AD**.

404. *A ratio of greater inequality is diminished by adding the same quantity to both its terms.*

Let $\frac{AB}{OA}$ represent a given ratio, where **OA** is the abscissa and **AB** the ordinate.



Produce **OA** to **C** and draw the ordinate **CE**. Make **CD** = **AB** and **DE** = **AC**. Then the ratio $\frac{AB}{OA}$ has been altered to $\frac{CE}{OC}$ by adding the same quantity **DE** or **AC** to both its terms.

The new ratio $\frac{CE}{OC}$ < the old ratio $\frac{AB}{OA}$, if **OE** cuts **AB** below **B**: *i. e.* if the $\angle$**EBD** < the $\angle$**BOA**.

But the $\angle$**EBD** = $45°$, for **DB** = **DE**.

$\therefore$ the new ratio < the old, if the $\angle$**BOA** > $45°$: *i. e.* if the ratio $\frac{AB}{OA}$ is one of greater inequality.

Exercise CXLVIII.

1. By means of squared paper compare the ratio $\frac{9}{11}$ with

$\frac{14}{13}$, $\frac{7}{8}$, $\frac{17}{30}$, $\frac{19}{23}$, $\frac{33}{57}$, $\frac{12}{25}$.

2. On squared paper represent the ratios $\frac{4}{5}$, $\frac{5}{9}$, $\frac{7}{11}$, and see which is greatest and which least.

3. Draw the ratio formed by the sum of the antecedents ÷ the sum of the consequents of the following ratios.

(i) $\frac{3}{5}$, $\frac{5}{8}$, $\frac{6}{11}$. (ii) $\frac{4}{5}$, $\frac{5}{9}$, $\frac{7}{11}$.

4. Draw the ratio formed by adding 5 to the numerator and denominator of $\frac{14}{17}$. Compare it with $\frac{14}{17}$.

5. By squared paper reduce $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$, $\frac{11}{12}$ to ratios whose denominator is 24.

6. Measure abscissæ OA, OC equal to 9 and 13. Measure ordinates AB, CD equal to 17 and 21. Find what angle the line BD makes with the line OC.

7. Find graphically what number must be added to each member of the ratio $\frac{16}{25}$ to make it $\frac{8}{11}$.

8 What number must be taken from each member of the ratio $\frac{13}{19}$ to make it $\frac{3}{5}$?

9. Two men share *Rs.*20 in the ratio 5 : 3. Find their shares graphically.

10. Divide 69 graphically so that the two parts may be in the ratio 9 : 14.

11. Show graphically that, if $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$ are unequal, $\frac{a+c+e}{b+d+f}$ lies in value between the greatest and least of them.

12. The marks gained in an examination-paper for which the maximum was 65 were 52, 40, 38. Find by a diagram what these would be if the maximum were 100.

III. PROPORTION.

405. When two ratios are *equal*, the four quantities composing them are said to be **proportional** to one another.

Thus, if $\frac{a}{b}=\frac{c}{d}$, then the four quantities a, b, c and d are *proportionals*. This is expressed by saying that ***a** is to **b** as **c** is to **d***, and denoted thus, $a : b :: c : d$ or $a : b = c : d$.

The first and last terms in a proportion (*viz.* a and d) are called the **Extremes,** and the other two (*viz.*, b and c) the **Means.** Also d is called a **fourth proportional** to a, b and c.

406. *When four quantities are proportionals, the product of the extremes is equal to the product of the means.*

For if $\frac{a}{b}=\frac{c}{d}$, then multiplying both sides by bd,

we have $\frac{a}{b}\times bd=\frac{c}{d}\times bd$, or $ad=bc$.

Hence, if three terms of a proportion are given, we can find the other ; thus

$$a=\frac{bc}{d},\ b=\frac{ad}{c},\ c=\frac{ad}{b},\ d=\frac{bc}{a}.$$

407. *If the product of two quantities be equal to that of two others, the four are proportionals, those of one product being the extremes, and of the other the means.*

For if $ad=bc$, then dividing both sides by bd,

we have $\frac{ad}{bd}=\frac{bc}{bd}$ or $\frac{a}{b}=\frac{c}{d}$;

and $\therefore$ $a:b::c:d$, in which proportion a, d are the extremes, and b, c the means.

408. Four quantities are said to be **inversely proportional**. when the first is to the second as the reciprocal of the third is to the reciprocal of the fourth, *i. e.* as the fourth is to the third.

Thus, a, b, c and d are *inversely proportional*,

when $a:b::\frac{1}{c}:\frac{1}{d}$ *i. e.* $d:c$, for $\frac{1}{c}:\frac{1}{d}=\frac{1}{c}/\frac{1}{d}=\frac{d}{c}=d:c$.

409. Quantities are said to be in **continued proportion**, when the first is to the second, as the second is to the third, as the third is to the fourth ; and so on.

Thus, a, b, c, d, &c. are in continued proportion,

when $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=$&c.

410. If three quantities a, b, c form what is called a **continued proportion**, so that $a:b=b:c$, we shall have $ac=b^2$; or *the product of the extremes is equal to the square of the mean*, and *conversely*.

In this case b is said to be a **mean proportional** between a and c; and c is said to be a **third proportional** to a and b.

411. *If three quantities are in continued proportion, the first has to the third the duplicate ratio of that which it has to the second.*

For, if $\frac{a}{b}=\frac{b}{c}$, then $\frac{a}{c}=\frac{a}{b}\times\frac{b}{c}=\frac{a}{b}\times\frac{a}{b}=\frac{a^2}{b^2}$;

$\therefore$ $a:c$ is the **duplicate ratio** of $a:b$.

Also, *if four quantities are in continued proportion, the first has to the fourth the triplicate ratio of that which it has to the second.*

For, if $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, then $\frac{a}{d}=\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}=\frac{a^3}{b^3}$;

$\therefore$ $a : d$ is the **triplicate ratio** of $a : b$.

412. Propositions regarding four quantities a, b, c and d in proportion, may be obtained, like those on ratios, by the use of fractions, and of these the most useful are the following :—

(i) If $\frac{a}{b}=\frac{c}{d}$, then $1\div\frac{a}{b}=1\div\frac{c}{d}$, or $\frac{b}{a}=\frac{d}{c}$;

that is, $b : a=d : c$. (**Invertendo**).

(ii) If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a}{b}\times\frac{b}{c}=\frac{b}{c}\times\frac{c}{d}$, or $\frac{a}{c}=\frac{b}{d}$;

that is, $a : c=b : d$. (**Alternando**).

(iii) If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a}{b}+1=\frac{c}{d}+1$, or $\frac{a+b}{b}=\frac{c+d}{d}$;

that is, $a+b : b=c+d : d$. (**Componendo**).

(iv) If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a}{b}-1=\frac{c}{d}-1$, or $\frac{a-b}{b}=\frac{c-d}{d}$.

that is, $a-b : b=c-d : d$. (**Dividendo**).

(v) Since $\frac{a+b}{b}=\frac{c+d}{d}$ (iii) and $\frac{a-b}{b}=\frac{c-d}{d}$ (iv),

$\therefore$ $\frac{a+b}{b}\times\frac{b}{a-b}=\frac{c+d}{d}\times\frac{d}{c-d}$, or $\frac{a+b}{a-b}=\frac{c+d}{c-d}$;

that is, $a+b : a-b=c+d : c-d$.

(**Componendo and Dividendo**).

(vi) Since $\frac{a-b}{b}=\frac{c-d}{d}$ (iv) and $\frac{b}{a}=\frac{d}{c}$ (i),

$\therefore$ $\frac{a-b}{b}\times\frac{b}{a}=\frac{c-d}{d}\times\frac{d}{c}$, or $\frac{a-b}{a}=\frac{c-d}{c}$;

that is, $a-b : a=c-d : c$. (**Convertendo**).

(vii) If $\frac{a}{b}=\frac{c}{d}$, then $\frac{m}{n}\times\frac{a}{b}=\frac{m}{n}\times\frac{c}{d}$, or $\frac{ma}{nb}=\frac{mc}{nd}$;

that is, $ma : nb=mc : nd$.

(viii) If $\frac{a}{b}=\frac{c}{d}$, then $\left(\frac{a}{b}\right)^n=\left(\frac{c}{d}\right)^n$, or $\frac{a^n}{b^n}=\frac{c^n}{d^n}$.

that is, $a^n : b^n = c^n : d^n$.

413. *If* $a : b = c : d$, *and* $b : e = d : f$, *then* $a : e = c : f$.

For $\frac{a}{b}=\frac{c}{d}$ and $\frac{b}{e}=\frac{d}{f}$, $\therefore \frac{a}{b}\times\frac{b}{e}=\frac{c}{d}\times\frac{d}{f}$, or $\frac{a}{e}=\frac{c}{f}$.

This is the proposition **ex æquali** referred to in Euc. V.

414. *If* $a : b = c : d$, *and* $e : f = g : h$, *then* $ae : bf = cg : dh$.

For $\frac{a}{b}=\frac{c}{d}$, and $\frac{e}{f}=\frac{g}{h}$; $\therefore \frac{a}{b}\times\frac{e}{f}=\frac{c}{d}\times\frac{g}{h}$ or $\frac{ae}{bf}=\frac{cg}{dh}$.

This is called **compounding** the two proportions, and so we may compound any number of such proportions.

415. If four quantities form a proportion, we may derive from them many other proportions all equally true.

Thus, if $\frac{a}{b}=\frac{c}{d}$, then $\frac{ma}{mb}=\frac{c}{d}$, or $ma : mb = c : d$;
similarly,

$$ma : b = mc : d,\ a : mb = c : md,\ a : b = mc : md;$$

and, in like manner, $\frac{a}{m} : \frac{b}{m} = c : d$, $a : \frac{b}{m} = c : \frac{d}{m}$, &c.;

that is, either the *first* or *fourth* terms of any proportion may be multiplied or divided by any quantity, provided that either the *second* or *third* be multiplied or divided by the same.

Hence we may get rid of fractions, when occurring in proportions, by multiplying the *first* and *second*, or *first* and *third*, &c., terms by the L. C. M. of their denominators.

Thus, if $\frac{1}{9}a : \frac{1}{12}b = \frac{3}{40} : \frac{2}{25}$, (multiplying 1st and 2nd by 36, 3rd and 4th by 200), we have $4a : 3b = 15 : 16$.

Ex. 1. Find a fourth proportional to $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$.

Since $d=\frac{bc}{a}$, (Art. 406), this is $\frac{\frac{1}{3}\times\frac{1}{4}}{\frac{1}{2}}=\frac{1}{6}$.

Ex. 2. Find a mean proportional to 4 and 16.

Since $b^2=ac$, (Art. 410), this is $\sqrt{(4\times 16)}=\sqrt{(64)}=8$.

Ex. 3. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, prove $\frac{a}{d}=\frac{a^3}{b^3}$. (C. E. 1887-1902).

We have $\left(\frac{a}{b}\right)^3=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}=\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}=\frac{a}{d}$, or $\frac{a^3}{b^3}=\frac{a}{d}$.

Ex. 4. If $a:b=c:d$, express $(a+d)-(b+c)$ in terms of a, b, and c only.

Since $\frac{a}{b}=\frac{c}{d}$, $\therefore$ $ad=bc$, and $\therefore$ $d=\frac{bc}{a}$.

Then $(a+d)-(b+c)=\left(a+\frac{bc}{a}\right)-(b+c)=\frac{a^2-ab-ac+bc}{a}$

$$=\frac{a(a-b)-c(a-b)}{a}=\frac{(a-b)(a-c)}{a}.$$

Ex. 5. If $a:b=c:d$, show that if a be the greatest of the four quantities a, b, c, d, then d is the least. Hence show that $a-b>c-d$ or $a+d>b+c$.

Since $a/b=c/d$ and a the greatest of the four,

$\therefore$ a/b is an improper fraction, as also c/d.

$\therefore$ c is greater than d. Similarly, it may be shewn that b is greater than d. Thus b and c are both greater than d, and $\therefore$ d is the least.

Again since from Art. 412, (iv), $\frac{a-b}{b}=\frac{c-d}{d}$,

$\therefore$ $\frac{a-b}{c-d}=\frac{b}{d}$ = an improper fraction.

$$\left.\begin{array}{l}\therefore\ a-b>c-d,\\ \text{and } b+d=b+d\end{array}\right\}\quad \begin{array}{l}\therefore \text{ by addition,}\\ a+d>b+c.\end{array}$$

Exercise CXLIX.

1. Find a fourth proportional to

(i) 3, 5, 6. (ii) 12, 5, 10. (iii) $\frac{2}{7}$, $\frac{3}{4}$, $\frac{5}{6}$. (iv) $2a$, $3b$, $5a^2c$.

2. Find a third proportional to

(i) 4, 6. (ii) 2, 3. (iii) $\frac{5}{3}$, $\frac{5}{7}$. (iv) $(a-b)^2$, a^3-b^3.

3. Find a mean proportional to

(i) 4, 9. (ii) 4, $\frac{19}{25}$. (iii) $1\frac{7}{9}$, $1\frac{9}{16}$. (iv) $3a^3b$, $12abc^2$.

4. If $a : b = b : c$, prove that

(1) $a^2 - b^2 : a = b^2 - c^2 : c$. (2) $a^3 + b^3 = a(a+b)(a-b+c)$.

(3) $a^2 + b^2 : a + c = a^2 - b^2 : a - c$.

(4) $a - 2b + c = \dfrac{(a-b)^2}{a} = \dfrac{(b-c)^2}{c}$.

(5) $ma^2 - nb^2 : ma - nc = pa^2 + qb^2 : pa + qc$.

(6) $a + b + c : a - b + c = (a+b+c)^2 : a^2 + b^2 + c^2$.

5. If $\dfrac{a}{b} = \dfrac{c}{d}$, prove that

(1) $(a+b)(c+d) = \dfrac{b}{d}(c+d)^2 = \dfrac{d}{b}(a+b)^2$.

(2) $\left(\dfrac{1}{a} + \dfrac{1}{d}\right) - \left(\dfrac{1}{b} + \dfrac{1}{c}\right) = \dfrac{(a-b)(a-c)}{abc}$.

6. If $a : b = c : d$ and $m : n = p : q$, shew that

$$ma + nb : ma - nb = pc + qd : pc - qd.$$

7. If $a : b = c : d = e : f$, then $a - c : b - f = c : d$.

8. If a, b, c be in continued proportion, shew that

$$a^{2n} + b^{2n} + c^{2n} = (a^n + b^n + c^n)(a^n - b^n + c^n).$$

9. If $(x^2 - y^2)z = (y^2 - z^2)x$, show that x is to z in the duplicate ratio of x and y. (C. F. A. 1867).

10. If y is a mean proportional between x and z, show that $xy + yz$ is a mean proportional between $x^2 + y^2$ and $y^2 + z^2$. (P. E. 1890).

11. Find in its simplest form a mean proportional between

$6 + \sqrt{27}$ and $8 - \sqrt{48}$. (P. E. 1902).

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416. Many questions in Proportion may neatly be solved by the 'k' method explained in Art. 401.

Ex. 1. If $a : b = c : d$, show that

$$ma + nb : mc + nd = \sqrt{(pa^2 + qb^2)} : \sqrt{(pc^2 + qd^2)}.$$

Let $\dfrac{a}{b} = \dfrac{c}{d} = k$, then $a = bk$, and $c = dk$.

$$\therefore \frac{ma + nb}{mc + nd} = \frac{mbk + nb}{mdk + nd} = \frac{b(mk + n)}{d(mk + n)} = \frac{b}{d}.$$

and $\dfrac{\sqrt{(pa^2+qb^2)}}{\sqrt{(pc^2+qd^2)}}=\dfrac{\sqrt{(pb^2k^2+qb^2)}}{\sqrt{(pd^2k^2+qd^2)}}=\dfrac{b\sqrt{(pk^2+q)}}{d\sqrt{(pk^2+q)}}=\dfrac{b}{d}.$

Hence, $\dfrac{ma+nb}{mc+nd}=\dfrac{\sqrt{(pa^2+qb^2)}}{\sqrt{(pc^2+qd^2)}}$, each being equal to $\dfrac{b}{c}$

Ex. 2. If $a:b=c:d$, prove that

$$7a+12b : 3a+5b=7c+12d : 3c+5d.$$

Let $\dfrac{a}{b}=\dfrac{c}{d}=k$, then $a=bk$, and $c=dk$.

$\therefore\ \dfrac{7a+12b}{3a+5b}=\dfrac{7bk+12b}{3bk+5b}=\dfrac{b(7k+12)}{b(3k+5)}=\dfrac{7k+12}{3k+5}$,

and $\dfrac{7c+12d}{3c+5d}=\dfrac{7dk+12d}{3dk+5d}=\dfrac{d(7k+12)}{d(3k+5)}=\dfrac{7k+12}{3k+5}$.

Hence, $\dfrac{7a+12b}{3a+5b}=\dfrac{7c+12d}{3c+5d}$, each being equal to $\dfrac{7k+12}{3k+5}$.

Ex. 3. If a, b, c and d are in continued proportion, prove that $(ab+bc+cd)^2=(a^2+b^2+c^2)(b^2+c^2+d^2)$. (C. E. 1887).

Let $\dfrac{a}{b}=\dfrac{b}{c}=\dfrac{c}{d}=k$.

Then $k=\dfrac{a^2}{ab}=\dfrac{b^2}{bc}=\dfrac{c^2}{cd}=\dfrac{a^2+b^2+c^2}{ab+bc+cd}$............ ......(1)

Again, $k=\dfrac{ab}{b^2}=\dfrac{bc}{c^2}=\dfrac{cd}{d^2}=\dfrac{ab+bc+cd}{b^2+c^2+d^2}$................(2)

$\therefore\ \dfrac{a^2+b^2+c^2}{ab+bc+cd}=\dfrac{ab+bc+cd}{b^2+c^2+d^2}$, from (1) and (2).

Hence $(ab+bc+cd)^2=(a^2+b^2+c^2)(b^2+c^2+d^2)$.

Otherwise thus: Let $\dfrac{a}{b}=\dfrac{b}{c}=\dfrac{c}{d}=k$.

Then $c=dk$; $b=ck=dk^2$; $a=bk=dk^3$.

Now, $(ab+bc+cd)^2=(dk^3\times dk^2+dk^2\times dk+dk\times d)^2$

$=\{kd^2(k^4+k^2+1)\}^2=k^2d^4(k^4+k^2+1)^2$,

and $(a^2+b^2+c^2)(b^2+c^2+d^2)=(d^2k^6+d^2k^4+d^2k^2)(d^2k^4+d^2k^2+d^2)$

$=d^2k^2(k^4+k^2+1).d^2(k^4+k^2+1)$

$=d^4k^2(k^4+k^2+1)^2$.

Hence $(ab+bc+cd)^2=(a^2+b^2+c^2)(b^2+c^2+d^2)$, for each is equal to the same quantity.

Exercise CL.

1. If $a : b = c : d$, prove that

(1) $a \pm b : a = c \pm d : c$. (C. E. 1862).

(2) $a : b = a \pm c : b \pm d$. (C. E. 1872).

(3) $ma - nb : a + b = mc - nd : c + d$. (C. E. 1893).

(4) $ma + nb : mc + nd = b^2c : ad^2$. (C. E. 1876).

(5) $2a + 3b : 4a + 5b = 2c + 3d : 4c + 5d$. (A. E. 189[illegible]).

(6) $a : a + c = a + b : a + b + c + d$. (A. E. 1894).

(7) $a^2 + b^2 : a^2 - b^2 = c^2 + d^2 : c^2 - d^2$. (A. E. 1889).

(8) $a^2 + c^2 : b^2 + d^2 = ac : bd$. (C. E. 1877).

(9) $a^2 + b^2 : a^2 - b^2 = ac + bd : ac - bd$. (C. E. 1879).

(10) $a^2d - bc^2 = ac(b - d)$. (C. E. 1890).

(11) $(a^2 + c^2)(b^2 + d^2) = (ab + cd)^2$. (A. E. 1890).

(12) $\sqrt{(a^2 + c^2)} : \sqrt{(b^2 + d^2)} = ma + nc : mb + nd$. (C. E. 1880).

(13) $a^2 \pm c^2 : b^2 \pm d^2 = (a \pm c)^2 : (b \pm d)^2$. (C. E. 1872).

(14) $(a + c)^3 : (b + d)^3 = a(a - c)^2 : b(b - d)^2$. (C. E. 1888).

(15) $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2 : (c^{\frac{1}{2}} + d^{\frac{1}{2}})^2 = a - b : c - d$. (C. E. 1895).

(16) $\sqrt{(3a^2 + 4c^2)} : \sqrt[3]{(5a^3 - 6c^3)} = \sqrt{(3b^2 + 4d^2)} : \sqrt[3]{(5b^3 - 6d^3)}$. (C.E. 1900)

(17) $4(a + b)(c + d) = bd\left(\dfrac{a + b}{b} + \dfrac{c + d}{d}\right)^2$. (C. E. 1874).

(18) $\dfrac{1}{ma} + \dfrac{1}{nb} + \dfrac{1}{pc} + \dfrac{1}{qd} = \dfrac{1}{bc}\left\{\dfrac{a}{q} + \dfrac{b}{p} + \dfrac{c}{n} + \dfrac{d}{m}\right\}$. (B. P. E. 1884).

(19) $\dfrac{(a - b)(a - c)}{a} = (a + d) - (b + c)$. (B. P. E. 1886).

(20) $\dfrac{a^2}{b} : \dfrac{c^2}{d}$ is inversely as $\dfrac{a}{b^2} : \dfrac{c}{d^2}$. (M. F. A. 1884).

(21) $a^2 + ab + b^2 : a^2 - ab + b^2 = c^2 + cd + d^2 : c^2 - cd + d^2$. (C. E. 1894).

(22) $pa^2 + qc^2 : pb^2 + qd^2 = ma^2 - nc^2 : mb^2 - nd^2$. (C. E. 1899).

(23) $a : c = \sqrt[4]{(a^4 + b^4)} : \sqrt[4]{(c^4 + d^4)}$.

(24) $(la^3 + mb^3)\left(\dfrac{l}{a^3} + \dfrac{m}{b^3}\right) = (ld^3 + mc^3)\left(\dfrac{l}{d^3} + \dfrac{m}{c^3}\right)$.

(25) $a^2b - 3ac^2 : b^3 - 3ad^2 = a^2 + 5c^2 : b^2 + 5d^2$. (A. E. 1902).

2. If a, b, c and d are in continued proportion, prove that
$\sqrt{(ab)} - \sqrt{(bc)} + \sqrt{(cd)} = \sqrt{\{(a-b+c)(b-c+d)\}}$. (M. F. A. 1890).

3. If $a : b = b : c = c : d$, prove that
$a : d = a^3 + b^3 + c^3 : b^3 + c^3 + d^3$.

4. If $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$, prove that $\frac{a}{d} = \frac{pa^3 + qb^3 + rc^3}{pb^3 + qc^3 + rd^3}$. (C. E. 1898).

5. If $x : a = y : b = z : c$, prove that
$\frac{x^3}{a^2} + \frac{y^3}{b^2} + \frac{z^3}{c^2} = \frac{(x+y+z)^3}{(a+b+c)^2}$. (C. E. 1901).

6. If $c : d = x : y$, then will $cd : xy = c^2 + d^2 : x^2 + y^2$. (P. E. 1892).

7. If $a : b = b : c$, shew that $a^2 + ab + b^2 : b^2 + bc + c^2 = a : c$. (A. E. 1895).

8. If $a : b :: c : d$ and $p : q :: r : s$, prove that
$ap + cr : bq + ds :: \sqrt{(acpr)} : \sqrt{(bdqs)}$. (A. E. 1896).

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417. The following examples will illustrate the *converse* theorem considered in Art. 407.

Ex. 1. If $(a+b+c+d)(a-b-c+d) = (a-b+c-d)(a+b-c-d)$, to prove that a, b, c, d are proportionals. (C. E. A. 1893).

We have $\{(a+d)+(b+c)\}\{(a+d)-(b+c)\} = \{(a-d)-(b-c)\}\{(a-d)+(b-c)\}$, or $(a+d)^2 - (b+c)^2 = (a-d)^2 - (b-c)^2$, Art. 124

or $(a+d)^2 - (a-d)^2 = (b+c)^2 - (b-c)^2$, by transposition,

$\therefore$ $4ad = 4bc$, or $ad = bc$. Hence $a : b = c : d$.

Otherwise thus :—Writing the equation in a fractional form,

we have $\frac{a+b+c+d}{a-b+c-d} = \frac{a+b-c-d}{a-b-c+d}$.

Hence, by *Comp. & Divd.*, $\frac{a+c}{b+d} = \frac{a-c}{b-d}$, or $\frac{a+c}{a-c} = \frac{b+d}{b-d}$.

Again, by *Comp. & Divd.*, $\frac{a}{c} = \frac{b}{d}$, and $\therefore$ $a : b = c : d$.

Ex. 2. If $a+b : b+c = c+d : d+a$, prove that
$a = c$ or $a+b+c+d = 0$. (C. E. 1891).

We have $(a+b)(d+a)=(b+c)(c+d)$. Art. 406.

$\therefore\ a^2+(b+d)a+bd=c^2+(b+d)c+bd.$

Transposing and reducing,

$$a^2-c^2+(b+d)(a-c)=0, \text{ or } (a-c)(a+b+c+d)=0.$$

$\therefore$ either $a-c=0$ *i.e.* $a=c$, or $a+b+c+d=0$.

418. The following is an example of 'k' theorem explained in Art. 401.

Ex. If $a:b=c:d=e:f$, show that

Each ratio$=\sqrt[3]{(a^3+c^3+e^3)} : \sqrt[3]{(b^3+d^3+f^3)}$. (C. E. 1882)

Since $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, $\therefore \frac{a^3}{b^3}=\frac{c^3}{d^3}=\frac{e^3}{f^3}=\frac{a^3+c^3+e^3}{b^3+d^3+f^3}$.

$\therefore$ Each ratio$=\sqrt[3]{\left(\frac{a^3+c^3+e^3}{b^3+d^3+f^3}\right)}$.

Exercise CLI.

1. If $3a+4b : 5a+6b=3c+4d : 5c+6d$, then will $a:b=c:d$. (C. E. 1897).

2. If $(2a+3b+5c+4d)(2a-3b-5c+4d)=(2a+3b-5c-4d)\times(2a-3b+4c-5d)$, then will $a:b=c:d$.

3. If $(pa+qb+rc+sd)(pa-qb-rc+sd)=(pa-qb+rc-sd)\times(pa+qb-rc-sd)$, shew that bc, ad, ps, qr are in proportion (B. P. E. 1890).

4. If $a:b=c:d=e:f$, show that

(1) Each ratio$=(pa^3+qc^3+re^3)^{\frac{1}{3}} : (pb^3+qd^3+rf^3)^{\frac{1}{3}}$. (A. E. 1893).

(2) Each ratio$=(a^3-3ace+e^3)^{\frac{1}{3}} : (b^3-3bdf+f^3)^{\frac{1}{3}}$. (M. F. A. 1887).

(3) $\left(\frac{a+2c+3e}{b+2d+3f}\right)^2=\frac{ac+ce}{bd+df}$. (B. P. E. 1887).

(4) $a^2+c^2+e^2 : b^2+d^2+f^2=ce : df$. (C. E. 1876).

(5) $a^3+c^3+e^3 : b^3+d^3+f^3=ace : bdf$.

5. If $ab=cd=ef$, show that

$$\frac{ab+ce+ea}{bdf(b+d+f)}=\frac{a^2+c^2+e^2}{d^2f^2+f^2b^2+b^2d^2}.$$ (B. P. E. 1889).

6. If $x : y = y : z$, find the simplest value of

$$\frac{xyz(x+y+z)^3}{(xy+yz+zx)^3}. \text{ (C. E. 1892).}$$

7. What number must be added to each of the numbers 3, 5, 7, 10 to give four numbers in proportion. (C. E. 1893).

8. If x and y be unequal, and x have to y the duplicate ratio of $x+z$ to $y+z$, prove that z is a mean proportional between x and y.

9. If a, b, c and d are in continued proportion, show that $b+c$ is a mean proportional between $a+b$ and $c+d$.

10. If b is a mean proportional between a and c, prove that

$$\frac{a^2-b^2+c^2}{a^{-2}-b^{-2}+c^{-2}}=b^4.$$

11. If $a : b = c : d$, prove that $\dfrac{a^2+b^2+c^2+d^2}{a^{-2}+b^{-2}+c^{-2}+d^{-2}}=abcd.$

12. If $\dfrac{x}{a}=\dfrac{y}{b}$, prove that $\dfrac{x^2+a^2}{x+a}+\dfrac{y^2+b^2}{y+b}=\dfrac{(x+y)^2+(a+b)^2}{(x+y)+(a+b)}$.

(A. E. 1899).

13. If $a : b :: c : d$, and if x be homogeneous with a, b, c and d, then $a^3+x^3 : b^3c^3 :: 1+x^3/a^3 : d^3$. (M. M. 1860).

14. If $x+2y : a+3b = y+3x : a+4b$, prove that $x : y = a+5b : 2a+5b$, and that $y+2x : x+3y = 4a+15b : 7a+20b$.

IV. PROBLEMS IN RATIO AND PROPORTION.

419 The following are illustrative examples.

Ex. 1. Divide 39 into two such parts that the greater increased by 6 shall be to the less diminished by 3 as 5 to 2. (C. E. 1859).

Let x be the greater part,

then $39-x$ is the less part.

$\therefore$ By the question, $x+6 : 39-x-3 = 5 : 2$.

$\therefore$ $2(x+6)=5(36-x)$, or $2x+12=180-5x$.

$\therefore$ $7x=168$; $\therefore$ $x=24$, and $39-x=15$.

Hence the parts are 24 and 15.

Ex. 2. A certain number consists of two digits ; the left-hand digit is double the right-hand digit, and if the digits be inverted,

the ratio of the number thus formed to 60 is $4:5$. Find the number. (C. E. 1874).

Let x be the right-hand digit, then $2x$ is the left-hand digit, and the number$=x+2x\times 10=21x$, and the number formed by inverting the digits$=2x+10\times x=12x$.

$\therefore$ by the question, $\frac{12x}{60}=\frac{4}{5}$; $\therefore$ $x=4$.

Hence the number$=21\times 4=84$.

Ex. 3. Two vessels contain mixtures of wine and water in the ratios of 8 to 3 and 5 to 1 respectively. In what ratio must liquid be drawn from each vessel to give a mixture in the ratio of 4 to 1?

Let x be the number to be drawn from the first,

and y .... second.

Since $8+3=11$, $\therefore$ in first, wine$=\frac{8}{11}$ and water$=\frac{3}{11}$,

and since $5+1=6$, $\therefore$ in second, wine$=\frac{5}{6}$ and water$=\frac{1}{6}$.

Now, x quantity drawn from first, will contain $\frac{8}{11}x$ wine and $\frac{3}{11}x$ water, and y quantity drawn from second, will contain $\frac{5}{6}y$ wine and $\frac{1}{6}y$ water.

$\therefore$ By the question, $\frac{8}{11}x+\frac{5}{6}y : \frac{3}{11}x+\frac{1}{6}y=4:1$.

$\therefore$ $\frac{8}{11}x+\frac{5}{6}y=\frac{12}{11}x+\frac{4}{6}y$, or $\frac{4}{11}x=\frac{1}{6}y$.

$\therefore$ $x=\frac{1}{6}\times\frac{11}{4}y=\frac{11}{24}y$ and $\therefore$ $x:y=11:24$.

Exercise CLII.

1. Solve the following equations :—

(1) $6x-a : 4x-b :: 3x+b : 2x+a$. (M. M. 1859).

(2) $x : 27 :: y : 9 :: 2 : x-y$.

(3) $\left.\begin{array}{l} x+y+1 : x+y+2 :: 6 : 7 \\ y+2x : y-2x :: 12x+6y-3 : 6y-12x-1 \end{array}\right\}$

(4) $\frac{ax+by}{cz}=\frac{cz+ax}{by}=\frac{by+cz}{ax}=x+y+z$.

(5) $\frac{1}{a+b+c}=\frac{\frac{a}{x}+\frac{b}{y}}{c}=\frac{\frac{b}{y}+\frac{c}{z}}{a}=\frac{\frac{c}{z}+\frac{a}{x}}{b}$. (C. F. A. 1871).

2. What number is that to which if 1, 5 and 13 be severally added, the first sum shall be to the second as the second to the third?

3. Find two numbers in the ratio of $2\frac{1}{2} : 2$, such that, when diminished each by 5, they shall be in that of $1\frac{1}{3} : 1$.

4. A's present age is to B's present age as $8 : 7$; 27 years ago their ages were as $5 : 4$. Find their present ages. (A. E. 1900).

5. Three numbers are in the ratios $2 : 3 : 5$, and the sum of their cubes is 4320. Find them. (P. E. 1900).

6. Two numbers each consisting of the same two digits are in the ratio of $4 : 7$. Find the numbers. (P. E. 1896).

7. Find two numbers in the ratio $1\frac{1}{2} : 2\frac{2}{3}$, such that when increased by 15, they shall be in the ratio $1\frac{2}{3} : 2\frac{1}{2}$. (P. E. 1899).

8. A and B trade with different sums; A gains *Rs.*2000, B loses *Rs.*500, and now A's stock : B's :: $2 : \frac{1}{2}$; but, if A had gained *Rs.*1000, and B lost *Rs.*850, their stocks would have been as $15 : 3\frac{1}{7}$. Find the original stock of each.

9. In a certain examination the number of those who passed was 3 times the number of those who failed. If there had been 16 fewer candidates and if 6 more had failed, the numbers would have been as 2 to 1. Find the number of candidates.

10. A quantity of milk is increased by watering in the ratio of $5 : 4$, and then 12 seers are sold; the rest, being mixed with 3 seers of water, is increased in the ratio of $7 : 6$. How many seers of milk were there at first?

V. HARDER RATIO AND PROPORTION.

420. The following are typical solutions of some harder examples in Ratio and Proportion.

Ex. 1. If $\dfrac{x}{b+c-a} = \dfrac{y}{c+a-b} = \dfrac{z}{a+b-c}$, find the value of $(b-c)x+(c-a)y+(a-b)z$. (C. E. 1878).

Let each of the given ratios $=k$; then

$$x=(b+c-a)k,\ y=(c+a-b)k,\ z=(a+b-c)k.$$

$$\therefore \text{ the given expression} = \{(b-c)(b+c-a)+(c-a)(c+a-b) + (a-b)(a+b-c)\}k$$

$$=\{b^2-c^2-a(b-c)+c^2-a^2-b(c-a)+a^2-b^2-c(a-b)\}k$$

$$=[b^2-c^2+c^2-a^2+a^2-b^2-\{a(b-c)+b(c-a)+c(a-b)\}]k$$

$$=0\times k=0.$$

Ex. 2. If $\dfrac{ay-bx}{c} = \dfrac{cx-az}{b} = \dfrac{bz-cy}{a}$, then $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$.

Let each of the given ratios $=k$; then

$$ay-bx=ck; \quad \therefore acy-bcx=c^2k, \text{ multiplying by } c,$$
$$cx-az=bk; \quad bcx-abz=b^2k, \quad \ldots\ldots\ldots\ldots \quad b$$
$$bz-cy=ak; \quad abz-acy=a^2k, \quad \ldots\ldots\ldots\ldots \quad a$$

Now, adding the equations, we have

$$0=(a^2+b^2+c^2)k, \qquad \therefore k=0.$$

$$\therefore \frac{ay-bx}{c}=0, \text{ and } \therefore ay-bx=0, \text{ or } ay=bx; \ \therefore \frac{x}{a}=\frac{y}{b}.$$

Also $\dfrac{cx-az}{b}=0$, and $\therefore cx-az=0$, or $cx=az$; $\therefore \dfrac{x}{a}=\dfrac{z}{c}$.

$$\text{Hence } \frac{x}{a}=\frac{y}{b}=\frac{z}{c}.$$

Ex. 3. If $a(y+z)=b(z+x)=c(x+y)$, prove that

$$\frac{y-z}{a(b-c)}=\frac{z-x}{b(c-a)}=\frac{x-y}{c(a-b)}.$$

Let each of the given ratios $=k$; then

$$a=\frac{k}{y+z}; \qquad \therefore b-c=k\left(\frac{1}{z+x}-\frac{1}{x+y}\right)=k\frac{y-z}{(x+y)(z+x)}$$
$$b=\frac{k}{z+x}; \qquad \therefore c-a=k\left(\frac{1}{x+y}-\frac{1}{y+z}\right)=k\frac{z-x}{(x+y)(y+z)}$$
$$c=\frac{k}{x+y}; \qquad \therefore a-b=k\left(\frac{1}{y+z}-\frac{1}{z+x}\right)=k\frac{x-y}{(y+z)(z+x)}$$

$$\therefore a(b-c)=k^2\cdot\frac{y-z}{(x+y)(y+z)(z+x)}, \text{ or } \frac{y-z}{a(b-c)}=\frac{(x+y)(y+z)(z+x)}{k^2}$$

Similarly, $\dfrac{z-x}{b(c-a)}=\dfrac{(x+y)(y+z)(z+x)}{k^2}$

and $\dfrac{x-y}{c(a-b)}=\dfrac{(x+y)(y+z)(z+x)}{k^2}$.

$$\text{Hence } \frac{y-z}{a(b-c)}=\frac{z-x}{b(c-a)}=\frac{x-y}{c(a-b)}.$$

Otherwise thus :—

We have $y+z=\dfrac{k}{a}$, $z+x=\dfrac{k}{b}$, $x+y=\dfrac{k}{c}$.

$$\therefore y-z=k\left(\frac{1}{c}-\frac{1}{b}\right)=\frac{k(b-c)}{bc}; \quad \therefore \frac{y-z}{a(b-c)}=\frac{k}{abc}.$$

Similarly, the others $=\dfrac{k}{abc}$.

Ex. 4. If $\frac{2x-y}{2a+b}=\frac{2y-z}{2b+c}=\frac{2z-x}{2c+a}$, shew that

$$3(a+b+c)(x+4y+3z)=(7a+8b+9c)(x+y+z).$$

Let each of the given ratios $=k$; then

$$k=\frac{\text{sum of numerators}}{\text{sum of denominators}}=\frac{x+y+z}{3(a+b+c)}\ldots\ldots\ldots(1)$$

$$\text{Also } k=\frac{p(2x-y)+q(2y-z)+r(2z-x)}{p(2a+b)+q(2b+c)+r(2c+a)}$$

$$=\frac{x(2p-r)+y(2q-p)+z(2r-q)}{a(2p+r)+b(2q+p)+c(2r+q)}.\ \ldots\ldots\ (2)$$

where p, q and r are any quantities.

Now, give such values to p, q, r, as to make $2p-r=1$, $2q-p=4$ and $2r-q=3$, so that the numerator of (2) becomes $x+4y+3z$.

Solving the above equations, we find $p=2$, $q=3$ and $r=3$.

$$\therefore \text{ from (2) } k=\frac{x+4y+3z}{a(4+3)+b(6+2)+c(6+3)}=\frac{x+4y+3z}{7a+8b+9c}\ldots(3)$$

$$\text{Hence, } \frac{x+y+z}{3(a+b+c)}=\frac{x+4y+3z}{7a+8b+9c}, \text{ from (1) and (3).}$$

$$\therefore\ 3(a+b+c)(x+4y+3z)=(7a+8b+9c)(x+y+z).$$

Exercise CLIII.

1. Assuming that $\frac{a+b-c}{a+b}=\frac{b+c-a}{b+c}=\frac{c+a-b}{c+a}$, and that $a+b+c$ is not $=0$, show that $a=b=c$. (C. E. 1873).

2. If $\frac{x}{(b-c)(b+c-2a)}=\frac{y}{(c-a)(c+a-2b)}=\frac{z}{(a-b)(a+b-2c)}$, find the value of $x+y+z$. (C. E. 1889).

3. If $\frac{a-b}{ay+bx}=\frac{b-c}{bz+cx}=\frac{c-a}{cy+az}=\frac{a+b+c}{ax+by+cz}$, then each of these ratios $=\frac{1}{x+y+z}$, supposing $a+b+c$ not to be zero.

4. Shew that if $\frac{a-b}{c}+\frac{b-c}{a}+\frac{c+a}{b}=1$, and $a-b+c$ is not $=0$, then $\frac{1}{a}=\frac{1}{b}+\frac{1}{c}$. (C. E. 1875).

5. If $x : ax+by+cz = y : bx+cy+az = z : cx+ay+bz$, shew that each of these ratios $=\frac{1}{a+b+c}$, supposing $x+y+z$ is not $=0$. (C. E. 1902).

6. If $a(y+z)=b(z+x)=c(x+y)$, prove that

$$\frac{a-b}{x^2-y^2}=\frac{b-c}{y^2-z^2}=\frac{c-a}{z^2-x^2}.$$

7. If $\frac{a}{y+z}=\frac{b}{z+x}=\frac{c}{x+y}$, prove that

$$\frac{a(b-c)}{y^2-z^2}=\frac{b(c-a)}{z^2-x^2}=\frac{c(a-b)}{x^2-y^2}. \quad \text{(M. F. A. 1887).}$$

8. If $\frac{a-a'}{a'-a''}=\frac{b-b'}{b'-b''}=\frac{c-c'}{c'-c''}$, prove that each of these

$$=\frac{ab'-a'b}{a'b''-a''b'}=\frac{bc'-b'c}{b'c''-b''c'}=\frac{ca'-c'a}{c'a''-c''a'}.$$

9. If $\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}$, prove that

$$(a+b+c)(yz+zx+xy)=(x+y+z)(ax+by+cz).$$

10. If $x : (b+c) = y : c+a = z : (a+b)$, prove that

$$a : (y+z-x) = b : (z+x-y) = c : (x+y-z). \quad \text{(C. E. 1903).}$$

11. If $(a+b+c)x=(b+c-a)y=(c+a-b)z=(a+b-c)w$,

show that $\frac{1}{y}+\frac{1}{z}+\frac{1}{w}=\frac{1}{x}$. (C. E. 1905).

12. If $\frac{x+2y}{3a-c}=\frac{y+2z}{3b-a}=\frac{z+2x}{3c-b}$, show that

$$3(x+y+z)(9a+8b-c)=2(a+b+c)(6x+11y+7z).$$

13. If $\frac{x-\frac{yz}{x}}{1-yz}=\frac{y-\frac{zx}{y}}{1-zx}$, and x and y be unequal, then each of the ratios is equal to $x+y+z$ or $x^{-1}+y^{-1}+z^{-1}$. (B. P. E. 1892).

14. If $\frac{4x+5y-6z}{8a+15b-12c}=\frac{2x-y+z}{4a-3b+4c}=\frac{x-3y+2z}{3(a-6b+2c)}$, prove that $(7x+y-3z)(9a-75b+38c)=(3x-16y+14z)(15a-6b-2c)$.

15. If $\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)=\frac{4}{ac}-\frac{3}{b^2}$, prove that $\frac{a+c}{c}=\frac{2a}{b}$.

16. If $(b+c)x=(c+a)y=(a+b)z$, show that

$$\frac{x-y}{a^2-b^2}=\frac{y-z}{b^2-c^2}=\frac{z-x}{c^2-a^2}.$$

17. If $(a+b)(y+z-x)=(b+c)(z+x-y)=(c+a)(x+y-z)$, then $\frac{x-y}{c^2-a^2}=\frac{y-z}{a^2-b^2}=\frac{z-x}{b^2-c^2}$.

18. If $\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}$, show that

$$(a+b+c)\{x(y+z)+y(z+x)+z(x+y)\}=2(x+y+z)(ax+by+cz).$$

19. If $a(x-y)+a^2=b(y-z)+b^2=c(z-x)+c^2$, then

$$\text{each}=\frac{a+b+c}{a^{-1}+b^{-1}+c^{-1}}.$$

20. If $a+\frac{y^2-z^2}{b-c}=b+\frac{z^2-x^2}{c-a}$, then each $=c+\frac{x^2-y^2}{a-b}$.

21. If $alx+bmy+cnz=apx+bqy+crz=ax^2+by^2+cz^2=0$, prove that $x(mr-nq)+y(np-lr)+z(lq-mp)=0$. (P. E. 1887).

VI. VARIATION.

421. When two quantities are such, that their ratio is *constant*, that is, remains the same, whatever values we give to the letters they contain, one of them is said to **vary as** the other.

The sign used to denote variation is $\propto$ (read **varies as**).

422. Hence if $A \propto B$, (where A and B are used to denote, not numerical or *constant*, but algebraical or *variable* quantities, such as admit of different values by giving different values to the letters they contain) then, according to the above definition, the value of the ratio $A : B$ will remain *constant*, whatever may be the values of the quantities A and B themselves. If then we put m to denote this *constant* value,

$$\text{we have } \frac{A}{B}=m, \text{ or } A=mB\ ;$$

so that, *when one quantity varies as another, they are connected by a constant multiplier.*

423. Hence also if $A \propto B$, and a, b be any pair of values of A and B, then for any *other* values of A and B,

$$\text{we have } A : B=m=a : b$$

that is, *when one quantity varies as another, if any two pairs of values be taken of them, the four will be proportionals :*

or since $A : a :: B : b$,

we may state this by saying that if one of them be changed from any one value (A) to any other value (a), the other will be changed *in the same proportion* from the value (B) corresponding to the first to the value (b) corresponding to the second.

424. The following terms are used in Variation :—

(1) If $A = mB$, then A is said to vary **directly** as B.

(2) If $A = \dfrac{m}{B}$, then A is said to vary **inversely** as B.

(3) If $A = mBC$, then A is said to vary **jointly** as B and C.

(4) If $A = m\dfrac{B}{C}$, then A is said to vary **directly** as B and **inversely** as C.

425. The following results in Variation are deserving of notice.

(i) If $A \propto B$ and $B \propto C$, then $A \propto C$.

For let $A = mB$, $B = nC$; then $A = mnC$ and $\therefore A \propto C$, since m, n, being constant, so also is mn.

So also, if $A \propto B$ and $B \propto \dfrac{1}{C}$, then $A \propto \dfrac{1}{C}$.

(ii) If $A \propto C$ and $B \propto C$, then $A \pm B \propto C$ and $\sqrt{(AB)} \propto C$.

For let $A = mC$, $B = nC$;

then $A \pm B = mC \pm nC = (m \pm n)C$, and $\therefore A \pm B \propto C$;

and $\sqrt{(AB)} = \sqrt{(mC \times nC)} = \sqrt{(mnC^2)} = \sqrt{(mn)}C$,

and therefore $\sqrt{(AB)} \propto C$.

(iii) If $A \propto BC$, then $B \propto \dfrac{A}{C}$, and $C \propto \dfrac{A}{B}$.

For let $A = mBC$, then $B = \dfrac{1}{m} \cdot \dfrac{A}{C}$, or $B \propto \dfrac{A}{C}$; so $C \propto \dfrac{A}{B}$.

(iv) If $A \propto B$, and $C \propto D$, then $AC \propto BD$.

For let $A = mB$, $C = nD$; then $AC = mnBD$, or $AC \propto BD$.

(v) If $A \propto B$, then $A^n \propto B^n$.

(vi) If $A \propto B$, and P be any other quantity,

then $AP \propto BP$, and $\dfrac{A}{P} \propto \dfrac{B}{P}$.

426. *If $A \propto B$ when C is constant, and $A \propto C$ when B is constant, then $A \propto BC$ when both B and C are variable.*

Since $A \propto B$, when C is constant;

$\therefore$ $A = mB$, (where m does not contain B as factor)......(1)

Again since $A \propto C$, when B is constant;

$\therefore$ $m\,B \propto C$, when B is constant;

$\therefore$ $m \propto C$;

$\therefore$ $m = nC$, (where n does not contain C as factor)... ....(2)

Also n, being a factor of m, does not contain C.

Hence, from (1) and (2), we get

$\therefore$ $A = nCB$, (when n is independent of B and C).

$\therefore$ $A \propto BC$.

Note. The following is an illustration of the above.

In a triangle, the area $\propto$ the base when the altitude is constant, and $\propto$ as the altitude when the base is constant. When both base and altitude are variable, the area $\propto$ base $\times$ altitude.

Ex. 1. If $a \propto b$, and $a = 10$ when $b = 4$, find b when $a = \frac{15}{4}$.

Here $a = mb$, m being a constant.

The statement, that $a = 10$ when $b = 4$, enables us to find m.

From (1) $10 = m \times 4$; $\therefore$ $m = \frac{5}{2}$.

Thus the relation between a and b is $a = \frac{5}{2}b$.

Hence, when $a = \frac{15}{4}$, we have $\frac{15}{4} = \frac{5}{2}b$.

$\therefore$ $b = \frac{15}{4} \times \frac{2}{5} = \frac{3}{2}$.

Ex. 2. If y = the sum of two quantities, one of which $\propto x$, and the other $\propto x^2$, and when $x = 1$, $y = 6$, when $x = 2$, $y = 20$; express y in terms of x.

Let $y = p + q$, of which $p \propto x$ and $q \propto x^2$.

Also let $p = mx$ and $q = nx^2$, where m and n are constants.

$\therefore$ $y = mx + nx^2$.

$$\left.\begin{array}{l}\text{Putting } y = 6,\ x = 1, \text{ we have } 6 = m + n \ldots\ldots\ldots\ldots\ldots(1)\\ \text{Also, putting } y = 20,\ x = 2, \text{ we have } 20 = 2m + 4n \ldots\ldots(2)\end{array}\right\}$$

Dividing (2) by 2 and subtracting (1), we get

$$n=4 \text{ and } \therefore\ m=2.$$

Hence $y=2x+4x^2$ is the required relation.

Ex. 3. The wages of 100 men for 6 months amount to *Rs.*10800. How many men can be employed for 7 months for *Rs.*4536.

Denoting the number of men by a, of wages by w, and of months by d, we know that $w \propto a$, when d is given, and $w \propto d$, when a is given.

$$\therefore\ w \propto ad\ ;\ \therefore\ w=mad.$$

The first statement gives $10800=m\times 100\times 6$;

$$\therefore\ m=18 \text{ and } \therefore\ w=18ad$$

Hence when $w=4536$ and $d=7$.

$$\text{The no. of men}=\frac{4536}{18\times 7}=36.$$

Exercise CLIV.

1. If $a \propto b^2c$, and when $a=1$, $b=2$, $c=3$, express a in terms of b and c.

2. If $xy \propto x^2+y^2$, and 3, 4 be contemporaneous values of x and y, express xy in terms of x^2+y^2.

3. If $a \propto b$, and when $b=3$, $a=4$, find the value of a when $b=5$

4. If $y \propto 1/x^2$, and when $x=8$, $y=10$, find the value of y when $x=9$.

5. If $x \propto p+q$, $p \propto y$ and $q \propto 1/y$, and if when $y=1$, $x=18$, when $y=2$, $x=19\frac{1}{2}$, find x when $y=11$.

6. If $y=$the sum of two quantities, whereof one is constant and the other $\propto x$ inversely, and when $x=2$, $y=0$, when $x=3$, $y=1$, find the value of y, when $x=6$.

7. If $y=$the sum of two quantities, whereof one is constant, and the other $\propto xy$, and when $x=2$, $y=-2\frac{1}{3}$, when $x=-2$, $y=1$, express y in terms of x.

8. If $y=$the sum of three quantities, which vary as x, x^2, x^3 respectively, and when $x=1, 2, 3$, $y=6, 22, 54$, respectively, express y in terms of x.

9. If $y=$the sum of three quantities, of which the first $\propto x^2$, the second $\propto x$, and the third is constant: and when $x=1, 2, 3$, $y=6, 11, 18$, respectively, express y in terms of x.

10. Given that $z \propto x+y$, and $y \propto x^2$, and that when $x=\frac{1}{2}$, the values of y and z are $\frac{1}{3}$ and $\frac{1}{4}$; express z in terms of x.

11. If x varies directly as y and inversely as z, and $x=a$, when $y=b$ and $z=c$, find the value of x, when $y=b^2$ and $z=c^2$. (C. F. A. 1877).

12. If $x \propto \frac{z}{y^2}$ and $z^2 \propto \frac{y}{x}$, shew that $x \propto \frac{1}{y} \propto \frac{1}{z}$.

13. If $a^2+b^2 \propto a^2-b^2$ prove that $a+b \propto a-b$.

14. Given that $x+y$ varies as $z+\frac{1}{z}$, and $x-y$ varies as $z-\frac{1}{z}$, find the relation between x and z, if $z=2$, when $x=3$ and $y=1$.
(B. P. E. I

15. If x, y, z be variable quantities such that $y+z-x$ is constant, and that $(x+y-z)(x+z-y)$ varies as yz, prove that $x+y+z$ varies as yz.

16. If $x+y$ varies as z when y is constant, and if $x+z$ varies as y when z is constant, shew that when both y and z vary, then $x+y+z$ varies as yz. (C. F. A. 1871).

17. Find how soon 20 men will earn *Rs.*30, if 3 men earn *Rs.*9 in 16 days. (M. F. A. 1885).

18. Nine horses having 4 feeds a day can be kept for 3 weeks for £12. 8*s.* $0\frac{3}{4}$*d.*; what will be the cost of 15 horses for 36 days with 3 feeds a day?

19. With a capital of *Rs.*450 a man gains *Rs.*99 in 11 months. What profit does he make in 10 months on a capital of *Rs.*1000?

20. Prove that the volume of a sphere whose radius is 6 inches is equal to the sum of the volumes of three spheres whose radii are 3, 4, 5 inches respectively. [Given the volume of a sphere $\propto$ (radius)^3].

21. If z varies as $(x+a)(y+b)$, and is equal to $(a+b)^2$ when $x=b$ and $y=a$, find the value of z when $x=a+2b$, $y=2a+b$.

22. The area of any triangle varies jointly as any side, and the perpendicular let fall upon it from the opposite angle; express the area of the right-angled triangle ABC in terms of the sides AC, BC, containing the right-angle, it being found that, when the sum of the two sides is 14 feet and hypotenuse 10 feet, the area is 24 square feet.

CHAPTER XIX.

ELIMINATION AND SPECIAL ARTIFICES.

I. ELIMINATION.

427. Elimination is the process of obtaining from a given system of equations containing one or more algebraical quantities, a new equation free from those quantities and involving only the others. The result thus obtained is called the **Eliminant** of the given equations.

Thus, to *eliminate* x from the equations

$$ax+b=0\ldots\ldots(1) \text{ and } a'x+b'=0,$$

we have, from (1) $x=-b/a$, and from (2) $x=-b'/a'$.

Equating these values of x, we get

$$-\frac{b}{a}=-\frac{b'}{a'}, \text{ or } a'b=ab' \text{ ; } i.\ e.\ a'b-ab'=0.$$

Thus, we obtain an equation free from the quantity x and involving only the other quantities which occur in the given equations. The result $a'b-ab'=0$ is called the *Eliminant*.

Ex. 1. Eliminate x from the equations

$$\left.\begin{array}{l} Ax^2+Bx+C=0\ldots\ldots(1) \\ A'x^2+B'x+C'=0\ldots\ldots(2) \end{array}\right\} \text{ (C. F. A. 1864).}$$

From (1) and (2), By the *Rule of Cross Multiplication*, we have

$$\frac{x^2}{BC'-B'C}=\frac{x}{CA'-C'A}=\frac{1}{AB'-A'B},$$

$$\therefore \frac{x^2}{BC'-B'C}\times\frac{1}{AB'-A'B}=\left(\frac{x}{CA'-C'A}\right)^2,$$

$$\therefore (BC'-B'C)(AB'-A'B)=(CA'-C'A)^2,$$

which is the required *Eliminant.*

Ex. 2. Eliminate x and y from the equations

$$ax+by=c,\ a'x+b'y=c',\ a''x+b''y=c''.$$

From (1) and (2), we have $\left.\begin{array}{l} ax+by-c=0 \\ a'x+b'y-c'=0 \end{array}\right\}$

By the *Rule of Cross Multiplication*,

$$\frac{x}{-bc'+b'c}=\frac{y}{-a'c+ac'}=\frac{1}{ab'-a'b}.$$

Hence $x=\dfrac{b'c-bc'}{ab'-a'b}$ and $y=\dfrac{ac'-a'c}{ab'-a'b}$.

Substituting these values of x and y in (3), we have

$$a''\left(\frac{b'c-bc'}{ab'-a'b}\right)+b''\left(\frac{ac'-a'c}{ab'-a'b}\right)=c'',$$

or $a''(b'c-bc')+b''(ac'-a'c)=c''(ab'-a'b)$.

$\therefore\quad a''(b'c-bc')+b''(ac'-a'c)+c''(a'b-ab')=0.$

i.e. $a''(bc'-b'c)+b''(ca'-c'a)+c''(ab'-a'b)=0.$

Ex. 3. Eliminate x, y and z from the equations

$$\left.\begin{array}{lr} ax+by+cz=0 & \ldots\ldots\ldots (1) \\ a'x+b'y+c'z=0 & \ldots\ldots\ldots (2) \\ a''x+b''y+c''z=0 & \ldots\ldots\ldots (3) \end{array}\right\}$$

From (1) and (2) by the *Rule of Cross Multiplication*, we have

$$\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{z}{ab'-a'b}=k, \text{ suppose.}$$

Hence $x=k(bc'-b'c)$, $y=k(ca'-c'a)$, $z=k(ab'-a'b)$.

Substituting these values of x, y and z in (3),

$$a''(bc'-b'c)k+b''(ca'-c'a)k+c''(ab'-a'b)k=0.$$

$\therefore\quad a''(bc'-b'c)+b''(ca'-c'a)+c''(ab'-a'b)=0$, dividing out by k.

Ex. 4. Eliminate x and y from the equations

$$ax+by=c,\ bx-ay=d,\ x^2+y^2=1.$$

Squaring (1) and (2) and adding the two results, we get

$$a^2x^2+b^2y^2+b^2x^2+a^2y^2=c^2+d^2.$$

$\therefore\quad (a^2+b^2)x^2+(a^2+b^2)y^2=c^2+d^2$; $\therefore\ (a^2+b^2)(x^2+y^2)=c^2+d^2$.

But from (3) $x^2+y^2=1$, $\therefore\ a^2+b^2=c^2+d^2$.

Ex. 5. Eliminate x from the following equations

$$x^3+\frac{1}{x^3}+3\left(x+\frac{1}{x}\right)=m,\ x^3-\frac{1}{x^3}-3\left(x-\frac{1}{x}\right)=n. \text{ (C. F. A. 1865).}$$

$$\left.\begin{aligned} \text{From (1)}\ \left(x+\frac{1}{x}\right)^3=m,\ \therefore\ x+\frac{1}{x}=m^{\frac{1}{3}}\ \ldots\ \ldots(3) \\ (2)\ \left(x-\frac{1}{x}\right)^3=n,\ \therefore\ x-\frac{1}{x}=n^{\frac{1}{3}}.\ \ldots\ \ldots\ (4) \end{aligned}\right\}$$

Adding and subtracting (3) and (4), we have

$$2x=m^{\frac{1}{3}}+n^{\frac{1}{3}}\ \text{and}\ 2/x=m^{\frac{1}{3}}-n^{\frac{1}{3}}.$$

Hence $2x\times\frac{2}{x}=(m^{\frac{1}{3}}+n^{\frac{1}{3}})(m^{\frac{1}{3}}-n^{\frac{1}{3}})$, or $4=m^{\frac{2}{3}}-n^{\frac{2}{3}}$.

Ex. 6. Eliminate x and y from the equations

$$x+y=a,\ x^2+y^2=b,\ x^3+y^3=c.$$

Squaring (1) and subtracting (2), we get

$$2xy=a^2-b,\ \text{or}\ xy=\tfrac{1}{2}(a^2-b).\ \ldots\ldots\ldots\ldots(4)$$

Again, cubing (1), $x^3+y^3+3xy(x+y)=a^3\ldots\ldots\ldots(5)$

Substituting (1), (3) and (4) in (5), we have

$$c+3\times\tfrac{1}{2}(a^2-b)a=a^3,\ \text{or}\ a^3-3ab+2c=0.$$

Ex. 7. Eliminate x, y and z from the equations

$$x^2(y+z)=a,\ y^2(z+x)=b,\ z^2(x+y)=c,\ xyz=a.$$

Adding (1), (2), (3) and twice (4), we have

$$\begin{aligned} a+b+c+2d&=x^2(y+z)+y^2(z+x)+z^2(x+y)+2xyz \\ &=(y+z)(z+x)(x+y)\ldots\ldots\ \ldots\ldots\ldots(5) \end{aligned}$$

Again, multiplying (1), (2) and (3), we have

$$\begin{aligned} abc&=x^2y^2z^2(y+z)(z+x)(x+y) \\ &=d^2(a+b+c+2d),\ \text{from (4) and (5).} \end{aligned}$$

Ex. 8. Eliminate x and y from the equations

$$x+y+z=a,\ yz+zx+xy=b,\ xyz=c.$$

We have *identically*, $(z-x)(z-y)(z-z)=0$,

$$\text{or}\ z^3-(x+y+z)z^2+(yz+zx+xy)z-xyz=0.$$

Now, substituting from the given equations, we have

$$z^3-az^2+bz-c=0.$$

Exercise CLV.

1. Eliminate x from the equations :—

(1) $\left.\begin{array}{l} x+\dfrac{1}{x}=a+b \\ x-\dfrac{1}{x}=a-b \end{array}\right\}$ (2) $\left.\begin{array}{l} x^3+\dfrac{3}{x}=4(a^3+b^3) \\ 3x+\dfrac{1}{x^3}=4(a^3-b^3) \end{array}\right\}$ (3) $\left.\begin{array}{l} a+c=\dfrac{b}{x}-dx \\ a-c=\dfrac{d}{x}-bx \end{array}\right\}$

2. Eliminate x and y from the equations :—

(1) $x+y=a,\ x^2-y^2=b^2,\ xy=c^2.$

(2) $ax+by=c,\ a'x+b'y=c',\ x^2+y^2=1.$

3. Eliminate x and y from the equations

$$ax+by=x+y+xy=x^2+y^2-1=0.$$

4. Eliminate x, y, z from the equations :—

(1) $\dfrac{y-z}{y+z}=a,\ \dfrac{z-x}{z+x}=b,\ \dfrac{x-y}{x+y}=c.$

(2) $y^2+z^2=ayz,\ z^2+x^2=bzx,\ x^2+y$

(3) $\dfrac{x}{y+z}=a,\ \dfrac{y}{z+x}=b,\ \dfrac{z}{x+y}=c.$

(4) $\dfrac{y}{z}+\dfrac{z}{y}=a,\ \dfrac{z}{x}+\dfrac{x}{z}=b,\ \dfrac{x}{y}+\dfrac{y}{x}=c.$

(5) $\dfrac{ax}{by+cz}=\dfrac{by}{cz+ax}=\dfrac{z}{x+y}=\frac{1}{2}.$

5. Eliminate x, y, z from the equations

$$ax+by=z,\ by+cz=x,\ cz+ax=y.$$

6. Eliminate x from the equations

$$ax^3+bx+c=0,\ a'x^3+b'x+c'=0.$$

7. Eliminate x from the equations

$$32\frac{c}{a}=\left(\frac{x}{a}\right)^5+10\frac{x}{a}+5\left(\frac{a}{x}\right)^3,\quad 32\frac{a}{c}=\left(\frac{a}{x}\right)^5+10\frac{a}{x}+5\left(\frac{x}{a}\right)^3.$$

8. Shew that if $ax^2+by^2+cz^2=ax+by+cz=yz+zx+xy=0$, then $abc=(b+c-a)(c+a-b)(a+b-c)$.

9. Eliminate x, y, z from the equations :—

(1) $x^2(y-z)=a,\ y^2(z-x)=b,\ z^2(x-y)=c,\ xyz=d.$

(2) $x+y+z=a,\ yz+zx+xy=b,\ x^3+y^3+z^3=c,\ xyz=d.$

10. Eliminate a, b, c, from the equations

$bz+cy=a$, $az+cx=b$, $ay+bx=c$. (C.F.A. 1870 & A.I.E. 1893).

11. Eliminate l and m from the equations :—

$l^2x+m^2y=a$, $l^2+m^2=1$ and $-lx+my=0$. (P. E. 1902).

II. SPECIAL ARTIFICES.

428. The following are typical examples with their solutions.

Ex. 1. If $x(b-c)+y(c-a)+z(a-b)=0$,

prove that $\dfrac{bz-cy}{b-c}=\dfrac{cx-az}{c-a}=\dfrac{ay-bx}{a-b}$.

We have $a(b-c)+b(c-a)+c(a-b)=0$, (identically)

and $x(b-c)+y(c-a)+z(a-b)=0$, (given)

$\therefore$ by the *Rule of Cross Multiplication*, we have

$$\frac{b-c}{bz-cy}=\frac{c-a}{cx-az}=\frac{a-b}{ay-bx};$$

Hence $\dfrac{bz-cy}{b-c}=\dfrac{cx-az}{c-a}=\dfrac{ay-bx}{a-b}$.

Ex. 2. If $x=cy+bz$, $y=az+cx$, and $z=bx+ay$,

shew that $\dfrac{x^2}{1-a^2}=\dfrac{y^2}{1-b^2}=\dfrac{z^2}{1-c^2}$.

From the given relations, we have

$x-cy-bz=0$...(1), $cx-y+az=0$...(2), $bx+ay-z=0$...(3)

From (1) and (2), by the *Rule of Cross Multiplication*,

$$\frac{x}{ac+b}=\frac{y}{bc+a}=\frac{z}{1-c^2}\text{......................}(4)$$

Similarly, from (2) and (3), $\dfrac{x}{1-a^2}=\dfrac{y}{ab+c}=\dfrac{z}{ac+b}$........(5)

and from (1) and (3) $\dfrac{x}{ab+c}=\dfrac{y}{1-b^2}=\dfrac{z}{bc+a}$........(6)

Hence, from (4) and (5) we have

$$\frac{x^2}{(ac+b)(1-a^2)}=\frac{z^2}{(1-c^2)(ac+b)}, \text{ or } \frac{x^2}{1-a^2}=\frac{z^2}{1-c^2}.$$

Similarly, from (5) and (6), we have

$$\frac{x^2}{(1-a^2)(ab+c)} = \frac{y^2}{(ab+c)(1-b^2)}, \text{ or } \frac{x^2}{1-a^2} = \frac{y^2}{1-b^2}.$$

Hence the result.

Ex. 3. If $ax+by+cz=0$, and $a/x+b/y+c/z=0$, prove that

$$ax^3+by^3+cz^3+(a+b+c)(y+z)(z+x)(x+y)=0.$$

From the given relations, we have

$$\left.\begin{aligned} ax+by+cz&=0 \\ \text{and } ayz+bzx+cxy&=0 \end{aligned}\right\}$$

By the *Rule of Cross Multiplication*, we have

$$\frac{a}{x(y^2-z^2)} = \frac{b}{y(z^2-x^2)} = \frac{c}{z(x^2-y^2)} = k, \text{ suppose.}$$

$\therefore\ a=kx(y^2-z^2)$, and $\therefore\ ax^3=kx^4(y^2-z^2)$.
$b=ky(z^2-x^2)$, and $\therefore\ by^3=ky^4(z^2-x^2)$.
$c=kz(x^2-y^2)$, and $\therefore\ cz^3=kz^4(x^2-y^2)$.

By addition, $a+b+c=k\{x(y^2-z^2)+y(z^2-x^2)+z(x^2-y^2)\}$
$=k(y-z)(z-x)(x-y)$.

and $ax^3+by^3+cz^3=k\{x^4(y^2-z^2)+y^4(z^2-x^2)+z^4(x^2-y^2)\}$
$=-k(y^2-z^2)(z^2-x^2)(x^2-y^2)$.

$$\therefore\ \frac{ax^3+by^3+cz^3}{a+b+c} = -(y+z)(z+x)(x+y).$$

Hence $ax^3+by^3+cz^3+(a+b+c)(y+z)(z+x)(x+y)=0$.

Ex. 4. Having given $x=by+cz+du$, $y=ax+cz+du$,
$z=ax+by+du$, $u=ax+by+cz$,

show that $\dfrac{a}{1+a}+\dfrac{b}{1+b}+\dfrac{c}{1+c}+\dfrac{d}{1+d}=1.$

Assume $ax+by+cz+du=k, \ldots\ldots (5)$ then,
from (1) $x+ax$ or $x(1+a)=k$; from (2) $y(1+b)=k$;
from (3) $z(1+c)=k$ and from (4) $u(1+d)=k$.

$\therefore\ x(1+a)=y(1+b)=z(1+c)=u(1+d)=k.$

$$\therefore\ x=\frac{k}{1+a},\quad y=\frac{k}{1+b},\quad z=\frac{k}{1+c} \text{ and } u=\frac{k}{1+d}.$$

Hence $\frac{ak}{1+a}+\frac{bk}{1+b}+\frac{ck}{1+c}+\frac{dk}{1+d}=k$, from (5)

or $\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}+\frac{d}{1+d}=1.$

Ex. 5. If x, y, z, are unequal, and if $2a-3y=\frac{(z-x)^2}{y}$, and $2a-3z=\frac{(x-y)^2}{z}$ then will $2a-3x=\frac{(y-z)^2}{x}$, and $x+y+z=a$.

We have from (1) $2ay-3y^2=(z-x)^2$,
and from (2) $2az-3z^2=(x-y)^2$.

$\therefore$ by subtr. $2a(y-z)-3(y^2-z^2)=(z-x)^2-(x-y)^2$
$$=-(y-z)(y+z-2x).$$

Dividing by $y-z$, which is not $=0$, we have
$$2a-3(y+z)=-(y+z-2x);$$
$\therefore$ $2a=3(y+z)-(y+z-2x)=2(x+y+z)$; $\therefore$ $a=x+y+z$. (a)

Again, since, $(z-x)^2$ or $z^2-2zx+x^2=2ay-3y^2$, from (1)
$$=2(x+y+z)y-3y^2, \text{ from } (a$$
$$=2xy-y^2+2yz.$$

$\therefore$ $y^2-2yz+z^2$ or $(y-z)^2=2xy+2xz-x^2=2x(x+y+z)-3x^2$
$$=2ax-3x^2, \text{ from } (a)$$
$$=x(2a-3x).$$
$$\therefore \frac{(y-z)^2}{x}=2a-3x.$$

Ex. 6. If $\frac{x}{l(mb+nc-la)}=\frac{y}{m(nc+la-mb)}=\frac{z}{n(la+mb-nc)}$

prove that $\frac{l}{x(by+cz-ax)}=\frac{m}{y(cz+ax-by)}=\frac{n}{z(ax+by-cz)}$.

We have $\frac{\frac{x}{l}}{mb+nc-la}=\frac{\frac{y}{m}}{nc+la-mb}=\frac{\frac{z}{n}}{la+mb-nc}=k$, suppose ;

then $k=\frac{\frac{y}{m}+\frac{z}{n}}{2la}=\frac{\frac{z}{n}+\frac{x}{l}}{2mb}=\frac{\frac{x}{l}+\frac{y}{m}}{2nc}$

$$=\frac{ny+mz}{2lmna}=\frac{lz+nx}{2lmnb}=\frac{mx+ly}{2lmnc},$$

$$=\frac{x(ny+mz)}{2lmnax}=\frac{y(lz+nx)}{2lmnby}=\frac{z(mx+ly)}{2lmncz},$$

$$=\frac{nxy+mzx}{ax}=\frac{lyz+nxy}{by}=\frac{mxz+lyz}{cz},$$

$$=\frac{2lyz}{by+cz-ax}=\frac{2mxz}{cz+ax-by}=\frac{2nxy}{ax+by-cz},$$

$$=\frac{2lxyz}{x(by+cz-ax)}=\frac{2mxyz}{y(cz+ax-by)}=\frac{2nxyz}{z(ax+by-cz)}.$$

Hence dividing by $2xyz$, we have

$$\frac{l}{x(by+cz-ax)}=\frac{m}{y(cz+ax-by)}=\frac{n}{z(ax+by-cz)}.$$

Otherwise thus : Let each of the given fractions $=k$,

then $k=\dfrac{by+cz-ax}{l^2a^2-(mb-nc)^2}=\dfrac{by+cz-ax}{(la+nc-mb)(la+mb-nc)}$,

also $k=\dfrac{x}{l(mb+nc-la)}$,

$\therefore\ k^2=\dfrac{x(by+cz-ax)}{l(la+nc-mb)(la+mb-nc)(mb+nc-la)}$,

$\therefore\ \dfrac{l}{x(by+cz-ax)}=\dfrac{1}{k^2(mb+nc-la)(nc+la-mb)(la+mb-nc)}$

similarly $\dfrac{m}{y(cz+ax-by)}=$ same thing, &c.

Exercise CLVI.

1. If $(bx-ay)^2=(b^2-ac)(x^2-az)$, prove that
$(by-cx)^2=(b^2-ac)(y^2-cz)$.

2. If $x(y^2-z^2)=(b-c)yz$ and $y(z^2-x^2)=(c-a)zx$,
prove that $z(x^2-y^2)=(a-b)xy$.

3. If $ax+by+cz=0$, and $a/x+b/y+c/z=0$,
then will $ax^2+by^2+cz^2+(a+b+c)(yz+zx+xy)=0$.

4. If $bz+cy=a$, $az+cx=b$ and $ay+bx=c$,
prove that $\dfrac{a^2}{1-x^2}=\dfrac{b^2}{1-y^2}=\dfrac{c^2}{1-z^2}$.

5. If $\dfrac{1}{1+l+lm}+\dfrac{1}{1+m+mn}+\dfrac{1}{1+n+nl}=1$, prove that either $lmn=1$ or $(1+l)(1+m)(1+n)=-1$. (B. P. E. 1889).

6. If $(a+b+c)(a+b+d)=(c+d+a)(c+d+b)$, prove that each of these quantities is equal to

$$\frac{(a-c)(a-d)(b-c)(b-d)}{(a+b-c-d)^2}.$$

7. If $ab-\frac{1}{2}(a+b)(p+q)+pq=0$, and $cd-\frac{1}{2}(c+d)(p+q)+pq=0$, shew that $\left(\frac{p-q}{2}\right)^2=\frac{(a-c)(a-d)(b-c)(b-d)}{(a+b-c-d)^2}$.

8. If $\frac{(2x-y-z)^3}{x}=\frac{(2y-z-x)^3}{y}$, then each $=\frac{(2z-x-y)^3}{z}$.

9. If $a^2-b^2=b^2-c^2=c^2-a^2$, shew that

$$\frac{ab-c^2}{a-b}+\frac{bc-a^2}{b-c}+\frac{ca-b^2}{c-a}=0. \text{ (B. M. 1891).}$$

10. If $a(by+cz-ax)=b(cz+ax-by)=c(ax+by-cz)$,

prove that $\frac{y+z-x}{a}=\frac{z+x-y}{b}=\frac{x+y-z}{c}$.

11. If $ax^3=by^3=cz^3$ and $1/x+1/y+1/z=1$, find the value of $\sqrt[3]{(ax^2+by^2+cz^2)}$.

12. If $\frac{b-c}{1+bc}+\frac{c-a}{1+ca}+\frac{a-b}{1+ab}=0$, prove that two of the quantities a, b, c must be equal to each other.

13. If $(a+b+c)^3=a^3+b^3+c^3$, shew that

$(a+b+c)^{2n+1}=a^{2n+1}+b^{2n+1}+c^{2n+1}$, n being any positive integer.

14. If $x^2-yz=y^2-zx$, and if x and y be unequal, then will each of the expressions $=\frac{1}{2}(x^2+y^2+z^2)=z^2-xy$.

15. If $\frac{xyz}{y+z}-x^2=\frac{xyz}{z+x}-y^2$, prove that each $=\frac{xyz}{x+y}-z^2$, being given that x and y are unequal.

16. Having given $\frac{x}{1-x^2}=\frac{y+z}{m+nyz}$, $\frac{y}{1-y^2}=\frac{z+x}{m+nzx}$, prove that, if x, y be unequal, $\frac{z}{1-z^2}=\frac{x+y}{m+nxy}$.

17. If a, b, c be unequal, and $a+\frac{bc-a^2}{a^2+b^2+c^2}=b+\frac{ca-b^2}{a^2+b^2+c^2}$, then each $=c+\frac{ab-c^2}{a^2+b^2+c^2}$, and if $a+b+c=1$, then each of these expressions vanishes.

CHAPTER XX.

INEQUALITIES AND MISCELLANEOUS THEOREMS.

I. INEQUALITIES.

429. Inequality is the method of determining which of the two given algebraical expressions is the *greater* of the two. This is best done by shewing that *if a and b be real quantities, and $a > b$, then $a-b$ is a positive quantity.*

430. Most of the results in *Inequalities* may be obtained by the application of the following general theorem :—

The sum of the squares of two unequal quantities is always greater than twice their product.

That is, $$a^2+b^2 > 2ab.$$

For, let a and b be two real unequal quantities ; then $a-b$ is positive or negative, according as $a >$ or $< b$.

But since the square of every quantity, whether positive or negative is always positive,

$\therefore$ $(a-b)^2$ or a^2+b^2-2ab is a positive quantity.

$$\therefore a^2+b^2 > 2ab.$$

Note. If $a=b$, then $a^2+b^2=2ab$. Hence a^2+b^2 is *never less than* $2ab$.

Ex. 1. If a, b and c be any unequal positive quantities, prove that $(b+c)(c+a)(a+b) > 8abc$.

Since $b^2+c^2 > 2bc$, $\therefore$ b^2+c^2+2bc or $(b+c)^2 > 4bc$.

Similarly, $(c+a)^2 > 4ca$ and $(a+b)^2 > 4ab$.

Hence, by multiplication, we obtain

$$(b+c)^2(c+a)^2(a+b)^2 > 64a^2b^2c^2.$$

Ext. the sq. root, $(b+c)(c+a)(a+b) > 8abc$.

Ex. 2. A man receives $\frac{x}{y}$ths of *Rs.*10 and afterwards $\frac{y}{x}$ths of *Rs.*10. He then gives away *Rs.*20. Shew that he cannot lose by the transaction. (C. E. 1881).

The total sum received $=\left(\frac{x}{y}+\frac{y}{x}\right)$ of *Rs.*10.

Since $10\left(\frac{x}{y}+\frac{y}{x}\right)-20=10\left(\frac{x}{y}+\frac{y}{x}-2\right)=10\left(\sqrt{\frac{x}{y}}-\sqrt{\frac{y}{x}}\right)^2$

$=$ a positive quantity.

$\therefore$ he cannot lose by the transaction.

Ex. 3. Which is the greater, $\sqrt{5}+\sqrt{7}$ or $\sqrt{3}+\sqrt{10}$?

$\sqrt{5}+\sqrt{7} >$ or $< \sqrt{3}+\sqrt{10}$,

according as $12+2\sqrt{(35)} >$ or $< 13+2\sqrt{(30)}$, squaring,

" " $2\sqrt{(35)} >$ or $< 1+2\sqrt{(30)}$,

" " $140 >$ or $< 121+4\sqrt{(30)}$, squaring,

" " $19 >$ or $< 4\sqrt{(30)}$,

" " $361 >$ or < 480, squaring.

Now, we see that $480 > 361$, therefore $\sqrt{3}+\sqrt{10} > \sqrt{5}+\sqrt{7}$.

Exercise CLVII.

(All the quantities below are real, positive and unequal).

1. Prove that $\frac{a+b}{2} > \frac{2ab}{a+b}$, and that $\frac{a}{b^2}+\frac{b^2}{a} > \frac{1}{b}+\frac{1}{a}$.

2. Shew that the sum of any fraction, and its reciprocal, is > 2.

3. Shew that

(1) $a^3+b^3 > a^2b+ab^2$. (2) $a^2+b^2+c^2 > bc+ca+ab$.

(3) $ab(a+b)+ac(a+c)+bc(b+c) > 6abc$.

(4) $a^3+b^3+c^3 > \frac{1}{2}\{ab(a+b)+ac(a+c)+bc(b+c)\}$.

(5) $a^6+a^4b^2+a^2b^4+b^6 > (a^3+b^3)^2$.

4. If $x^2=a^2+b^2$ and $y^2=c^2+d^2$, shew that $xy > ac+bd$ or $aa+bc$.

5. If $a > b$, shew that $a^4-b^4 < 4a^3(a-b)$ and $> 4b^3(a-b)$.

6. Shew that $(a+b+c)^3 > 27abc$.

7. Shew that $abc > (b+c-a)(c+a-b)(a+b-c)$.

8. Which is the greater :—

(1) $3+\sqrt{5}$ or $4+\sqrt{3}$? (2) $\sqrt{5}+\sqrt{14}$ or $\sqrt{3}+3\sqrt{2}$?

(3) x^3+1 or x^2+x? (4) $2(1+a^2+a^4)$ or $3(a+a^3)$?

9. A man receives $(x+2a)/y$ of *Rs*.5 and again $y/(x+2a)$ of *Rs*.5. He then gives away *Rs* 10. Show that he cannot lose by the transaction. (P. E. 1889).

II. MISCELLANEOUS THEOREMS.

431. Meaning of $A/0$. By actual division, we get

$$\frac{A}{1-x}=A+Ax+Ax^2+Ax^3+Ax^4+\ldots\text{to an infinite number of terms.}$$

If in this result x be made equal to *unity*, the left-hand side becomes $\frac{A}{1-1}$ or $\frac{A}{0}$, and the right-hand side becomes

$$A+A+A+A+A+\ldots\text{to an infinite number of terms}$$
$$=A\times\infty=\infty.$$

Hence $A/0=\infty$, (*an infinitely large number*).

432 *If the sum of the squares of any number of real quantities be zero, then each of the quantities is separately equal to zero.*

Let $A^2+B^2+C^2+\ldots=0$, where A, B, C,...are all real quantities.

Now the algebraical expression for which A stands may be either positive or negative, but its square is always *positive*; hence A^2 is essentially positive. Similarly B^2, C^2,...are all essentially positive. Now, since none of the quantities A^2, B^2, C^2,...is negative, their sum cannot be equal to zero, unless each of them be equal to zero. Hence $A^2=0$, $B^2=0$, $C^2=0$, &c. and $\therefore$ $A=0$, $B=0$, $C=0$, &c. Hence the result.

Ex. 1. If $a^2+b^2+c^2=bc+ca+ab$, and a, b, and c be all real, then $a=b=c$.

Multiplying by 2 and transposing, we get

$$2a^2+2b^2+2c^2-2bc-2ca-2ab=0.$$

Re-arranging terms, we get

$$(b^2-2bc+c^2)+(c^2-2ca+a^2)+(a^2-2ab+b^2)=0,$$
$$\text{or } (b-c)^2+(c-a)^2+(a-b)^2=0.$$

Hence $b-c=0$, $c-a=0$ and $a-b=0$, or $a=b=c$.

Ex. 2. If $(ax+by+cz)^2=(a^2+b^2+c^2)(x^2+y^2+z^2)$, prove that $x/a=y/b=z/c$. (C. F. A. 1869).

Multiplying out and cancelling like terms, we have

$$2abxy+2acxz+2bcyz=a^2(y^2+z^2)+b^2(z^2+x^2)+c^2(x^2+y^2).$$

Transposing and re-arranging terms, we have

$(a^2y^2-2abxy+b^2x^2)+(c^2x^2-2acxz+a^2z^2)+(b^2z^2-2bcyz+c^2y^2)=0,$

or $(ay-bx)^2+(cx-az)^2+(bz-cy)^2=0.$

Hence $ay-bx=0$, $cx-az=0$ and $bz-cy=0$.

$\therefore$ $ay=bx$, $cx=az$ and $bz=cy$.

$\therefore$ $y/b=x/a$, $x/a=z/c$ and $z/c=y/b$.

Thus $x/a=y/b=z/c$.

433. If $A\times B=0$, then either $A=0$ or $B=0$, but not necessarily $A=0$ and $B=0$.

$$\left.\begin{array}{l}\text{For, if } A=a, \text{ and } B=0\\ \text{or if } A=0 \text{ and } B=b\end{array}\right\}\text{ then } AB=0.$$

Also, if $A=0$ and $B=0$ then also $AB=0$.

434. If in finding the value of an expression, it assumes the form 0/0 (which is **indeterminate** in value) for any particular value of any symbol, we must by suitable transformation change the expression into another of equal value, such that, it may not take the form 0/0, for the proposed value. Fractions which assumes this form, are called **vanishing fractions.**

Ex. 1. Find the value of $\dfrac{x^3-a^3}{x-a}$, when $x=a$.

Here the expression assumes the form 0/0, when a is substituted for x. Now, to avoid this form, we proceed thus :—

$$\frac{}{x-a}=x^2+ax+a^2=a^2+a^2+a^2=3a^2.$$

Ex. 2. Find the value of $\dfrac{3x^2-8x+5}{x^3-4x^2+6x-3}$, when $x=1$.

$$\text{The expression}=\frac{(x-1)(3x-5)}{(x-1)(x^2-3x+3)}=\frac{3x-5}{x^2-3x+3}$$

$$=\frac{3-5}{1-3+3}=-2.$$

Exercise CLVIII.

1. If $a^2+b^2+2=2(a+b)$, prove that $a=b=1$.
2. If $(a+b)^2+(b+c)^2+(c+d)^2=4(ab+bc+cd)$, then $a=b=c=d$.
3. If $a^2+b^2+c^2=2(a+b-1)$ prove that $a=b=1$ and $c=0$.

4. What is the only solution of $(x+2a)^2+y^2=0$? (P. E. 1889).

5. If $1+xx'+yy'=\sqrt{\{(1+x^2+y^2)(1+x'^2+y'^2)\}}$

then $x=x'$ and $y=y'$.

6. If $(a-c)^2+(b-d)^2+(a^2+b^2-1)(c^2+d^2-1)=0$, prove that $(a^2+b^2)(c^2+d^2)=1$.

7. If $x^2+y^2+2=(1+x)(1+y)$, prove that $x=y=1$.

8. If $a^3+b^3+c^3=3abc$, show that either $a+b+c=0$ or $a=b=c$.

9. Find the values of, (when $x=a$)

(1) $\dfrac{x^2-a^2}{x-a}$. (2) $\dfrac{x^3-a^3}{x^2-a^2}$. (3) $\dfrac{x^4-a^4}{x^3-a^3}$. (4) $\dfrac{x^6-a^6}{x^4-a^4}$.

10. Find the values of

(1) $\dfrac{x^3+3x-36}{x^3-4x^2+8x-15}$, when $x=3$.

(2) $\dfrac{2x^3-3x^2+1}{(1-x)^2}$, when $x=1$.

REVISION PAPERS IV.

Paper I.

1. Resolve into factors :

(i) $12x^2+7x-12$. (ii) $4a^2+b^2-c^2-d^2+4ab+2cd$.

2. Divide $(a^2-b^2)^2-(a^2-3ab+2b^2)^2$ by $(a-b)^2$.

3. Find the G. C. M. of $15x^3-4x^2-53x+30$ and $15x^3-x^2-31x-15$. (C. F. A. 1882).

4. Simplify $\dfrac{x^2+(a-b)x-ab}{x^2+(a+b)x+ab}$. (C. F. A. 1863).

5. Prove that $(x+y)^7=x^7+y^7+7xy(x+y)(x^2+xy+y^2)^2$.

6. If $2s=a+b+c$, prove that

$(s-a)^3+(s-b)^3+(s-c)^3-3(s-a)(s-b)(s-c)$

$=\frac{1}{2}(a^3+b^3+c^3-3abc)$. (C. F. A. 1878).

7. Solve the equations :—

(1) $\frac{x-3}{2}+\frac{3x-2}{8}=\frac{2x}{5}+\frac{x+5}{10}$. (C. F. A. 1875).

(2) $\frac{3}{5}\left(\frac{x-1}{2}\right)+2\left(\frac{x-2}{6}\right)+\frac{5}{3}\left(\frac{x-3}{4}\right)=3\frac{1}{30}$. (C. F. A. 1879).

8. A number has three digits, the sum of which equals 10 ; the first and third exceed the second by 4, and the first and second exceed the third by 8. Find the number. (C. F. A. 1867).

Paper II.

1. Multiply $x^{p(q-1)}+y^{q(p-1)}$ by $x^{p(q-1)}-y^{q(p-1)}$.

2. Divide $(x^2+2xy-3y^2)^2-(x^2-4xy+3y^2)^2$ by $(x-y)^2$.

3. If $a+b+c=0$, prove that
$(bc+ca+ab)^3+(a^2-bc)(b^2-ca)(c^2-ab)=0$. (M. F. A. 1888).

4. Find the G. C. M. of $6x^4-7x^2+2$ and $2x^3+6x^2-x-3$. (C. F. A. 1878).

5. Resolve into factors the expression :—
$(y^3+1)(x^2+x+1)(x+1)-(x^3+1)(y^2+y+1)(y+1)$. (M. F. A. 1887).

6. Prove that $(y+az)^3+(z+ax)^3+(x+ay)^3-3(y+az)(z+ax)\times(x+ay)=(1+a^3)(x^3+y^3+z^3-3xyz)$.

7. Solve the equations :—

(1) $\frac{2x(x+1)}{x+2}-\frac{2x-7}{x-4}=2(x-2)$. (C. F. A. 1869).

(2) $\frac{1}{x-3}-\frac{2}{x-4}+\frac{1}{x-6}$. (C. F. A. 1872).

8. A bag contains 160 coins consisting of half-crowns, shillings, six-pences and four-pences, and the values of the sums of money represented by each denomination of coin are the same ; how many of each are there ? (C. F. A. 1874).

Paper III.

1. Find the value of $\sqrt{\left(\frac{1+a}{1-b}\right)}+\sqrt{\left\{\frac{3(1+2a^2)}{1-b^2}\right\}}+\sqrt{(a^2-2ab+4b^2)}$, when $a=\frac{1}{4}$, $b=\frac{1}{5}$.

2. Multiply together $x-1+\sqrt{2}$, $x+2+\sqrt{3}$, $x-1-\sqrt{2}$, and $x+2-\sqrt{3}$.

3. If $a^2=b^2+c^2$, prove that
$$(a+b+c)(b+c-a)(a+c-b)(a+b-c)=4b^2c^2.$$

4. Simplify $\dfrac{6x^2+x-1}{2x^2-5x-12}\times\dfrac{6x^2+11x+3}{9x^2-1}\div\dfrac{2x^2+9x+4}{x^2-16}$.

5. Divide $x^3-b(4a+b)x+(a+2b)(a^2+3b^2)$ by $x+a+2b$.

6. From the equation $\dfrac{3}{y-5}+\dfrac{4}{2-x}-\dfrac{14}{(x-2)(y-5)}=0$, find the value of x/y.

7. Solve the equations :—

(1) $\dfrac{x}{2}+\dfrac{y}{3}=5$, $\dfrac{x}{3}+\dfrac{y}{1}=7$. (C. F. A. 1887).

(2) $\dfrac{2}{x}+\dfrac{7}{y}=29$, $\dfrac{5}{x}-\dfrac{6}{y}=2$. (P. I. E. 1889).

(3) $3z-2y=15$, $z-2x=1$, $x+y+z=25$. (C. F. A. 1879).

8. Take any number, the one next to it, and a third equal to the product of the first two. Add together the squares of the three numbers and prove that the result will always be a perfect square, whatever the number you choose to start with. (A. I. E. 1893).

Paper IV.

1. Prove that $(x+y)^5=x^5+y^5+5xy(x+y)(x^2+xy+y^2)$.

2. If $\left(x+\dfrac{1}{x}\right)^2=3$, prove that $x^3+\dfrac{1}{x^3}=0$. (C. F. A. 1883).

3. If $x=\dfrac{1}{1-y}$, $y=\dfrac{1}{1-z}$, $z=\dfrac{1}{1-u}$ and $u=\dfrac{1}{1-v}$, find the relation between x and v.

4. Extract the square root of the expression
$$(x^{\frac{1}{3}}-y^{\frac{1}{3}})(x^{\frac{1}{3}}y^{\frac{2}{3}}-y+2x^{\frac{2}{3}}y-\tfrac{1}{6})+x^{\frac{4}{3}}y+x^{\frac{4}{3}}y^{-1}.\text{ (M. F. A. 1893).}$$

5. Eliminate x and y from the equations
$$x+\frac{1}{x}=a,\ y+\frac{1}{y}=b \text{ and } xy+\frac{1}{xy}=c.$$

6. Solve the equations :—

(1) $\dfrac{x-7}{x-9}-\dfrac{x-9}{x-11}=\dfrac{x-13}{x-15}-\dfrac{x-15}{x-17}$. (P. I. E. 1891).

(2) $\sqrt{(x-y)}=\sqrt{(y+1)}$, $x+y=6$. (C. F. A. 1883).

(3) $bz+cy=a$, $cx+az=b$, $ay+bx=c$. (C. F. A. 1870).

7. Prove that $x^4+ax^3+bx^2+cx+d$ will be a perfect square for all values of x, if $(a^2-4b)^2=64d$ and $c^2=a^2d$. (A. I. E. 1891).

8. A merchant buys goods at 24 guineas the cwt., and by retailing them at 5*s.* 3*d.* the lb. makes 10 per cent. more profit than if he had sold the whole for £240. What weight did he buy? (A. E. 1897).

Paper V.

1. Resolve into their simplest factors :—

(1) $8(2x+y)^3+(x-2y)^3$. (2) $(2a-b)^4-(a-2b)^4$.

2. Prove that $(x+y)(x+y-1)(x+y-2)$

$$=x(x-1)(x-2)+3x(x-1)y+3xy(y-1)+y(y-1)(y-2).$$

3. Determine m and n so that $x^4+ax^3+mx^2+cx+n$ may be an exact square. (M. F. A. 1895).

4. Resolve into four factors the expression $a^3(b+c)+b^3(c+a)+c^3(a+b)+2(b^2c^2+c^2a^2+a^2b^2)+4abc(a+b+c)$. (M. F. A. 1888).

5. Plot the points (10, 10), (15, 18), (30, 22), (39, 10). If the quadrilateral joining them represents a field, each square unit representing one-tenth of an acre, find the area of the field.

6. Solve the equations :—

(1) $\dfrac{3+2x}{1+2x}-\dfrac{5+2x}{7+2x}=1-\dfrac{4x^2-2}{7+16x+4x^2}$. (P. I. E. 1890).

(2) $\dfrac{198x+3}{4\cdot 5x+3}+\dfrac{4x+5}{\cdot 5x-1}-52=0$. (C. F. A. 1863).

(3) $\left.\begin{array}{l} x+y+z=0,\ a^2x+b^2y+c^2z=0 \\ \dfrac{x}{b^2-c^2}+\dfrac{y}{c^2-a^2}+\dfrac{z}{a^2-b^2}=3 \end{array}\right\}$

7. Eliminate x and y from the equations :—

(1) $ay+bx=bh$, $ky+hx=b^2$, and $x^2+y^2=b^2$.

(2) $x^2-y^2=ax-by$, $4xy=bx+ay$, and $x^2+y^2=1$.

8. A and B travel together 120 miles by rail. A takes a return ticket for which he has to pay one fare and a half. Coming back they find that A has travelled cheaper than B by 4*a.* 2*p.* for every 00 miles. Find the fare per mile. (P. E. 1890).

Paper VI.

1. Draw the graphs of $\frac{1}{12}x+\frac{1}{16}y=1$, $4x-3y=0$, $y-x=2$. What do you deduce as to the three simultaneous equations?

2. Plot the points given by the table below, and deduce the equation of the graph which passes through them.

$x=$	-5	-1	3	7	11	15
$y=$	7	4	1	-2	-5	-8

3. If $x=pa-b-c$, $y=pb-c-a$ and $z=pc-a-b$, prove that
$$x^3+y^3+z^3-3xyz=(p+1)^2(p-2)(a^3+b^3+c^3-3abc).$$

4. Prove that $(x+y)^8=x^8+y^8+8xy(x^2+xy+y^2)^3+$
$$4x^2y^2(x^2+xy+y^2)^2+2x^4y^4.$$

5. Simplify $\dfrac{2x^2-5x+3}{2x-3}-\dfrac{3x^2+x-4}{x-1}+\dfrac{2(3x^2-13x-10)}{3x+2}$.

6. Solve the equations :—

(1) $\dfrac{(x-a)(x-b)}{x-a-b}=\dfrac{(x-c)(x-d)}{x-c-d}$. (C. F. A. 1876).

(2) $\sqrt{(x+2)}+\sqrt{(x-3)}=5$. (P. I E. 1890).

7. If $2x=a+\dfrac{1}{a}$ and $2y=b+\dfrac{1}{b}$; find the value of
$$xy+\sqrt{(x^2-1)(y^2-1)}.$$

8. Eliminate x, y and z from the equations :—

(1) $y^2+z^2=2ayz$, $z^2+x^2=2bzx$, $x^2+y^2=2cxy$.

(2) $1/x+1/y+1/z=1/a$, $x+y+z=b$, $x^2+y^2+z^2=c^2$, $x^3+y^3+z^3=d^3$.

Paper VII.

1. Find the values of $4x-3x^2$ for integral values of x from -3 to 3. Tabulate your work.

2. Divide $(a+b)^4+(a^2-b^2)^2+(a-b)^4$ by $3a^2+b^2$.

3. Find the L. C. M. of $3(x^4-x^2y^2)$, $6(x^2y^2+y^4)$,
$$9(x^3-x^2y+xy^2-y^3).$$

4. Simplify $\left\{\frac{x^3}{y^3}-\frac{y^3}{x^3}-3\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)+5\right\}\div\left(\frac{x}{y}-1-\frac{y}{x}\right)$.

5. Solve the equation $\frac{2x^2+5x+4}{x+2}=\frac{4x^2+8x+6}{2x+3}$.

Test your solution.

6. Solve the equations $3x+4y+14=0$, $5x-2y+6=0$.

Deduce the solution of the equations

$$\frac{3}{x}+\frac{4}{y}+14=0,\ \frac{5}{x}-\frac{2}{y}+6=0.$$

7. If four positive numbers are in continued proportion, show that the difference between the extremes is at least three times as great as the difference between the means. (P. E. 1900).

8. I bought a horse and carriage for *Rs.*800. I sold the horse at a profit of 20 per cent., and the carriage at a loss of 4 per cent., and found that on the whole transaction I had gained 5 per cent. What was the original cost of the horse?

Paper VIII.

1. Multiply $a^3+4a^2b+8ab^2+8b^3$ by $a^3-4a^2b+8ab^2-8b^3$.

2. Extract the square root of
$x^2(x^2+y^2+z^2)+2x(y+z)(yz-x^2)+y^2z^2$. (M. M. 1890).

3. Resolve into factors :—
$x^2+6x-187$ and $x^4-5x^3+9x^2-7x+2$. (B. M. 1901).

4. Find the G. C. M. of
$x^3-2ax^2-5a^2x-12a^3$ and $x^3-7ax^2+13a^2x-4a^3$. (A. E. 1891).

5. Simplify $\frac{1}{x-y}\left\{\frac{(x-y)^3+(y-z)^3}{x-z}-(x+z-2y)^2\right\}$.

6. Solve the equations :—

(1) $\frac{(a^2-1)(ax+1)}{a^2(x+a)}+\frac{(a^2+1)(x-a)}{ax+1}=\frac{ax+1}{x+a}+\frac{a(ax-1)}{ax+1}$.

(2) $\frac{a+c}{x-2b}-\frac{b+c}{x-2a}=\frac{a-c}{x+2b}-\frac{b-c}{x+2a}$. (M. M. 1888).

7. A man rows to a place 48 miles distant and back in 14 hours. He finds that he can row 4 miles with the stream in the same time as 3 miles against the stream. Find the rate of the stream.

(P. E. 1897).

8. A man spends $Rs.700$ in 45 days; make a graph and read off from it his expenditure in 17, 32 and 41 days, to the nearest rupee.

Paper IX.

1. Divide $x^5+x^4+4x^3+21x^2+23x-40$ by x^2+4x+5, using the method of detached coefficients.

2. Find the H. C. F of $x^4-8x^3+13x^2-30x+8$ and $x^4-4x^3-11x^2-50x+16$.

3. Simplify $\dfrac{\dfrac{1}{x-a}-\dfrac{1}{x+a}-\dfrac{2a}{x^2+a^2}}{\dfrac{1}{x^3-a^3}-\dfrac{1}{x^3+a^3}}\left(\dfrac{1}{x^2+ax+a^2}+\dfrac{1}{x^2-ax+a^2}\right)$.

4. If $ab+bc+ca=0$, prove that $(a+b+c)^3=a^3+b^3+c^3-3abc$. (M. F. A. 1894).

5. If $\dfrac{a}{b+c}=\dfrac{b}{c+a}=\dfrac{c}{a+b}$, shew that each of these fractions is equal to $\frac{1}{2}$ or -1. (M. F. A. 1895).

6. Solve the equations :—

(1) $\left(\dfrac{x-a}{x+b}\right)^3=\dfrac{x-2a-b}{x+a+2b}$. (2) $\dfrac{ax}{x-b}+\dfrac{bx}{x-a}=a+b$.

(3) $3\sqrt{x}-1=\dfrac{5}{3\sqrt{x}+7}+6$. (P. I. E. 1891).

7. Simplify the expressions

(1) $\dfrac{a^2x^m-b^2x^{m+4}}{a-bx^2}$. (2) $\left(\dfrac{x^3}{y}-\dfrac{y^3}{x}\right)\left(\dfrac{3x+y}{x+y}-\dfrac{3x-y}{x-y}\right)$.

(3) $\dfrac{1}{(a^2-b^2)(a^2-c^2)}+\dfrac{1}{(b^2-c^2)(b^2-a^2)}+\dfrac{1}{(c^2-a^2)(c^2-b^2)}$.

8. A man travels part of a journey on a bicycle, and then for the last 72 miles takes a train which travels four times as fast as he did on his bicycle and arrives at his destination in $3\frac{1}{2}$ hours from the start. If he had travelled the whole way in the train he would have saved $1\frac{1}{2}$ hours. Find the length of the journey in miles. (B. M. 1902).

Paper X.

1. If the coefficients of x^4 and of x in the product of $2x^3+3x^2+ax-10$ and $3x^3-ax^2-10x+4$ are equal to one another, find the value of a.

2. Find the G. C. M. of $4x^6-209x^3+15$ and $15x^6-209x^3+4$. (M. F. A. 1891).

3. Prove that if $x+\frac{1}{y}=a$, $y+\frac{1}{z}=b$, $z+\frac{1}{x}=c$, then

$$(1-bc)x+(1-ab)x^{-1}+2b=(1-ca)y+(1-bc)y^{-1}+2c$$
$$=(1-ab)z+(1-ca)z^{-1}+2a. \quad \text{(B. P. E. 1891).}$$

4. By addition or subtraction, after multiplying by a numerical factor if necessary, prove that the expressions $5x^4+2x^3+2x^2+2x+5$ and $2x^4-5x^3-5x^2-5x+2$ have no common factor in x. (B. P. E. 1895).

5. In the same diagram draw the graphs of

$$y=x+3,\ 2y-x=8, \text{ and } 2y+5x=20.$$

What do you deduce as to the roots of the different pairs of equations?

6. Solve the equations :—

(1) $\dfrac{(x+a)(x+b)}{(x+c)(x+d)}=\dfrac{x-c-d}{x-a-b}$. (A. E. 1900).

(2) $x+2y+3z=a$, $y+2z+3x=b$, $z+2x+3y=c$. (M. F. A. 1894).

7. If $\dfrac{a^2+b^2-c^2-d^2}{a-b+c-d}=\dfrac{a^2-b^2-c^2+d^2}{a+b+c+d}$, show that $a+b=c+d$ and

$$\frac{ac-bd}{a-b+c-d}=\frac{ad-bc}{a-b-c+d}. \quad \text{(M. F. A. 1892).}$$

8. The denominator of a certain fraction exceeds its numerator by one. Two other fractions are formed. one of them by adding 9 to the denominator, and the other by subtracting 6 from the numerator, of the original fraction. These two fractions are equal. Find the original fraction.

9 An old clock increased uniformly in value from *Rs.*45 in the year 1890, to *Rs.*85 in 1899. Find graphically its value in 1893, 1894 and 1897, to the nearest half-rupee.

10. **A**, **B**, **C**, **D** are four railway stations. From **B** to **C** is $2\frac{2}{3}$ miles more, and from **C** to **D** $5\frac{2}{7}$ miles less than from **A** to **B**. A train starts from **A** and travels at the rate of 14 miles an hour. At **B** an accident happens to the engine, which causes a delay of 6 hours. After this the train proceeds to **C** at half speed. There another delay of $\frac{1}{2}$ an hour occurs and then the train moves on to **D** at a speed further diminished by one mile an hour. A man starts from **A** at the same time as the train, and travels straight across country to **D**, a distance of 58 miles. Including stoppages he averages 3 miles an hour and reaches **D** just with the train. What is the distance by rail from **A** to **D**? (M. M. 1880).

CHAPTER XXI.

QUADRATIC EQUATIONS.

435. Equations in which the square of the unknown quantity, and no higher power, is found, are called **Quadratic Equations** or **equations of the second degree**.

436. Quadratic Equations are of two kinds :—

(i) **Pure Quadratics**, in which the square only of the unknown quantity is found, without the first power.

Thus, $x^2-9=0$, $3x^2=12$ are *Pure Quadratics*.

(ii) **Adfected Quadratics**, where the first power enters, as well as the square.

Thus, $x^2-3x+2=0$ is an *Adfected Quadratic*.

I. PURE QUADRATIC EQUATIONS.

437. **Pure Quadratics** are solved, as in simple equations, by collecting the unknown quantities on one side, and the known quantities on the other. We shall thus find the value of x^2, and thence the value of x (taking the square root), to which we must prefix the double sign ($\pm$). Such equations therefore will have two equal roots, with contrary signs.

Ex. 1. Solve the equation $x^2-9=0$.

Here, by transposition, $x^2=9$, $\therefore x=\pm 3$.

Note. If we had put $\pm x=\pm 3$, we should still have had only these two different values of x, *viz.* $x=+3$, $x=-3$; since $-x=+3$ gives $x=-3$ and $-x=-3$ gives $x=+3$.

Ex. 2. Solve $\frac{1}{8}(3x^2+5)-\frac{1}{3}(x^2+21)=39-5x^2$.

Multiplying by 24, the L. C. M. of the denrs. 8 and 3, we have
$3(3x^2+5)-8(x^2+21)=936-120x^2$, or $x^2-153=936-120x^2$;

By transposition, $x^2+120x^2=936+153$ or $121x^2=1089$.

$\therefore x^2=\frac{1089}{121}=9$ and $\therefore x=\pm 3$.

Ex. 3. Solve $\dfrac{5x^2+17}{x^2-11}+\dfrac{14x^2-117}{2x^2-9}=12.$

By division, $\left(5+\dfrac{72}{x^2-11}\right)+\left(7-\dfrac{54}{2x^2-9}\right)=12.$

$\therefore\ \dfrac{72}{x^2-11}-\dfrac{54}{2x^2-9}=0$, or $\dfrac{4}{x^2-11}=\dfrac{3}{2x^2-9}$ (dividing by 18).

$\therefore\ 4(2x^2-9)=3(x^2-11)$, or $8x^2-36=3x^2-33$;

$\therefore\ 5x^2=3$; $\therefore\ x^2=\frac{3}{5}$ and $\therefore\ x=\pm\sqrt{\frac{3}{5}}$.

Exercise CLIX.

Solve the following equations :—

1. $\frac{1}{2}x^2=14-3x^2.$

2. $x^2+5=\frac{10}{3}x^2-16.$

3. $(x-5)^2=25.$

4. $\dfrac{3}{4x^2}-\dfrac{1}{6x^2}=\dfrac{7}{3}.$

5. $8x+\dfrac{7}{x}=\dfrac{65x}{7}.$

6. $\dfrac{3}{1+x}+\dfrac{3}{1-x}=8$

7. $(2x-5)^2=x^2-20x+73.$

8. $x^2+7x=7(x+3)+4.$

9. $(2x-3)(x+1)=3(2x-1).$

10. $(x-2)(x-5)=10.$

11. $\dfrac{2x^2+10}{15}=7-\dfrac{50+x^2}{25}.$

12. $\dfrac{3x^2}{4}-\dfrac{15x^2+8}{6}=2x^2-3.$

13. $\dfrac{4x^2+5}{10}-\dfrac{2x^2-5}{15}=\dfrac{7x^2-25}{20}.$

14. $\dfrac{14x^2+16}{21}-\dfrac{2x^2+8}{8x^2-11}=\dfrac{2x^2}{3}.$

15. $\dfrac{4}{x-3}-\dfrac{4}{x+3}=\dfrac{1}{3}.$

16. $\dfrac{x+3}{x+2}+\dfrac{x-3}{x-2}=\dfrac{2x-3}{x-1}.$

17. $\dfrac{x+1}{x-1}+\dfrac{x+2}{x-2}=2\dfrac{x+3}{x-3}.$

18. $\dfrac{x+1}{x-1}+\dfrac{x+2}{x-2}=2\dfrac{11x+18}{11x-18}.$

19. $x(x-5)(x-9)=(x-6)(x^2-27).$

20. $(x+7)(x^2-4)=(x+1)(x^2+14x+22).$

21. $\dfrac{x+4}{x-4}-\dfrac{x-4}{x+4}=\dfrac{9+x}{9-x}-\dfrac{9-x}{9+x}.$

22. $\dfrac{1}{1-x}-\dfrac{1}{1+x}=\dfrac{3x}{1+x^2}.$

23. $\dfrac{3x+8}{x-4}-\dfrac{5(12-x)}{2x+3}=11.$

24. $\dfrac{3x+1}{4x+3}-\dfrac{x-2}{4x-3}=\dfrac{5}{9}.$

II. PURE QUADRATICS INVOLVING SURDS.

438. The following are illustrative examples.

Ex. 1. Solve $\frac{1}{x+\sqrt{(2-x^2)}}+\frac{1}{x-\sqrt{(2-x^2)}}=ax.$

Simplifying the first member, we get

$$\frac{2x}{x^2-(2-x^2)} \text{ or } \frac{x}{x^2-1}=ax. \quad \text{Hence, } x=0\,;$$

$$\text{and } \frac{1}{x^2-1}=a \text{ or } x^2-1=\frac{1}{a}\,; \quad \therefore x^2=1+\frac{1}{a}=\frac{a+1}{a}.$$

$$\therefore x=\pm\sqrt{\left(\frac{a+1}{a}\right)}.$$

Ex. 2. Solve $\frac{\sqrt{(a^2+x^2)}+x}{\sqrt{(a^2+x^2)}-x}=\frac{b}{c}.$

By *Comp. & Divd.* $\frac{\sqrt{(a^2+x^2)}}{x}=\frac{b+c}{b-c}.$

Squaring, $\frac{a^2+x^2}{x^2}$ or $\frac{a^2}{x^2}+1=\left(\frac{b+c}{b-c}\right)^2$; $\therefore \frac{a^2}{x^2}=\left(\frac{b+c}{b-c}\right)^2-1=\frac{4bc}{(b-c)^2}$;

$$\therefore \frac{x^2}{a^2}=\frac{(b-c)^2}{4bc}\,; \quad \therefore \frac{x}{a}=\pm\frac{b-c}{2\sqrt{(bc)}} \text{ and } \therefore x=\pm\frac{a(b-c)}{2\sqrt{(bc)}}.$$

Exercise CLX.

Solve the following equations :—

1. $2x^2+2x+1=\sqrt{(x^4+8x^3+8x^2+4x+49)}.$

2. $\sqrt{(\frac{1}{4}x+3)}-\sqrt{(\frac{1}{4}x-3)}=\sqrt{(\frac{2}{3}x)}.$

3. $\sqrt{\left(\frac{x+5}{x-5}\right)}+\sqrt{\left(\frac{x-5}{x+5}\right)}=\frac{4}{\sqrt{3}}.$ 4. $\frac{2ax}{1+\sqrt{(x^2+1)}}=b.$

5. $\sqrt{(3x^2+16)}+\sqrt{(3x^2-16)}=8+4\sqrt{2}.$

6. $\frac{x-10}{\sqrt{(x-5)}+\sqrt{5}}+\frac{x-15}{\sqrt{(x+5)}-2\sqrt{5}}=\frac{2x-25}{\sqrt{(2x-5)}-2\sqrt{5}}.$

$\frac{a+\sqrt{(a^2-x^2)}}{a-\sqrt{(a^2-x^2)}}=4.$ 8. $\frac{\sqrt{(x^2+1)}+\sqrt{(x^2-1)}}{\sqrt{(x^2+1)}-\sqrt{(x^2-1)}}=$

9. $\dfrac{x-\sqrt{(x^2-1)}}{x+\sqrt{(x^2-1)}}=\frac{4}{5}$. **10.** $\dfrac{a+x+\sqrt{(2ax+x^2)}}{a+x-\sqrt{(2ax+x^2)}}=b^2$.

11. $\dfrac{\sqrt{(2a^2-x^2)}+b\sqrt{(2a-x)}}{\sqrt{(2a^2-x^2)}-b\sqrt{(2a-x)}}=\dfrac{\sqrt{a}+b}{\sqrt{a}-b}$.

12. $\dfrac{\sqrt{(x^2+1)}+\sqrt{(x^2-1)}}{\sqrt{(x^2+1)}-\sqrt{(x^2-1)}}+\dfrac{\sqrt{(x^2+1)}-\sqrt{(x^2-1)}}{\sqrt{(x^2+1)}+\sqrt{(x^2-1)}}=4\sqrt{(x^2-1)}$.

13. $\sqrt{(x^2+2bx+a^2)}+\sqrt{(x^2-2bx+a^2)}=2\sqrt{(x^2-b^2)}$.

14. $\dfrac{5}{x+\sqrt{(3-x^2)}}+\dfrac{5}{x-\sqrt{(3-x^2)}}=\dfrac{2x}{3}$. **15.** $\dfrac{ax+\sqrt{(a^2x^2-1)}}{ax-\sqrt{(a^2x^2-1)}}=16$.

III. ADFECTED QUADRATIC EQUATIONS.

439. An **Adfected Quadratic** may always be reduced to the form, $x^2+px+q=0$, where the coefficient of x^2 is $+1$, and p, q represent numbers or known quantities.

Now, in this equation, we have $x^2+px=-q$, and, adding $(\frac{1}{2}p)^2$ to each side, we get $x^2+px+\frac{1}{4}p^2=\frac{1}{4}p^2-q$: (by this step, the first side becomes a *complete square*), and taking the square root of each side, prefixing, as before, the double sign to that of the latter, we have

$$x+\tfrac{1}{2}p=\pm\sqrt{(\tfrac{1}{4}p^2-q)}, \text{ and } \therefore\ x=-\tfrac{1}{2}p\pm\sqrt{(\tfrac{1}{4}p^2-q)},$$

which expression gives us, according as we take the upper or lower sign, two roots of the quadratic.

440. From the preceding we obtain the following RULE (called **Completing squares**) for the solution of an adfected quadratic :—

Reduce it to its simplest form; place the terms involving x^2 and x on one side, (the coefficient of x^2 being $+1$,) and the known quantity on the other. Then, if we **add the square of half the coefficient of x to each side,** *the first side will become a complete square; and taking the square root of each, prefixing the double sign to the second, we shall obtain, as before, the two roots of the equation.*

Ex. 1. Solve $x^2+7x=8$.

Here, $x^2+7x+(\frac{7}{2})^2=8+\frac{49}{4}=\frac{81}{4}$.

Ext. the square root, $x+\frac{7}{2}=\pm\frac{9}{2}$,

whence $x=-\frac{7}{2}\pm\frac{9}{2}=1$ or -8.

Ex. 2. Solve $7x^2-13x=2$.

Dividing by 7, the coefficient of x^2, we have $x^2-\frac{13}{7}x=\frac{2}{7}$.

Completing the square, $x^2-\frac{13}{7}x+(\frac{13}{14})^2=\frac{2}{7}+\frac{169}{196}=\frac{225}{196}$.

Extracting the square root, $x-\frac{13}{14}=\pm\frac{15}{14}$.

$$\therefore\ x=\tfrac{13}{14}\pm\tfrac{15}{14}=2 \text{ or } -\tfrac{1}{7}.$$

Verification. When $x=2$,

the left-hand side $=7(2)^2-13\times 2=28-26=2$

$=$ the right-hand side.

$\therefore$ 2 is a root.

When $x=-\frac{1}{7}$, the left-hand side $=7(-\frac{1}{7})^2-13\times-\frac{1}{7}$

$=\frac{1}{7}+\frac{13}{7}=\frac{14}{7}=2$

$=$ the right-hand side.

$\therefore$ $-\frac{1}{7}$ is also a root.

441. The *general* form of an adfected quadratic is

$$ax^2+bx+c=0,$$

where a, b, and c are any quantities whatever, positive or negative, integral or fractional.

We now proceed to solve the above equation.

By transposition, $ax^2+bx=-c$.

Dividing both sides by a, $x^2+\frac{b}{a}x=-\frac{c}{a}$.

Completing the square, $x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}=\frac{b^2-4ac}{4a^2}$.

Extracting the square root, $x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$.

$$\therefore\ x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

The above formula may be used for the solution of any quadratic equation.

Thus, in the equation $25x^2-7x-86=0$,

$a=25$, $b=-7$, $c=-86$;

$$\therefore\ x=\frac{+7\pm\sqrt{7^2-4\times 25\times(-86)}}{50}$$

$$=\frac{7\pm\sqrt{8649}}{50}=\frac{7\pm 93}{50}=2 \text{ or } -\tfrac{43}{25}.$$

442. The above formula may be simplified when the coefficient of x is an *even* number, as $2b$.

Thus, $x=\dfrac{-2b\pm\sqrt{(2b)^2-4ac}}{2a}=\dfrac{-b\pm\sqrt{b^2-ac}}{a}$.

Exercise CLXI.

Solve the following equations :—

1. $x^2-10x+16=0$.	**2.** $x^2+6x+4=59$.	**3.** $x^2-x=210$.
4. $x^2-14=13x$.	**5.** $x^2-x=42$.	**6.** $x^2+4x=140$.
7. $x^2+19x=20$.	**8.** $x^2+32x=320$.	**9.** $x^2+13x=-12$.
10. $7x^2-26x=1008$.	**11.** $2x^2-3x=54$.	**12.** $13x^2-22x=1911$.
13. $x^2+111x=3400$.	**14.** $12x^2+x=1740$.	**15.** $3x^2-x=102$.
16. $x^2+3a^2=4ax$.	**17.** $x^2-3ax+2a^2=0$.	**18.** $x^2+2ab=b^2+2ax$

443. Solution by factorization. *Transpose all the terms to the left-hand side, and resolve into linear factors. Equate either of the factors to zero.*

Ex. 1. Solve $x^2+4x=5$.

Transposing all the terms to the left-hand side, we have

$$x^2+4x-5=0.$$

factorizing, $(x+5)(x-1)=0$.

Hence, $x+5=0$, $\therefore x=-5$ or $x-1=0$, $\therefore x=1$.

Ex. 2. Solve $2x^2-11x+12=0$.

Here, factorizing, $(x-4)(2x-3)=0$.

Hence, $x-4=0$, $\therefore x=4$ or $2x-3=0$, $\therefore x=\frac{3}{2}$.

Verification. When $x=4$,

$$2x^2-11x+12=2\times16-11\times4+12=32-44+12=0.$$

$\therefore$ 4 is a root of the equation.

When $x=\frac{3}{2}$, $2x^2-11x+12=2\times\frac{9}{4}-11\times\frac{3}{2}+12$

$$=\tfrac{9}{2}-\tfrac{33}{2}+12=-12+12=0.$$

$\therefore$ $\frac{3}{2}$ is also a root.

444. It is important to observe *that if $x-a$ is a factor of both sides of an equation, a is a root of the equation.*

Ex. 3. Solve $6x(4x+5)+7(4x+5)=0$.

$4x+5$ is a common factor ; $\therefore 4x+5=0$ gives a root.

whence $x=-\frac{5}{4}$.

Again, dividing by $4x+5$, we have left

$6x+7=0$; whence $x=-\frac{7}{6}$.

Hence the required roots are $-\frac{5}{4}$ and $-\frac{7}{6}$.

Exercise CLXII.

Solve the following equations :—

1. $2x^2-5x+2=0$. 2. $7x^2-3x=160$. 3. $x^2=4(x+8)$.
4. $2(5x-12)=x^2$. 5. $3x^2+10x=57$. 6. $5x^2=370-13x$.
7. $x^2-13x=68$. 8. $1+2x^2=3x$. 9. $x^2-4x=4(x-4)$.
10. $5x(2x-3)+7(2x-3)=0$. 11. $x^2+4{\cdot}8x+2{\cdot}87=0$.
12. $4x^2+13x=12$. 13. $acx^2+(bc-ab)x-b^2=0$.
14. $(a^2-b^2)x^2-2ax+1=0$. 15. $abx^2-(a^2+b^2)x+ab=0$.
16. $x^2+2(b-c)x+c^2=2bc$. 17. $abx^2-(a+b)cx+c^2=0$.

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445. Sridhar Acharja's or Hindu method. An equation of the general form $ax^2+bx+c=0$, may, however, be solved as follows, without dividing by the coefficient of x^2.

Transposing, we have $ax^2+bx=-c$.

Multiply every term by $4a$ and add b^2 to each side ;

then $4a^2x^2+4abx+b^2=b^2-4ac$;

Taking the square root, $2ax+b=\pm\sqrt{b^2-4ac}$

$\therefore\ 2ax=-b\pm\sqrt{b^2-4ac}$, and $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Ex. 1. Solve $2x^2+3x+1=0$.

By transposition, $2x^2+3x=-1$.

Multiplying both sides by 4×2 or 8 and adding the square of 3 or 9 to both sides, we have

$$16x^2+24x+9=9-8=1.$$

Extracting the square root, $4x+3=\pm 1$.

$\therefore\ 4x=-3\pm 1=-2$ or -4 ; $\therefore\ x=-\frac{1}{2}$ or -1.

Ex. 2. Solve $3x^2+14x+3=0$.

By transposition, $3x^2+14x=-3$.

Multiplying both sides by 3 (for here the coefficient of the second term is a multiple of 2), and adding the square of $\frac{14}{2}$ or 7, *i. e.* 49 to both sides, we have

$$9x^2+42x+49=49-9=40.$$

Extracting the square root, $3x+7=\pm 2\sqrt{(10)}$.

$\therefore\ 3x=-7\pm 2\sqrt{(10)}$ and $\therefore\ x=\frac{1}{3}(-7\pm 2\sqrt{10})$.

446. When the exact values of the roots of an equation cannot be found (as above), we may approximate them, to any degree of accuracy.

Ex. 3. Solve the equation $5x^2-9x-4=0$.

We have $x=\dfrac{9\pm\sqrt{81+4\times5\times4}}{10}$, Art. 441.

$=\dfrac{9\pm\sqrt{161}}{10}=\dfrac{9\pm12{\cdot}688}{10}$, approximately

$=\dfrac{21{\cdot}688}{10}$ or $-\dfrac{3{\cdot}688}{10}$,

$=2{\cdot}17$ or $-{\cdot}37$, correct to two decimal places.

Exercise CLXIII.

Solve the following equations :—

1. $9x^2-24x+16=0$. **2.** $21x^2-17x+2=0$. **3.** $-3x^2=2(x-4)$.

4. $5x^2+14x=55$. **5.** $3x^2-11x-20=0$. **6.** $8x^2-9x+1=0$.

7. $141x^2-88x-45=0$. (M. M. 1895).

8. $129x^2-34x-80=0$. (M. M. 1896).

9. $150x^2=299x+2$. **10.** $9x^2=18x+16$. **11.** $70x^2=35+x$.

12. $7x^2+32x=15$. **13.** $25x^2-7x=86$. **14.** $11x=3(2x^2+1)$.

15. $3(x-2)^2=8(x+2)+3$. **16.** $(x+10)^2=144(100-x^2)$.

17. $(x-1\frac{2}{3})(x-2\frac{1}{2})=\frac{1}{2}(x-1)(1+\frac{1}{3}x)$. **18.** $(5x-3)(3x+1)=1$.

19. $x^4+12x^3+3x^2+7x+1=(x^2+7x+4)(x^2+5x-4)$.

20. $(x+1)(3x-1)=28(x+1)-16(3x-1)$.

21. $x^2-ax+\frac{1}{2}ab+\frac{1}{2}b(x-b)=0$. **22.** $2(x-2a)^2=(3x-2b)(3a-b)$.

23. $a(b-c)x^2+b(c-a)x+c(a-b)=0$. **24.** $x^2-2ax+a^2-b=0$.

25. $4(a^2+b^2+x^2)+17ab=10(a+b)x$.

26. $(a^2-b^2)(x^2-1)=4abx$. **27.** $(b^2-a^2)(x^2+1)=2(a^2+b^2)x$.

Solve the following equations, giving the values of the roots *correct to two places of decimals* :—

28. $x^2=2(1-x)$. **29.** $8x^2-24x=5$. **30.** $3x^2-4x=156$.

31. $118x-2\frac{1}{2}x^2=20$ **32.** $(x+1)(x-4)+(x+2)(x-3)=0$.

IV. ADFECTED QUADRATICS INVOLVING FRACTIONS.

447. If the given equation involve fractions or brackets, they should first be cleared away, as in the following Examples.

Ex. 1. Solve the equation $\frac{3}{4}(x^2-3)=\frac{1}{8}(x-3)$.

Clearing of fractions, we have $6(x^2-3)=x-3$.

$\therefore\ 6x^2-x-15=0$; $\therefore\ (3x-5)(2x+3)=0$, (factorizing),

$\therefore\ 3x-5=0$, which gives $x=\frac{5}{3}$ or $2x+3=0$, which gives $x=-\frac{3}{2}$.

Verification. When $x=\frac{5}{3}$, the left-hand side $=\frac{3}{4}(\frac{25}{9}-3)=-\frac{1}{6}$.

...... the right-hand side $=\frac{1}{8}(\frac{5}{3}-3)=-\frac{1}{6}$.

$\therefore\ \frac{5}{3}$ is a root.

When $x=-\frac{3}{2}$, the left-hand side $=\frac{3}{4}(\frac{9}{4}-3)=-\frac{9}{16}$.

...... the right-hand side $=\frac{1}{8}(-\frac{3}{2}-3)=-\frac{9}{16}$.

$\therefore\ -\frac{3}{2}$ is also a root.

Ex. 2. Solve $\frac{7}{4}-\frac{2x-5}{x+5}=\frac{3x-7}{2x}$.

Multiplying every term by $4x$, we have

$$7x-\frac{8x^2-20x}{x+5}=6x-14. \quad \therefore\ x+14=\frac{8x^2-20x}{x+5}.$$

Multiplying crosswise, $(x+14)(x+5)=8x^2-20x$.

$\therefore\ x^2+19x+70=8x^2-20x$, or $7x^2-39x=70$.

$\therefore\ x^2-\frac{39}{7}x=10$; $\therefore\ x^2-\frac{39}{7}x+(\frac{39}{14})^2=10+\frac{1521}{196}=\frac{3481}{196}$.

Ext. the sq. root, $x-\frac{39}{14}=\pm\frac{59}{14}$; $\therefore\ x=\frac{39}{14}\pm\frac{59}{14}=7$ or $-1\frac{3}{7}$.

Ex. 3. Solve $\frac{2x-3}{3x-5}+\frac{3x-5}{2x-3}=\frac{5}{2}$.

Clearing of fractions, we have

$$2\{(2x-3)^2+(3x-5)^2\}=5(3x-5)(2x-3),$$

or $26x^2-84x+68=30x^2-95x+75$, or $4x^2-11x=-7$.

$\therefore\ x^2-\frac{11}{4}x=-\frac{7}{4}$; $\therefore\ x^2-\frac{11}{4}x+(\frac{11}{8})^2=(\frac{11}{8})^2-\frac{7}{4}=\frac{9}{64}$.

Ext. the sq. root $x-\frac{11}{8}=\pm\frac{3}{8}$; $\therefore\ x=\frac{11}{8}\pm\frac{3}{8}=1\frac{3}{4}$ or 1.

Otherwise thus: Let $\dfrac{2x-3}{3x-5}=y$, then $\dfrac{3x-5}{2x-3}=\dfrac{1}{y}$,

and the equation becomes $y+\dfrac{1}{y}=\dfrac{5}{2}$.

$\therefore\ 2y^2-5y+2=0$, or $(2y-1)(y-2)=0$; $\therefore\ y=2$ or $\frac{1}{2}$.

(i) When $\dfrac{2x-3}{3x-5}=y=2$,
then $2x-3=6x-10$, or $4x=7$,
$\therefore\ x=\frac{7}{4}=1\frac{3}{4}$.

(ii) When $\dfrac{2x-3}{3x-5}=y=\frac{1}{2}$,
then $4x-6=3x-5$,
$x=1$.

Exercise CLXIV.

Solve the following equations :—

1. $x=\frac{5}{3}+\frac{1}{12}x^2$.
2. $\frac{7}{11}x^2-\frac{2}{3}x=\frac{1}{33}(11x+18)$.
3. $\frac{1}{4}x^2=3x-8$.
4. $\dfrac{2x+11}{x}=5-\dfrac{x-5}{3}$.
5. $\dfrac{x}{2}+\dfrac{3}{x}=x-\frac{5}{2}$.
6. $x+\dfrac{24}{x-1}=3x-4$.
7. $\dfrac{x+22}{3}=\dfrac{4}{x}+\dfrac{9x-6}{2}$.
8. $\dfrac{40}{x-5}+\dfrac{27}{x}=13$.
9. $x-\dfrac{x^3-8}{x^2+5}=2$.
10. $\dfrac{4x+7}{19}+\dfrac{5-x}{3+x}=\dfrac{4x}{9}$.
11. $\dfrac{x+4}{x-3}-\dfrac{2x-3}{x+4}=7\frac{3}{8}$.
12. $\dfrac{2x+1}{x-1}+\dfrac{3x-2}{3x+2}=\frac{41}{12}$.
13. $\dfrac{4}{x+2}+\dfrac{5}{x+4}=\dfrac{12}{x+6}$.
14. $\dfrac{1}{x+2}+\dfrac{1}{x+4}=\dfrac{5}{x+10}$.
15. $\dfrac{1}{x+2}-\dfrac{1}{x+7}=\dfrac{1}{3x+1}$.
16. $\dfrac{12}{5-x}+\dfrac{4}{4-x}=\dfrac{32}{x+2}$.
17. $\dfrac{x+4}{3}-\dfrac{7-x}{x-3}=\dfrac{4x+7}{9}-1$.
18. $\dfrac{1}{x-1}-\dfrac{1}{x+3}=\frac{1}{35}$.
19. $\dfrac{5x}{x+4}\cdot\dfrac{3x-2}{2x-3}=2$.
20. $\dfrac{48}{x+3}=\dfrac{165}{x+10}-5$.
21. $\dfrac{3x-7}{x}+\dfrac{4x-10}{x+5}=3\frac{1}{2}$.
22. $\dfrac{2x}{x-4}+\dfrac{2x-5}{x-3}=8\frac{1}{3}$.
23. $\dfrac{2x+9}{9}+\dfrac{4x-3}{4x+3}=3+\dfrac{3x-16}{18}$.
24. $\dfrac{x^4+2x^3+8}{x^2+x-6}=x^2+x+8$.
25. $\dfrac{5x+36}{10x-81}+\frac{13}{25}=\dfrac{8x}{5x-8}$.
26. $\dfrac{2x-1}{3x-1}+3\,\dfrac{3x-1}{2x-1}=\dfrac{13}{2}$.

27. $\dfrac{x+5}{x+6}+\dfrac{x+6}{x+7}=2\left(\dfrac{x+8}{x+9}\right)$. 28. $\dfrac{7+x}{7-x}+\dfrac{7-x}{7+x}=\frac{29}{10}$.

29. $\left(\dfrac{8x-3}{4x-1}\right)^2=\dfrac{4x-5}{x-1}$. 30. $\dfrac{x+2}{x-2}+\dfrac{x-2}{x+2}=5\frac{1}{5}$.

31. $\dfrac{x+4}{x+6}+\dfrac{5}{2x+4}=\dfrac{3x+7}{3x+4}$. 32. $\dfrac{x+16}{5}+\dfrac{11}{x}=\dfrac{4x-17\frac{1}{3}}{3}$. (M.M. 1892).

33. $\dfrac{2+3x}{1-4x}-\dfrac{6-5x}{7x-25}=\dfrac{16-x}{28x-193}$. 34. $\dfrac{3}{5-x}+\dfrac{2}{4-x}=\dfrac{8}{x+2}$ (M.M. 1893).

35. $\dfrac{x-1}{x-2}-\dfrac{x-3}{x-4}=2\left(\dfrac{x-2}{x-3}-\dfrac{x-5}{x-6}\right)$.

36. $\dfrac{4}{2x+3}-\dfrac{18}{7x+12}=5$ (M. M. 1897). 37. $\dfrac{x-3}{x-2}+\dfrac{x-4}{x-1}+\frac{1}{4}=0$.

(M. M. 1898)

38. $\dfrac{x^2+3}{x-1}+\dfrac{x^2-x+1}{x-2}=2.\dfrac{x^2-2x+1}{x-3}$. 39. $\dfrac{x+1}{x-1}+\dfrac{x+2}{x-2}=\dfrac{2x+13}{x+1}$.

(M. M. 1899).

40. $\dfrac{3x-1}{3x-9}+\dfrac{x+1}{x-7}=\dfrac{x-10}{x^2-10x+21}$. (M. M. 1899).

Solve the following equations, (giving the values of x *correct to two places of decimals*) :—

41. $\dfrac{x+1}{x+2}+\dfrac{x-3}{x-4}=0$. 42. $2(x-1)=\dfrac{4-5x}{x+1}$ 43. $\dfrac{3x+1}{3x-1}-\dfrac{3x-1}{3x+1}=2$.

44. $\dfrac{1}{x+3}+\dfrac{1}{x+6}+\dfrac{1}{x+9}=0$. 45. $\dfrac{x-1}{x^2+3x+2}+\dfrac{x-3}{x^2+5x+6}=\dfrac{1}{x+2}$.

448. Literal Equations. The following are typical examples vith their solutions.

Ex. 1. Solve the equation $\dfrac{x}{a}+\dfrac{a}{x}=\dfrac{b}{a}+\dfrac{a}{b}$.

By transposition, $\dfrac{x-b}{a}=a\left(\dfrac{1}{b}-\dfrac{1}{x}\right)=a.\dfrac{x-b}{bx}$.

Hence, $x-b$ (being a common factor)$=0$; $\therefore x=b$,

and dividing by $x-b$, we have $\dfrac{1}{a}=\dfrac{a}{bx}$.

$\therefore bx=a^2$, and $\therefore x=a^2/b$.

Ex. 2. Solve $\frac{a^2}{x-b}+\frac{b^2}{x-a}=a+b$.

Clearing of fractions, we have

$$a^2(x-a)+b^2(x-b)=(a+b)(x-a)(x-b),$$

or $(a^2+b^2)x-a^3-b^3=(a+b)x^2-(a+b)^2x+ab(a+b)$.

Transposing, $(a+b)x^2-2(a^2+ab+b^2)x+(a+b)(a^2+b^2)=0$.

Factorizing, $\{(a+b)x-(a^2+b^2)\}\{x-(a+b)\}=0$.

$\therefore\ x-(a+b)=0$, and $\therefore\ x=a+b$.

Also $(a+b)x-(a^2+b^2)=0$, and $\therefore\ x=\frac{a^2+b^2}{a+b}$.

Exercise CLXV.

Solve the following equations :—

1. $\frac{x}{a}+\frac{a}{x}=\frac{x}{b}+\frac{b}{x}$.
2. $\frac{b}{x-c}+\frac{c}{x-b}=2$.
3. $\frac{b+c}{x-a}+\frac{c+a}{x-b}=2$.
4. $\frac{1}{x-a}+\frac{1}{x-b}=\frac{1}{a}+\frac{1}{b}$.
5. $\frac{1}{x-a-b}=\frac{1}{x}-\frac{1}{a}-\frac{1}{b}$.
6. $\frac{a}{x-a}+\frac{b}{x-b}=\frac{a}{b}+\frac{b}{a}$.
7. $cx+\frac{ac}{a+b}=(a+b)x^2$.
8. $\left(\frac{x-a}{x+a}\right)^2-5\left(\frac{x-a}{x+a}\right)+6=0$.
9. $\frac{x+a}{x+b}-\frac{x+b}{x+a}=(a-b)^2$.
10. $2\frac{a-bx}{ax-b}=\frac{a+bx}{ax+b}+1$.
11. $\frac{x}{x+a}+\frac{x}{x+b}=\frac{c}{c+a}+\frac{c}{c+b}$
12. $\frac{x+a}{x-a}+\frac{x+b}{x-b}-\frac{x-a}{x+a}-\frac{x-b}{x+b}=0$.

V. ADFECTED QUADRATICS INVOLVING SURDS.

449. The following are illustrative examples.

Ex. 1. Solve $\sqrt{(x+3)}+\sqrt{(x+8)}=5\sqrt{x}$.

Squaring, $x+3+x+8+2\sqrt{(x+3)(x+8)}=25x$.

Transposing, $2\sqrt{(x+3)(x+8)}=23x-11$.

Squaring again, $4x^2+44x+96=529x^2-506x+121$,

$\therefore\ 525x^2-550x+25=0$, or $21x^2-22x+1=0$.

Factorizing, $(x-1)(21x-1)=0$; $\therefore x=1$ or $\frac{1}{21}$.

Ex. 2. Solve $\sqrt{(5a+x)}+\sqrt{(5a-x)}=\dfrac{12a}{\sqrt{(5a+x)}}$.

Multiplying every term by $\sqrt{(5a+x)}$, we have

$$5a+x+\sqrt{(25a^2-x^2)}=12a, \text{ or } \sqrt{(25a^2-x^2)}=7a-x.$$

Squaring, $25a^2-x^2=49a^2-14ax+x^2$,

$\therefore\ 2x^2-14ax=-24a^2$, or $x^2-7ax+12a^2=0$.

Factorizing, $(x-3a)(x-4a)=0$; $\therefore\ x=3a$ or $4a$.

Ex. 3. Solve $(19+8\sqrt{6})x^2+2(17+6\sqrt{6})x=53+20\sqrt{6}$.

Multiplying both sides by the coefficient of x^2 (for the coefficient of x is even), we have

$$(19+8\sqrt{6})^2x^2+2(17+6\sqrt{6})(19+8\sqrt{6})x$$
$$=(53+20\sqrt{6})(19+8\sqrt{6}).$$

Add the square of $17+6\sqrt{6}$ to both sides; then

$$(19+8\sqrt{6})^2x^2+2(17+6\sqrt{6})(19+8\sqrt{6})x+(17+6\sqrt{6})^2$$
$$=(53+20\sqrt{6})(19+8\sqrt{6})+(17+6\sqrt{6})^2=4(618+252\sqrt{6}).$$

Extracting the square root of both sides, we have

$$(19+8\sqrt{6})x+(17+6\sqrt{6})=\pm 2(18+7\sqrt{6}).$$
$$\therefore (19+8\sqrt{6})x=-(17+6\sqrt{6})\pm 2(18+7\sqrt{6}),$$
$$=19+8\sqrt{6} \text{ or } -(53+20\sqrt{6}).$$
$$\therefore\ x=1 \text{ or } -\frac{53+20\sqrt{6}}{19+8\sqrt{6}}.$$

Exercise CLXVI.

Solve the following equations :—

1. $x+\sqrt{(x-2)}=1\frac{3}{4}$.
2. $\sqrt{(2x+3)}\times\sqrt{(3x+7)}=12$.
3. $3\sqrt{(112-8x)}=19+\sqrt{(3x+7)}$.
4. $\sqrt{(x+3)}+\sqrt{(x+6)}=3\sqrt{x}$.
5. $\sqrt{(3x-3)}+\sqrt{(5x-19)}=\sqrt{(2x+8)}$.
6. $\sqrt{(2x+1)}+\sqrt{(7x-27)}=\sqrt{(3x+4)}$.
7. $2\sqrt{(4x+5)}-\sqrt{(8x-4)}=\sqrt{(2x+11)}$.

8. $7\sqrt{(\frac{8}{2}x-5)}-\sqrt{(\frac{1}{5}x+45)}=\frac{7}{4}\sqrt{(10x+56)}$.

9. $2\sqrt{x}+\sqrt{4x+\sqrt{(7x+2)}}=1$. **10.** $\sqrt{(x+4)}+2\sqrt{(x-8)}=$.

11. $\dfrac{\sqrt{x}+9}{\sqrt{x}}=\dfrac{3\sqrt{x}-3\frac{4}{5}}{9-\sqrt{x}}$. **12.** $\dfrac{\sqrt{(12-x)}}{5}=\dfrac{3}{2+\sqrt{(12-x)}}$

13. $\dfrac{123+41\sqrt{x}}{5\sqrt{x}-x}=\dfrac{20\sqrt{x}+4x}{3-\sqrt{x}}-\dfrac{2x^2}{(5\sqrt{x}-x)(3-\sqrt{x})}$.

14. $(4+2\sqrt{3})x^2+(\sqrt{3}+1)x=2$. **15.** $(3+2\sqrt{2})x^2+(\sqrt{2}+1)x=$

16. $(6-2\sqrt{5})x^2+\frac{5}{2}(\sqrt{5}-1)x+1=0$.

17. $\sqrt{(x^2-8x+15)}+\sqrt{(x^2+2x-15)}=\sqrt{(4x^2-18x+18)}$.

18. $(x-c)\sqrt{ab}-(a-b)\sqrt{cx}=0$. **19.** $(a^2-b)x^2+2\sqrt{b}.x-1=0$.

20. $a+x+\sqrt{(2ax+x^2)}=\sqrt{(ax-x^2)}+\sqrt{(2a^2-ax-x^2)}$.

21. $b-x+\sqrt{(3b^2-2bx-x^2)}=\sqrt{(3b^2+4bx+x^2)}-\sqrt{(b^2-x^2)}$.

22. $\sqrt{\left(\dfrac{x+a}{x-2a}\right)}+\sqrt{\left(\dfrac{x-2a}{x+a}\right)}=2\frac{1}{2}$.

450. Nature of the roots of a Quadratic Equation.

From the general form of the quadratic $ax^2+bx+c=0$,

we have $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ (Art. 441.)

1. Hence four cases may arise as to the nature of the roots.

(i) When $b^2=4ac$, the quantity within the radical sign is *zero*, and the roots become $\dfrac{-b\pm 0}{2a}$, *i.e.*, $-\dfrac{b}{2a}$, $-\dfrac{b}{2a}$.

Hence, in this case, the equation has a **pair** of equal roots.

(ii) When b^2-4ac is a *positive* quantity and a *perfect square*, the roots are **real, unequal** and **rational.**

(iii) When b^2-4ac is a *positive* quantity, but *not a perfect square*, the roots are **real, unequal** and **irrational.**

(iv). When b^2-4ac is a *negative* quantity, the roots are **imaginary.**

Hence, if any equation be expressed in the form $ax^2+bx+c=0$, the roots will be **real and different, real and equal,** or **impossible,** according as $b^2 >, =,$ or $< 4ac$.

So also, in the form $x^2+px+q=0$, the roots will be **real and different, real and equal**, or **impossible**, according as $p^2 >, =$, or $< 4q$.

2. Also it is clear that if $b=0$, *the roots are equal in magnitude but opposite in sign.*

VI. EQUATIONS SOLVED LIKE QUADRATICS.

451. Many equations, though not actually quadratics themselves, may be put into the form of quadratics, and thus solved.

452. Equations reducible to the form $ax^{2n}+bx^n+c=0$, where n is either positive or negative, integral or fractional.

Assume $x^n=y$, then $x^{2n}=y^2$ and the equation becomes

$$ay^2+by+c=0\text{ ; whence } y \text{ and } \therefore x.$$

Ex. 1. Solve $x^4-26x^2+25=0$.

Let $x^2=y$, then $y^2-26y+25=0$, or $(y-25)(y-1)=0$.

$\therefore y-25=0$ or $y-1=0$, $\therefore y=25$ or 1.

Hence $x=\sqrt{y}=\pm 5$ or ± 1.

Ex. 2. Solve $\sqrt[4]{x}+7\sqrt{x}=116$.

Let $\sqrt[4]{x}=y$, then $\sqrt{x}=y^2$, and we have

$$y+7y^2=116 \text{ or } 7y^2+y-116=0.$$

$\therefore (y-4)(7y+29)=0$; whence $y=4$ or $-\frac{29}{7}$.

Hence $x=y^4=4^4$ or $(-\frac{29}{7})^4=256$ or $294\frac{1387}{2401}$.

Ex. 3. Solve $x^{-4}-9x^{-2}+20=0$.

Let $x^{-2}=y$, then $y^2-9y+20=0$, or $(y-4)(y-5)=0$.

$\therefore y=4$ or 5. Hence x^{-2} or $1/x^2=4$ or 5.

$$\therefore x^2=\tfrac{1}{4} \text{ or } \tfrac{1}{5} \text{ and } \therefore x=\pm\tfrac{1}{2} \text{ or } \pm 1/\sqrt{5}.$$

Exercise CLXVII.

Solve the following equations :—

1. $x^4-25x^2+144=0$. **2.** $x^4-5x^2+4=0$. **3.** $x^6-6x^3=16$.

4. $x^{-2}-2x^{-1}=8$. **5.** $3x+2\sqrt{x}-1=0$. **6.** $x-5\sqrt{x}-14=0$.

7. $3\sqrt{x}=x-10.$ **8.** $x^{-4}-4x^{-2}-5=0.$ **9.** $2x^{\frac{1}{2}}+2x^{-\frac{1}{2}}=5.$

10. $3x^{\frac{1}{4}}-5x^{\frac{1}{2}}+14=0.$ **11.** $x^{-\frac{3}{2}}-x^{-3}=-6.$ **12.** $x^{\frac{4}{3}}+7x^{\frac{2}{3}}=44.$

13. $x^{\frac{6}{5}}+x^{\frac{3}{5}}=756.$ **14.** $x^{\frac{5}{2}}-x=56x^{-\frac{1}{2}}.$ **15.** $x^{-\frac{3}{4}}+27x^{\frac{3}{4}}=28.$

16. $(x^2-9)^2=3+11(x^2-2).$ **17.** $(x^2-1)(x^2-2)+(x^2-3)(x^2-4)=x^4+5.$

18. $4x^{\frac{2}{3}}(x^{\frac{2}{3}}-2)=9x^{\frac{2}{3}}-4.$ **19.** $(x^{-\frac{1}{3}}+2)(x^{-\frac{1}{3}}+5)=x^{-1}+8.$

20. $x^{-3}+x^{-\frac{3}{2}}=2.$ **21.** $(x+1)^3-x^2(x^2-1)=(x-1)^3+2(x^2+3).$

453. Any equation which can be put into the form

$$\mathrm{A}(ax^{2n}+bx^n+c)^2+\mathrm{B}(ax^{2n}+bx^n+c)+\mathrm{C}=0,$$

may be treated as a quadratic, of which the roots are the values of $ax^{2n}+bx^n+c$; whence x may be found.

Ex. 1. Solve $x^2+6\sqrt{(x^2-2x+5)}=11+2x.$

Observing the quantity under the radical sign, we may put the equation in the form,

$$(x^2-2x+5)+6\sqrt{(x^2-2x+5)}=11+5=16.$$

Assume $\sqrt{(x^2-2x+5)}=y$, then $y^2+6y=16$;

$$\therefore y^2+6y-16=0, \text{ or } (y-2)(y+8)=0; \therefore y=2 \text{ or } -8.$$

(i) Let $x^2-2x+5=y^2=4$
$\therefore x^2-2x+1=0$
$\therefore (x-1)^2=0$ and $x=1, 1.$

(ii) Let $x^2-2x+5=y^2=64,$
$\therefore x^2-2x+1=60,$
$\therefore x-1=\pm2\sqrt{15}$, & $x=1\pm2\sqrt{15}.$

Ex. 2. Solve $9x^4+42x^3+55x^2+14x=728.$

Here, completing the square with the first two terms,

$$(9x^4+42x^3+49x^2)+6x^2+14x=728,$$

or $(3x^2+7x)^2+2(3x^2+7x)-728=0.$

Assume $3x^2+7x=y$, then $y^2+2y-728=0$;

$$\therefore (y+28)(y-26)=0; \therefore y=-28 \text{ or } 26.$$

(i) Let $3x^2+7x=y=26$
then $3x^2+7x-26=0$
$\therefore (x-2)(3x+13)=0$
$\therefore x=2$ or $-4\frac{1}{3}.$

(ii) Let $3x^2+7x=y=-28.$
$\therefore 3x^2+7x+28=0,$
$\therefore x=\dfrac{-7\pm\sqrt{(-287)}}{6}.$

Ex. 3. Solve $(1+x)^{\frac{2}{5}}+\frac{3}{16}(1-x)^{\frac{2}{5}}=(1-x^2)^{\frac{1}{5}}$.

Dividing both sides by $(1-x)^{\frac{2}{5}}$, we have

$$\left(\frac{1+x}{1-x}\right)^{\frac{2}{5}}+\frac{3}{16}=\left(\frac{1+x}{1-x}\right)^{\frac{1}{5}}, \text{ or } \left(\frac{1+x}{1-x}\right)^{\frac{2}{5}}-\left(\frac{1+x}{1-x}\right)^{\frac{1}{5}}+\frac{3}{16}=0.$$

Assume $\left(\frac{1+x}{1-x}\right)^{\frac{1}{5}}=y$, then $y^2-y+\frac{3}{16}=0$; $\therefore\ y=\frac{3}{4}$ or $\frac{1}{4}$.

(i) Let $\frac{1+x}{1-x}=y^5=(\frac{3}{4})^5=\frac{243}{1024}$.

By *Comp. & Divd.* $\frac{1}{x}=\frac{1267}{-781}$,

$\therefore\ x=-\frac{781}{1267}$.

(ii) Let $\frac{1+x}{1-x}=y^5=(\frac{1}{4})^5=\frac{1}{1024}$.

By *Comp. & Divd.* $\frac{1}{x}=\frac{1025}{-1023}$,

$\therefore\ x=-\frac{1023}{1025}$.

Exercise CLXVIII.

Solve the following equations :—

1. $\sqrt[4]{(x+13)}+5\sqrt{(x+13)}=22$.

2. $\sqrt[4]{(x+7)}+\frac{3}{4}\sqrt{(x+7)}=5$.

3. $5x-7x^2-8\sqrt{(7x^2-5x+1)}=8$.

4. $x^2-3x+7\sqrt{(11x-2x^2+2)}=\frac{5}{2}x+21$.

5. $2x^2-2x+2\sqrt{(2x^2-7x+6)}=5x-6$.

6. $3x(3-x)=11-4\sqrt{(x^2-3x+5)}$.

7. $x^2-x+5\sqrt{(2x^2-5x+6)}=\frac{3}{2}(x+11)$.

8. $9x-4x^2+\sqrt{(4x^2-9x+11)}=5$.

9. $x^2+x^{-2}+x+x^{-1}=4$.

10. $x^2+x^{-2}+x-x^{-1}=2$.

11. $15x-3x^2+4\sqrt{(x^2-5x+5)}=16$.

12. $\sqrt[3]{(1+x)^2}-\sqrt[3]{(1-x)^2}=3\sqrt[3]{(1-x^2)}$.

13. $x^2-2x\sqrt{x}+2x-\sqrt{x}=6$.

14. $x^4-2x^3+x^2=36$.

15. $\dfrac{\sqrt{(x^2+x+6)}}{3}=\dfrac{18-\{\frac{4}{3}\sqrt{(x^2+x+6)}-2\}}{\sqrt{(x^2+x+6)}}$.

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454. Consider the equation

$$(x+p)(x+q)(x+r)(x+s)=k,$$

where p, q, r, s are quantities such that the sum of *two* of them is equal to that of the *other two*, *i.e.*, $p+r=q+s$.

We have $(x+p)(x+r)\times(x+q)(x+s)=k$,

$\therefore\ \{x^2+(p+r)x+pr\}\times\{x^2+(q+s)x+qs\}=k$,

or $(x^2+mx+pr)\times(x^2+mx+qs)=k$, for $p+r=q+s=m$, suppose.

$\therefore\ (x^2+mx)^2+(pr+qs)(x^2+mx)+pqrs=k$.

Assuming $x^2+mx=y$, the equation becomes

$$y^2+(pr+qs)y+prqs=k, \text{ or } y^2+(pr+qs)y+prqs-k=0,$$

which is a quadratic in y. Whence x can be found.

Ex. 1. Solve $(x+1)(x+2)(x+3)(x+4)=24$.

Here, $\quad (x+1)(x+4)\times(x+2)(x+3)=24$,

or $\quad (x^2+5x+4)\times(x^2+5x+6)=24$.

Assume $x^2+5x=y$, then $(y+4)(y+6)=24$.

$\therefore\ y^2+10y+24=24$, or $y(y+10)=0$; $\therefore\ y=0$ or -10.

(i) Let $x^2+5x=y=0$
then $x(x+5)=0$, $\therefore\ x=0$ or -5

(ii) Let $x^2+5x=-10$,
then $x^2+5x+10=0$.
$\therefore\ x=\frac{1}{2}(-5\pm\sqrt{-15})$.

455. Next, consider the equation

$$(x+a)^4+(x+b)^4=2c^4.$$

Let $x+a=y$, then $x+b=y+b-a=y+p$, suppose.

Hence the equation becomes $y^4+(y+p)^4=2c^4$,

or $y^4+(y^4+4y^3p+6y^2p^2+4yp^3+p^4)=2c^4$,

Adding p^4, $2(y^4+2y^3p+3y^2p^2+2yp^3+p^4)=p^4+2c^4$,

or $2(y^2+py+p^2)^2=p^4+2c^4$.

Whence $y^2+py+p^2=\pm\sqrt{\frac{1}{2}(p^4+2c^4)}$, a quadratic in y.

Hence four values of y, and therefore of x, are determined.

Ex. 2. Solve $(x+2)^4+(x-1)^4=257$.

Here, expanding and adding, and dividing by 2, we obtain

$$x^4+2x^3+15x^2+14x-120=0, \text{ or } (x^2+x)^2+14(x^2+x)-120=0.$$

Assume $x^2+x=y$, then $y^2+14y-120=0$; $\therefore\ y=6$ or -20.

(i) Let $x^2+x=y=6$.
$\therefore\ x^2+x-6=0$,
$\therefore\ (x+3)(x-2)=0$, $\therefore\ x=2$ or -3.

(ii) Let $x^2+x=y=-20$.
$\therefore\ x^2+x+20=0$,
$\therefore\ x=\frac{1}{2}(-1\pm\sqrt{-79})$.

Exercise CLXIX.

Solve the following equations :—

1. $x^4-4x^3+2x^2+4x=15.$

2. $x^3-2x\sqrt{x}+\sqrt{x}-6=0.$

3. $x^4+4x^2+3x\sqrt{(x^2+4)}=4.$

4. $2\sqrt{(x^2+x+9)}-3x^2-3x=6.$

5. $4b\sqrt{(x^2+2ax+b^2)}+2x^2+4ax=\frac{1}{2}b^2.$

6. $\frac{x^2+1}{x}+\frac{1}{2}\cdot\frac{x+1}{\sqrt{x}}=13.$

7. $\frac{x^4}{2}+\frac{63}{x}=\frac{220\frac{1}{2}}{x^2}+49x^2-1196$

8. $\sqrt{(x+5)}+\sqrt{x}+2\sqrt{(x^2+5x)}=25-2x.$

9. $\sqrt{(x+a)}+\sqrt{x}=2\sqrt{(x^2+ax)}+2x+a.$

10. $2x+2\sqrt{\{(x+a)(x+b)\}}+\sqrt{(x+a)}+\sqrt{(x+b)}=c-a-b.$

11. $\frac{1}{8x^2-30x+27}+\frac{1}{12x^2-35x+18}=\frac{1}{2}(12x^2-29x+17).$

12. $x^2-4x\sqrt{x}+5x-2\sqrt{x}=6.$

13 $(x-2)^4+(x-4)^4=272.$

14. $(12x-1)(6x-1)(4x-1)(3x-1)=\frac{1}{96}.$

15. $x^2-5x-2-\frac{24}{x^2-5x}=0.$

16. $x^2+3x-\frac{9}{2}+\frac{2}{x^2+3x}=0.$

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456. Resolution into Factors. Many equations of higher powers than the second can conveniently be solved by using factors.

Ex. 1. Solve $2x^3-x^2=1.$

By suitable transposition, we have,

$$x^3-x^2=-x^3+1 \text{ or } x^2(x-1)=-(x^3-1)=-(x-1)(x^2+x+1).$$

$$\therefore\ x-1=0, \text{ or } x=1 \text{ and } x^2=-(x^2+x+1).$$

$$\therefore\ 2x^2+x+1=0, \text{ which solved gives } x=\tfrac{1}{4}(-1\pm\sqrt{-7}).$$

Ex. 2. Solve $x^3-2px^2+2px=1.$

By transposition, $(x^3-1)-2px(x-1)=0.$

Now $x-1$ being a common factor, $x-1=0$ and $\therefore\ x=1.$

And $(x^2+x+1)-2px=0$, $\therefore\ x^2-(2p-1)x+1=0,$

whence $x=\frac{1}{2}\{2p-1\pm\sqrt{(2p-1)^2-4}\}=\frac{1}{2}\{2p-1\pm\sqrt{4p^2-4p-3}\}.$

Ex. 3. Solve $x^4+2ax^3=2x+\frac{1}{a^2}$. (C. F. A. 1874).

By transposition, $\left(x^4-\frac{1}{a^2}\right)+2ax\left(x^2-\frac{1}{a}\right)=0$,

or $\left(x^2-\frac{1}{a}\right)\left\{\left(x^2+\frac{1}{a}\right)+2ax\right\}=0$; $\therefore\ x^2-\frac{1}{a}=0$ *i.e.* $x=\pm\frac{1}{\sqrt{a}}$,

or $x^2+\frac{1}{a}+2ax=0$, which solved gives $x=-a\pm\sqrt{\left(a^2-\frac{1}{a}\right)}$.

Exercise CLXX.

Solve the following equations :—

1. $x^3-8x+3=0$. **2.** $3x^3-22x=15$. **3.** $x^3-6ax^2+12a^2x=8a^3$.

4. $25x^4+5x^3-x-1=0$. **5.** $(x-1)^3+(2x+3)^3=27x^3+8$.

6. $(x+1)^2(x+4)=2$. **7.** $x^3+7x=49(x+1)$. **8.** $x^3+x^2+4=0$.

9. $x^2+\frac{1}{x^2}=a^2+\frac{1}{a^2}$. **10.** $x^2-\frac{7}{4}=\frac{3}{4x}$. **11.** $x^2-\frac{3}{4x}=1\frac{9}{16}$.

12. $\frac{3}{x-1}+\frac{6}{x+2}+\frac{1}{x-3}=\frac{5}{x+1}+\frac{2}{x-2}+\frac{7}{x+3}$. **13.** $\frac{x^2}{2x-5}=\frac{3}{x^2-17}$.

14. $27x^3+21x+8=0$. **15.** $x^3-76x+240=0$.

16. $\sqrt{2(a+x)}+7\sqrt{(ax-x^2)}=\sqrt{2(a+x)}+\sqrt{2(a-x)}$.

17. $(a-x)^3+(b-x)^3=(a+b-2x)^3$. **18.** $(a-x)^4+(b-x)^4=(a+b-2x)^4$.

19. $\sqrt[3]{(a-x)}+\sqrt[3]{(b-x)}=\sqrt[3]{(a+b-2x)}$.

20. $\sqrt[4]{(a-x)}+\sqrt[4]{(b-x)}=\sqrt[4]{(a+b-2x)}$.

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457. Reciprocal Equations. An equation is said to be **reciprocal** which is not altered by changing x into $1/x$.

Ex. 1. Solve $x^4+6x^3-23x^2+12x+4=0$.

Dividing by x^2, $x^2+6x-23+\frac{12}{x}+\frac{4}{x^2}=0$.

$\therefore\ \left(x^2+\frac{4}{x^2}\right)+6\left(x+\frac{2}{x}\right)-23=0$, or $\left(x+\frac{2}{x}\right)^2+6\left(x+\frac{2}{x}\right)-27=0$.

Whence $x+\frac{2}{x}=-3\pm 6=3$ or -9, and $\therefore\ x^2-3x$ or $x^2+9x=-2$.

(i) Take $x^2-3x=-2$.
$\therefore\ x^2-3x+2=0$.
$\therefore\ (x-1)(x-2)=0$, and $\therefore\ x=1$ or 2.

(ii) Take $x^2+9x=-2$.
$\therefore\ x=-\frac{9}{2}\pm\sqrt{(\frac{81}{4}-2)}$.
$=\frac{1}{2}(-9\pm\sqrt{73})$.

Ex. 2. Solve $x^6+2x^4-3x^3-3x^2+2x+1=0$.

The equation may be written $(x^6+1)+2x(x^3+1)-3x^2(x+1)=0$, where each bracket is plainly divisible by $x+1$;

Dividing, the original equation by $x+1$, we have
$x^4+x^3-4x^2+x+1=0$, a reciprocal equation of the 4th degree.

Dividing by x^2, $x^2+x-4+\frac{1}{x}+\frac{1}{x^2}=0$,

$$\therefore\ \left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)-4=0, \text{ or } \left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)-6=0$$

$$\therefore\ x+\frac{1}{x}=-\tfrac{1}{2}\pm\tfrac{5}{2}=2 \text{ or } -3, \text{ and } \therefore\ x^2+1=2x \text{ or } -3x.$$

(i) Take $x^2+1=2x$
$\therefore\ x^2-2x+1=0$
$\therefore\ (x-1)^2=0$ and $\therefore\ x=1, 1$.

(ii) Take $x^2+1=-3x$
$\therefore\ x^2+3x+1=0$
$\therefore\ x=\frac{1}{2}(-3\pm\sqrt{5})$.

Also corresponding to the factor $x+1$, we have $x+1=0$ or $x=-1$.

458. Exponential Equations. The following are illustrative examples.

Ex. 1. Solve $2^{2x+8}+1=32.2^x$.

We have $2^{2x+8}-2.2^{x+4}+1=0$, since $32=2.2^4$.
Taking the sq. root, $2^{x+4}-1=0$: $\therefore\ 2^{x+4}=1=2^0$.
Hence $x+4=0$ and $\therefore\ x=-4$.

Ex. 2. Solve $3^x+3^{-x}=9\frac{1}{9}$.

We have $3^x+\frac{1}{3^x}=\frac{82}{9}$, or $3^{2x}-\frac{82}{9}.3^x+1=0$.

$\therefore\ 3^x=\frac{41}{9}\pm\frac{40}{9}=9$ or $\frac{1}{9}=3^2$ or 3^{-2}; $\therefore\ x=\pm 2$.

459. Binomial Equations. The following are illustrative examples.

Ex. 1. Solve $x^3+1=0$.

Here, $(x+1)(x^2-x+1)=0$; $\therefore$ either $x+1=0$, *i.e.* $x=-1$, or $x^2-x+1=0$, which solved gives $x=\frac{1}{2}(1\pm\sqrt{-3})$

Ex. 2. Solve $x^4+1=0$.

Dividing by x^2, $x^2+\frac{1}{x^2}=0$, $\therefore$ $\left(x+\frac{1}{x}\right)^2=2$.

Taking the sq. root, $x+\frac{1}{x}=\pm\sqrt{2}$, and $\therefore$ $x^2\mp\sqrt{2}.x+1=0$,

whence $x=\frac{\pm 1\pm\sqrt{-1}}{\sqrt{2}}$.

Exercise CLXXI.

Solve the following equations :—

1. $x^4+6x^3+6x+1=\frac{77}{4}x^2$. **2.** $4x^4-4x^3-7x^2-4x+4=0$.

3. $x^4+4x^3+3x^2+4x+1=0$. **4.** $x^4+7x^3+8x^2+7x+1=0$.

5. $12(x^4+1)-56(x^3+x)+89x^2=0$. **6.** $4x^4-16x^3+23x^2-16x+4=0$.

7. $x^2+\frac{4}{x^2}+6x+\frac{12}{x}=23$. **8.** $4x^2-20-5\left(x+\frac{3}{x}\right)=-\frac{36}{x^2}$.

9. $6^{x-1}+6^{-x}=1\frac{1}{6}$. **10.** $x^6+1=0$. **11.** $x^6-1=0$. **12.** $x^{12}=1$.

Miscellaneous Equations.

Solve the following equations :—

1. $x^4-2x^3-2x^2+3x=108$. **2.** $\sqrt[4]{(x+21)}+\sqrt{(x+21)}=12$.

3. $\frac{x^2+4}{x^2-4}+\frac{x^2-4}{x^2+4}=\frac{34}{15}$. **4.** $x(x+4)+\frac{1}{x}\left(\frac{1}{x}+4\right)=10$.

5. $\frac{x-18}{\sqrt{x}-\sqrt{18}}+\frac{\sqrt{(x-18)}}{\sqrt{x}}=\sqrt{x}-\frac{\sqrt{x}}{\sqrt{(x-18)}}$.

6. $27x^2-\frac{841}{3x^2}+\frac{17}{3}=\frac{232}{3x}-\frac{1}{3x^2}+5$. **7.** $x^2+4x+\frac{4}{x^2}=\frac{17x-8}{x}$.

8. $\frac{5}{x-1}+\frac{4}{x+2}+\frac{21}{x-3}=\frac{5}{x+1}+\frac{4}{x-2}+\frac{21}{x+3}$.

9. $(a+x)^{\frac{2}{3}}-5(a^2-x^2)^{\frac{1}{3}}=-4(a-x)^{\frac{2}{3}}$. **10.** $4x^4+\frac{1}{2}x=4x^3+33$.

11. $6x\sqrt{x}-11x+6\sqrt{x}-1=0$. **12.** $2x^2-x-2x\sqrt{(1-x^2)}=1\frac{1}{2}$.

13. $3\{(x-1)^2-x\}^2+2x=341+2(x-1)^2$. **14.** $\frac{1}{2}x^4+\frac{17}{4}x^2-17x=8$.

15. $\frac{x^{-n}}{1+x^{-n}}+\frac{1-x^{-n}}{x^{-n}}=\frac{1}{6}$. **16.** $8\sqrt{(3x)}+\frac{243+324\sqrt{(3x)}}{16x-3}=16x+3$.

17. $\frac{3}{5}\left(\frac{a^4+a^2x^2+x^4}{a^4-a^2x^2+x^4}\right)=\left(\frac{ax}{a^2-x^2}\right)^2$. **18.** $\frac{x^2-4x-8}{\sqrt{(x^2+2x+11)}}=2\sqrt{2}$.

19. $x^3-3x=a^3+\frac{1}{a^3}$. **20.** $x^4+\frac{1}{x^4}=a^4+\frac{1}{a^4}$. **21.** $x^3+\frac{a^4}{x}=2a^2x$.

22. $\frac{7x+10}{x^2-4x+5}-\frac{2x+4}{x^2-2x+2}=x^3$.

23. $\sqrt[4]{\left(\frac{23+x}{23-x}\right)}+\sqrt[4]{\left(\frac{23-x}{23+x}\right)}=\sqrt{5}$ (P. E. 1901).

24. $x^6+(x+1)^6+1=2(x^2+x+1)^3$. **25.** $2^{x+1}+4^x=288$.

26. $(x+1)^6+(x-1)^6=19\{(x+1)^3+(x-1)^3\}$. **27.** $3^{x+1}+9^x=108$.

28. $(2x-1)(x+1)(2x+5)(x+4)=70$.

29. $\sqrt{(2x^2+9x-1)}+\sqrt{(2x^2-7x+7)}=6$.

30. $x(x-1)(x-2)=9.8.7$. **31.** $(x+2)(x+3)(x+4)=3.4.5$.

VII. PROBLEMS PRODUCING QUADRATIC EQUATIONS.

460. In the solution of Problems, depending on Quadratic and Higher Equations, there may be two or more values of the root, and these may be *real* quantities or *impossible*. In the former case, we must consider if any of the roots are excluded by the nature of the question, which may altogether reject *fractional*, or *negative*, or *surd* answers ; in the latter case, we conclude that the solution of the proposed question is arithmetically impossible.

Ex. 1. Divide 12 into two parts so that the square of one of them may be four times as great as the square of the other.

Let x be one of the parts ; then $12-x$ is the other part.

$\therefore$ By the question, $x^2=4(12-x)^2$.

Taking, the sq. root, $x=\pm 2(12-x)=24-2x$ or $=-24+2x$.

The former gives $x=8$ and $\therefore$ $12-x=4$.

Hence the parts are 8 and 4, which is the *arithmetical* solution.

The latter gives $x=24$ and $\therefore$ $12-x=-12$, which answers the condition *symbolically*.

Ex. 2. What number, when added to 30, will be less than its square by 12?

Let x be the number.

$\therefore$ By the question, $30+x=x^2-12$; or $x^2-x-42=0$.

$\therefore$ $(x-7)(x+6)=0$; whence $x=7$ or -6.

Hence the reqd. number is 7. Here the latter root is excluded.

Ex. 3. A person bought sheep for £33. 15*s*. which he sold again at £2. 8*s*. a head, gaining thereby as much as one sheep cost him. How many sheep did he buy?

Let x be the number of sheep bought.

Then $\dfrac{33\frac{3}{4}}{x}$ is the price of a sheep in £,

and $2\frac{2}{5}x$ is the selling price of the sheep in £.

$\therefore$ By the question, $2\frac{2}{5}x-33\frac{3}{4}=\dfrac{33\frac{3}{4}}{x}$,

which reduces to $48x^2-675x-675=0$; whence $x=15$ or $-\dfrac{15}{16}$.

Hence, the number of sheep bought $=15$.

Ex. 4. A person bought a number of oxen for *Rs.*120; if he had bought 3 more for the same money, he would have paid *Rs.*2 less for each. How many did he buy?

Let x be the number of oxen bought.

Then $\dfrac{120}{x}$ is the price actually given for each in *Rs.*

and $\dfrac{120}{x+3}$ is the price of each, if 3 more be bought.

$\therefore$ By the question, $\dfrac{120}{x+3}=\dfrac{120}{x}-2$,

which reduces to $x^2+3x-180=0$; whence $x=12$ or -15.

Hence the number of oxen bought $=12$.

Ex. 5. **A**'s rate of travelling is one mile an hour less than **B**'s, and **B** can go 21 miles in 20 minutes less than it takes **A** to go 20 miles. How many miles an hour can **A** travel?

Let x be **A**'s rate of travelling in miles per hour.

Then $x+1$ is **B**'s rate......................

Now **A** takes $\frac{20}{x}$ hrs. to travel 20 miles, and **B** takes $\frac{21}{x+1}$ hrs. to travel 21 miles.

$\therefore$ By the question, $\frac{20}{x}-\frac{1}{3}=\frac{21}{x+1}$, (for 20 min. $=\frac{1}{3}$ hr.)

which reduces to $x^2+4x-60=0$; whence $x=6$ or -10.

Hence **A**'s rate of travelling per hour is 6 miles.

Ex. 6. A reduction of 2*d.* a dozen in the price of eggs will give 6 more for 3*s.* 6*d.*; find the price per dozen.

Let x be the price of a dozen eggs in *d.*

Then we can obtain $\frac{12}{x}\times 42$ eggs, for 3*s.* 6*d.* or 42*d.*

Also, when $(x-2)d.$ is the price of a dozen eggs,

we obtain $\frac{12}{x-2}\times 42$ eggs, for 3*s.* 6*d.* or 42*d.*

$\therefore$ By the question, $\frac{12}{x-2}\times 42-\frac{12}{x}\times 42=6$,

which reduces to $x^2-2x-168=0$; whence $x=14$ or -12.

Hence the price of a dozen eggs is 14*d.*

Ex. 7. A person invested *Rs.* 10660 in the $3\frac{1}{2}$ per cent. Government Securities when they were at a certain rate per cent. premium. If he had invested them at a price $1\frac{1}{2}$ per cent less, his annual income would have been increased by *Rs.* 5. 4*a.* Find the price of the Government Securities.

Let x be the price of the Government Securities in *Rs.*

Then, the first income $=$ *Rs.* $\frac{10660}{x}\times 3\frac{1}{2}$,

and the second...... $=$ *Rs.* $\frac{10660}{x-1\frac{1}{2}}\times 3\frac{1}{2}$.

$\therefore$ By the question, $\frac{10660}{x-1\frac{1}{2}}\times 3\frac{1}{2}-\frac{10660}{x}\times 3\frac{1}{2}=5\frac{1}{4}$,

which reduces to $2x^2-3x-21320=0$; whence $x=104$ or $-102\frac{1}{2}$.

Hence, the required price is *Rs.* 104.

Ex. 8. A number of two digits is less than four times the product of its digits by 11, and the digit in the tens' place exceeds the digit in the units' place by four. Find the number.

Let x be the digit in the units' place.

Then $x+4$ is the digit in the tens' place.

The number $=10(x+4)+x=11x+40$.

Four times the product of the digits $=4x(x+4)$.

$\therefore$ By the question, $4x(x+4)-(11x+40)=11$,

which reduces to $4x^2+5x-51=0$; whence $x=3$ or $-\frac{17}{4}$.

Hence 3 is the digit in the units' place, and $3+4$ or 7 the digit in the tens' place, and therefore 73 is the required number.

Exercise CLXXII.

1. There are two numbers, one of which is $\frac{4}{5}$ of the other, and the difference of their squares is 81. Find them.

2. The difference of two numbers is $\frac{3}{8}$ of the greater, and the sum of their squares is 356. Find them.

3. Determine two magnitudes whose difference is $\frac{1}{6}$ and the sum of whose squares is $(\frac{5}{6})^2$.

4. There are two numbers, one of which is triple of the other, and the difference of their squares is 128. Find them.

5. What number is that, the sum of whose third and fourth parts is less by 2 than the square of its sixth part?

6. What two numbers are those whose difference is 5 and their sum multiplied by the greater 228?

7. What two numbers make up 14, so that the quotient of the less divided by the greater is $\frac{9}{16}$ of the quotient of the greater divided by the less?

8. A labourer dug two trenches, one 6 yards longer than the other, for *Rs.* 356, and the digging of each cost as many rupees per yard, as there were yards in its length. Find the length of each.

9. Bought two flocks of sheep for *Rs.* 360, in one of which there were 5 more than in the other: each sheep in each flock cost as many rupees as there were sheep in the other flock. How many were there of each?

10. By selling a horse for *Rs.* 240, I lose as much per cent. as it cost me. What was the prime cost of it?

11. There is a number such that the product of the numbers obtained by adding 3 and 5 to it respectively is less by 1 than the square of its double. Find it.

12. There is a rectangular field, whose length exceeds its breadth by 16 yards, and it contains 960 square yards. Find its dimensions.

13 The plate of a looking glass is 18 inches by 12, and it is to be framed with a frame of uniform width, whose area is to be equal to that of the glass. Find the width of the frame.

14. Two partners, **A** and **B**, gained *Rs.* 140 by trade; **A**'s money was 3 months in trade and his gain was *Rs* 60 less than his stock, and **B**'s money, which is *Rs.* 50 more than **A**'s, was in trade 5 months. What was **A**'s stock?

15. A person bought 38 sheep for *Rs.* 71. 4*a.*; but having lost a certain number n of them, he sold the remainder for n annas a head more than they cost him, and so gained upon the whole Re. 1. How many sheep did he lose?

16. **A** and **B** distribute *Rs.* 50 each in charity; **A** relieves 5 persons more than **B**, and **B** gives to each 8*a.* more than **A**. How many did each relieve?

17. **A** and **B** take shares in a concern to the amount altogether of *Rs.* 5000; they sell out at *par*, **A** at the end of 2 years, **B** of 8, and each receives in capital and profit *Rs.* 2970. How much did each embark?

18. Two trains each run a distance of 330 miles. One o them, whose average speed exceeds that of the other by 5 miles an hour, takes half an hour less to travel the whole distance. Find their average speeds.

19. In 100 minutes a boat's crew row $3\frac{1}{2}$ miles down a river and back again. If the river runs at 2 miles an hour, what is the pace of the boat in still water?

20. A battalion of soldiers when formed into a solid square present 16 men fewer in the front than they do when formed into a hollow square 4 deep. Find the number of men.

21. Mr. Gladstone was born in the year A. D. 1809. In the year A. D. x^2 he was $x-3$ years old: find x.

22. **A** and **B** set out at the same time; **A** from **C** to go to **D** and **B** from **D** to go to **C**; they meet on the road, when it appears that **A** has travelled 30 miles more than **B**, and that at the rate he is travelling, he will reach **D** in 4 days, and **B** will arrive at **C** in 9 days. Find the distance of **C** from **D**.

23 Ten minutes after the departure of an express train a slow train is started, travelling on the average 20 miles less per hour, which reaches a station 250 miles distant $3\frac{1}{3}$ hours after the arrival of the express. Find the rate at which each train travels.

24. A rides from P to Q in one hour at a uniform speed. B rides for one-third of the way 2 miles an hour faster than A, and for the rest of the journey 1 mile an hour slower, thus taking 40 seconds longer. Find the distance from P to Q.

25. The men in a regiment can be arranged in a hollow square 4 deep; if the number of men be increased by 129 they can be arranged in a solid square having on each side 10 men less than were on each outer side of the hollow square. Find how many men

VIII. THEORY OF QUADRATIC EQUATIONS.

461. *A quadratic equation cannot have more than two roots.*

If possible, let the general quadratic equation

$$ax^2+bx+c=0$$

have *three different* roots, α, β and γ.

Since, each of these values of x satisfies the equation,

we have
$$\left.\begin{aligned} a\alpha^2+b\alpha+c&=0\ldots\ldots\ldots(1)\\ a\beta^2+b\beta+c&=0\ldots\ldots\ldots(2)\\ a\gamma^2+b\gamma+c&=0\ \ \ldots\ldots(3)\end{aligned}\right\}$$

Subtracting (2) from (1), $a(\alpha^2-\beta^2)+b(\alpha-\beta)=0$.

Dividing by $\alpha-\beta$, which by supposition is not equal to zero,

$$a(\alpha+\beta)+b=0\ldots\ldots\ldots(4)$$

In the same way, subtracting (3) from (1) and dividing by $\alpha-\gamma$,

$$a(\alpha+\gamma)+b=0\ldots\ldots\ldots(5)$$

Subtracting (5) from (4), $a(\beta-\gamma)=0$;

$$\therefore \text{ either } a=0 \text{ or } \beta-\gamma=0,$$

which is impossible, for a is not equal to zero, nor is β equal to γ, by hypothesis.

Hence the quadratic cannot have more than two roots.

462. *If α and β are the two roots of the quadratic equation $ax^2+bx+c=0$, then*

$$\alpha+\beta=-\frac{b}{a} \text{ and } \alpha\beta=\frac{c}{a}.$$

Solving the equation, we have

$$x=\frac{-b+\sqrt{(b^2-4ac)}}{2a} \text{ or } \frac{-b-\sqrt{(b^2-4ac)}}{2a}$$

Let $\alpha=\dfrac{-b+\sqrt{(b^2-4ac)}}{2a}$, and $\beta=\dfrac{-b-\sqrt{(b^2-4ac)}}{2a}$.

$$\therefore\ \alpha+\beta=\frac{-b+\sqrt{(b^2-4ac)}}{2a}+\frac{-b-\sqrt{(b^2-4ac)}}{2a}=\frac{-2b}{2a}=-\frac{b}{a}.$$

$$\text{and } \alpha\beta=\frac{(-b)^2-(b^2-4ac)}{4a^2}=\frac{4ac}{4a^2}=\frac{c}{a}.$$

Since the equation can also be written in the form $x^2+\frac{b}{a}x+\frac{c}{a}=0$, we may express these results as follows :—

When the coefficient of x^2 in a quadratic equation is unity,

(i) **the sum of the roots is equal to the coefficient of x with the sign changed ;**

(ii) **the product of the roots is equal to the constant term.**

Note. *If α and β be the roots of the equation $x^2+px+q=0$, then*

$$\alpha+\beta=-p \text{ and } \alpha\beta=q.$$

463. *If α and β are the roots of $ax^2+bx+c=0$, then*

$$ax^2+bx+c=a(x-\alpha)(x-\beta).$$

We have $ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$

$$=a[x^2-(\alpha+\beta)x+\alpha\beta], \text{ Art. 462.}$$
$$=a(x-\alpha)(x-\beta).$$

Note. *If α and β are the roots of $x^2+px+q=0$, then*

$$x^2+px+q=(x-\alpha)(x-\beta).$$

Ex. 1. The quadratic whose roots are -4 and 7 is

$$(x+4)(x-7)=0, \text{ or } x^2-3x-28=0.$$

Ex. 2. If α and β are the roots of $x^2+px+q=0$, find the values of (i) $\alpha-\beta$, (ii) $\alpha^2+\beta^2$, (iii) $\alpha^3+\beta^3$.

We have $\alpha+\beta=-p$ and $\alpha\beta=q$. (Art. 462).

(i) Since $(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta=(-p)^2-4q=p^2-4q$,

$$\therefore\ \alpha-\beta=\pm\sqrt{(p^2-4q)}.$$

(ii) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=(-p)^2-2q=p^2-2q.$

(iii) $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=(-p)^3+3pq=-p^3+3pq.$

Ex. 3. If α and β are the roots of $ax^2+bx+c=0$, form the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

We have $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$.

The sum of the roots of the reqd. equation

$$=\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\left(\frac{b^2}{a^2}-\frac{2c}{a}\right)\div\frac{c}{a}$$

$$=\frac{b^2-2ac}{ac}.$$

The product of the roots $=\frac{\alpha}{\beta}\times\frac{\beta}{\alpha}=1.$

Hence the required equation is

$$x^2-\left(\frac{b^2-2ac}{ac}\right)x+1=0 \text{ or } acx^2-(b^2-2ac)x+ac=0.$$

464. *To find the condition that the equations $ax^2+bx+c=0$ and $a'x^2+b'x+c'=0$ may have a common root.*

Let α be the common root of the equations.

Then by supposition, $$\left.\begin{aligned} a\alpha^2+b\alpha+c&=0\ldots\ldots(1)\\ a'\alpha^2+b'\alpha+c'&=0\ldots(2)\end{aligned}\right\}$$

From (1) and (2) by the *Rule of Cross Multiplication*, we have

$$\frac{\alpha^2}{bc'-b'c}=\frac{\alpha}{ca'-c'a}=\frac{1}{ab'-a'b}.$$

Hence $\alpha^2=\frac{bc'-b'c}{ab'-a'b}$ and $\alpha=\frac{ca'-c'a}{ab'-a'b}$.

Therefore $\frac{bc'-b'c}{ab'-a'b}=\left(\frac{ca'-c'a}{ab'-a'b}\right)^2$

or $(ab'-a'b)(bc'-b'c)=(ca'-c'a)^2$, the reqd. condition.

465. *If a is positive and α, β are real roots of the equation $ax^2+bx+c=0$, the expression ax^2+bx+c vanishes when $x=\alpha$ or β, and*

14. *Graph corresponding to the equation*

$$3x^2-4y^2+6x-8y-7=0.$$

The equation may be written thus :—

$$3(x+1)^2-4(y+1)^2=6,$$

$\therefore \frac{(x+1)^2}{2}-\frac{(y+1)^2}{3/2}=1$, which represents a hyperbola, whose centre is the point $(-1, -1)$, semi-transverse axis $\sqrt{2}$ and semi-conjugate axis $\sqrt{\frac{3}{2}}$.

15. *Graph of* $\frac{a^2}{x}$.

Let $y=\frac{a^2}{x}$; then $xy=a^2$, which represents a rectangular hyperbola, whose centre is the origin and whose asymptotes are the axes of x and y, (as shewn in Fig. 8).

The transverse axis is the line **AA′** which bisects the angle between the axes of co-ordinates.

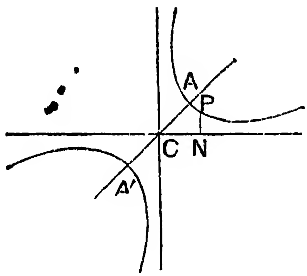


Fig. 8.

N.B. In a rectangular hyperbola, **CN.PN** = a constant, which gives the equation $xy=a^2$.

16. *Graph corresponding to the equation* $xy-2x-3y=10$.

We have $(x-3)(y-2)=16$, which represents a rectangular hyperbola, of which the centre is the point (3, 2) and asymptotes parallel to the axes of co-ordinates (as shewn in Fig. 9).

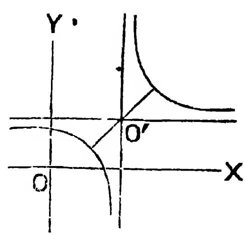


Fig. 9.

17. *Graph of* $2\sqrt{(ax)}$.

Let $y=2\sqrt{(ax)}$; then $y^2=4ax$;
which represents a parabola, whose vertex is the origin, the axis the axis of x, the tangent at the vertex the axis of y and $4a$ the latus rectum, (as shewn in Fig. 10).

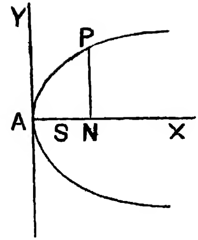


Fig. 10.

The curve is symmetrical with respect to the axis of x and lies on the right side of the axis of y and extends to infinity.

N.B. In the parabola, $PN^2=4AS.AN$, which gives $y^2=4ax$.

Note. The graphs of $\sqrt{x}$, $2\sqrt{x}$, &c. are all parabolas.

18. *Graph of* $2\sqrt{\{a(x-b)\}}$.

Let $y=2\sqrt{\{a(x-b)\}}$;
then $y^2=4a(x-b)$.

If we transfer the origin to the point $(b, 0)$, the equation becomes $y^2=4ax$, and the curve is a parabola, (as shewn in Fig. 11).

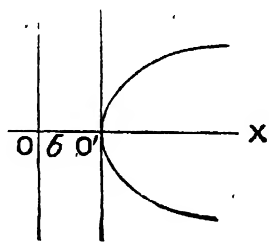


Fig. 11.

19. *Graph of* $\frac{x^2}{4a}$.

Let $y=\frac{x^2}{4a}$, then $x^2=4ay$; this represents a parabola whose vertex is the origin, the axis the axis of y and latus rectum $4a$, (as shewn in Fig. 12).

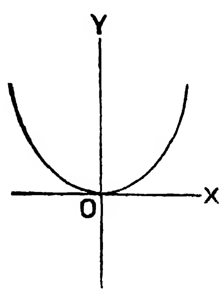


Fig. 12

N.B. The curve is symmetrical with respect to the axis of y.

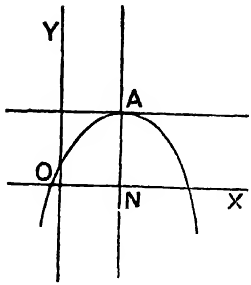


Fig. 13.

20. *Graph of* $4x-2x^2$.

Let $y=4x-2x^2=-2(x-1)^2+2$;

then $y-2=-2(x-1)^2$.

If we transfer the origin to the point (1, 2), the equation becomes $y=-2x^2$, and the curve is a parabola, (as shewn in Fig. 13).

N.B. Here the maximum value of y is 2 and then $x=1$. The axis of the parabola is the line AN and the vertex A.

21. *Graph of* $3x^2-8x+10$.

Let $y=3x^2-8x+10=3(x^2-\frac{8}{3}x+\frac{10}{3})$

$=3\{(x-\frac{4}{3})^2-\frac{16}{9}+\frac{10}{3}\}=3(x-\frac{4}{3})^2+\frac{14}{3}$;

then $y-\frac{14}{3}=3(x-\frac{4}{3})^2$.

Transfer the origin to the point $(\frac{4}{3}, \frac{14}{3})$ and the equation becomes $y=3x^2$, which represents a parabola (as shewn in Fig. 14).

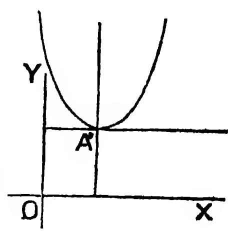


Fig. 14.

N.B. The curve does not cut the axis of x, because the roots of the equation $3x^2-8x+10=0$ are imaginary. In this figure the scales of x and y are different.

The minimum value of y is $\frac{14}{3}$ and then $x=\frac{4}{3}$.

II. GRAPHS OF QUADRATIC EQUATIONS.

467. We shall now consider some graphs of functions of a degree higher than the first.

Ex. 1. Draw the graph of $y=x^2$.

Corresponding values of x and y may be tabulated as follows :—

x	...	-5	-4	-3	-2	-1	0	1	2	3	4	5	..
y	...	25	16	9	4	1	0	1	4	9	16	25	...

If the above determined points are plotted and joined by a continuous line drawn freehand, we shall obtain the graph required, as shewn in Fig. 15, which we see to be a curve. Such a curve is called a **parabola.**

In the above Example we notice the following facts :—

(i) We have from the equation $x=\pm\sqrt{y}$; thus for every value of the ordinate y there are two values of the abscissa x, *equal in magnitude but opposite in sign.* Hence the curve is symmetrical with respect to the axis of y; so that plotting sufficient points in the first quadrant to determine the form of the graph, its form in the second quadrant may easily be obtained by joining corresponding points without actually potting them.

(ii) We notice that the graph lies wholly in the first and second quadrants; for, whatever value we may give to x, we can never have y negative, and threfore no point on the curve will be found *below* the axis of x.

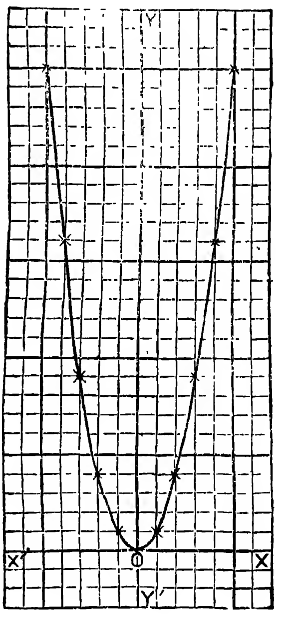


Fig. 15.

(iii) As the numerical value of x increases, so that of y increases rapidly. Hence the parts of the curve on either side of OY meet only at the origin and then extends upwards and outwards to an infinite distance in both the first and second quadrants.

Ex. 2. Draw the graph of $y = 2x - \frac{x^2}{4}$.

Tabulate the values of x and y as follows :—

x	10	9	8	7	6	5	4	3	2	1	0	−1	−2	3	−4
$2x$	20	18	16	14	12	10	8	6	4	2	0	−2	−4	6	8
$\frac{1}{4}x^2$	25	20·25	16	12·25	9	6·25	4	2·25	1	·25	0	·25	1	2·25	4
y	−5	−2·25	0	1·75	3	3·75	4	3·75	3	1·75	0	−2·25	−5	−8·25	−12

Plot the points (10, −5), (9, −2·25), (8, 0), &c...(0, 0), (−1, −2·25) ..&c. ; the resulting graph is shewn in Fig. 16, below.

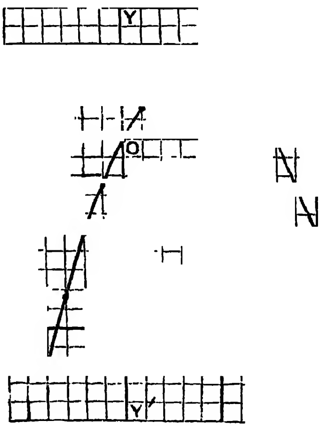


Fig. 16.

In the present case only three or four negative values of x and y need be plotted, but more attention must be paid to the results arising out of positive values of x.

N. B. The equation can be written in the form $(x-4)^2 = -4(y-4)$ which shews that the curve is a parabola whose vertex is the point (4, 4) and latus rectum 4.

468. The curve obtained from the equation $y = ax^2$, where a is constant, is called a **parabola.** If a is positive, the curve lies wholly

above the x-axis; but if a is negative, the curve will lie wholly below the x-axis. The greater a is, the more rapidly does the graph recede from the x-axis.

Exercise CLXXIV.

1. Draw the graph of $y=x^2$.

(i) Taking 0·4″ as unit for x, 0·1″ as unit for y.

(ii) Taking 1″ as unit for x, 0·1″ as unit for y.

(iii) Taking the x unit five times as large as the y unit.

2. Trace the graph of the equation $x^2=y$.

(i) When the x unit is equal to the y unit.

(ii) When the x unit is five times as large as the y unit.

3. Plot the graph of the equation $y=4x^2$.

(i) Taking the x unit equal to the y unit.

(ii) Taking the x unit four times the y unit.

4. Draw the graphs of :—

(i) $y=-x^2$. (ii) $y^2=x$. (iii) $y^2=-x$. (iv) $2y=x^2$

5. Draw the graphs of the following equations :—

(i) $y=8x^2$. (ii) $y^2=-4x$. (iii) $y=\frac{1}{4}x^2$.

6. Plot the following :—

(i) $y=2x+\dfrac{x^2}{4}$. (ii) $y=1-\frac{1}{2}x^2$. (iii) $y=\dfrac{x^2}{4}+x-2$.

7. Trace the graphs of :—

(i) $y=\frac{1}{4}x^2-x+2$. (ii) $y=x^2+x+3$. (iii) $y=2x^2-x+1$.

8. Draw the graphs of $y=x^2$ and $x=y^2$, and shew that they have only one common chord. Find its equation.

III. GRAPHIC SOLUTION OF QUADRATIC EQUATIONS.

469. If $f(x)$ represent a function of x, we can obtain an approximate solution of the equation $f(x)=0$ by plotting the graph of $y=f(x)$ and then measuring the intercepts made on the x-axis. These intercepts which make $y=0$ are the values of x and are therefore the roots of the equation $f(x)=0$.

Ex. 1. Solve graphically the equation $2x^2+x-15=0$.

Put $y=2x^2+x-15$, and find the graph of this equation.

x	4	3				-1	-2	-3	-4
$2x^2$	32	18				2	8	18	32
$x-15$	-11	-12	-13	-14	-15	16	-17	-18	-19
y	21		5	-12	-15	-14	-9	0	13

Take the unit for x twice as great as that for y.

Join the points (4, 21), (3, 6), (2, −5), (1, −12), (0, −15), (−1, −14) (−2, −11), (−3, 0), (−4, 13). The resulting curve is the graph of $y=2x^2+x-15$ and is shown in Fig. 17, below.

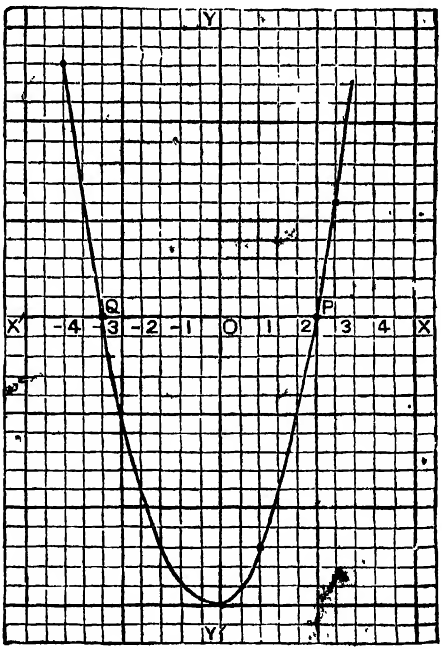


Fig. 17.

N. B. The equation can be written in the form $y+\frac{121}{8}=2(x+\frac{1}{4})^2$ which shews that the curve is a parabola whose vertex is $(-\frac{1}{4}, -\frac{121}{8})$

and latus rectum $\frac{1}{2}$. Generally when the terms of the second degree form a perfect square, the curve is a parabola.

Now the equation $2x^2+x-15=y$ is the same as the equation $2x^2+x-15=0$, if $y=0$.

Hence to find the roots of the latter equation we note the points on the curve where $y=0$, and these will give us the value of x which we have to find.

At the points P and Q where this curve meets the x-axis, $y=0$; therefore at these points $2x^2+x-15=0$.

But OP and OQ are the values of x at these points;

$\therefore$ they are the roots of the given equation.

At P, $x=2{\cdot}5$, and at Q, $x=-3$.

Hence we see that the roots are $2{\cdot}5$ and -3.

Verification. $2x^2+x-15=0$, or $(2x-5)(x+3)=0$.

i.e., $2x-5=0$ and $\therefore\ x=2{\cdot}5$

$x+3=0$ and $\therefore\ x=-3$.

Hence the solution is correct.

Ex. 2 Find graphically, correct to one decimal place, the roots of the equation $2x^2+6x-5=0$.

Trace the graph of $y=2x^2+6x-5$.

x	3	2	1	0	-1	-2	-3	-4	-5	-6
$2x^2$	18	8	2	0	2	8	18	32	50	72
$6x-5$	13	7	1	-5	-11	-17	-23	-29	-35	-41
y	31	15	[illegible]	-5	-9	-9	-5	3	15	31

Take the unit [illegible] twice as great as that for y.

Plot out the c[illegible] from the points thus obtained as shewn in Fig. 18 below.

The equation is satisfied when $2x^2+6x-5=0$, i.e. when $y=0$, *i.e.*, where the curve cuts the x-axis.

It will be seen that there are two points on the curve where $y=0$, namely the points P and Q.

At P, $x={\cdot}6$ (nearly), and at Q, $x=-3{\cdot}6$ (nearly).

Thus the roots required are ${\cdot}6$ and $-3{\cdot}6$.

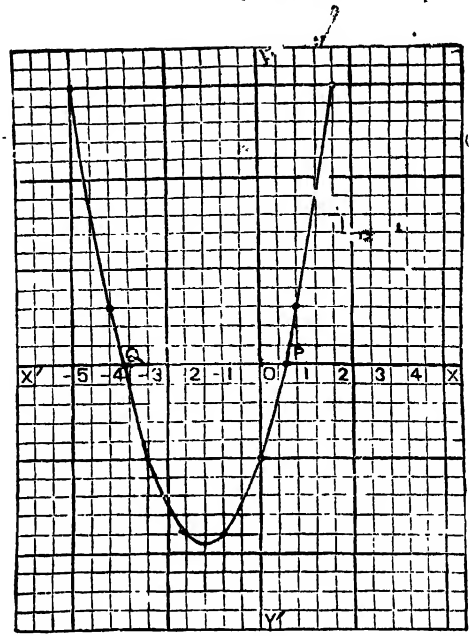


Fig. 18

Verification. When $x = ·6$,

$2x^2+6x-5=2(·36)+3·6-5=·72+3·6-5=-·68.$

Thus, when $x=·6$, $2x^2+6x-5$ is nearly zero.

Hence ·6 is an approximate root. I[illegible] same way we can verify the fact that $-3·6$ is an approximate r[illegible]

Or thus: Since $2x^2+6x-5=0$,

$\therefore\ x=\frac{1}{2}(-3\pm\sqrt{19}.)$

If we take 4·358 as the approximate value of $\sqrt{19}$, the values of x will be found to be $-3·68$ and ·68.

N. B. The equation to the curve is $y+\frac{19}{2}=2(x+\frac{3}{2})^2$ which shews that the curve is a parabola whose vertex is $(-\frac{3}{2}, -\frac{19}{2})$ and latus rectum $\frac{1}{2}$.

Ex. 3. Draw the graph of $y=x^2+3x$ and from it deduce the roots of the equation $x^2+3x-4=0$.

Tabulate the values of x and y as follows :—

x	3	2	1	0	−1	−2	−3	−4	−5	−6
x^2	9	4	1	0	1	4	9	16	25	36
$3x$	9	6	3	0	−3	−6	−9	−12	−15	−18
y	18	10	4	0	−2	−2	0	4	10	18

Take the unit for x twice as great as that for y.

Plot the points thus obtained and we obtain the graph required, as shewn in Fig. 19, below.

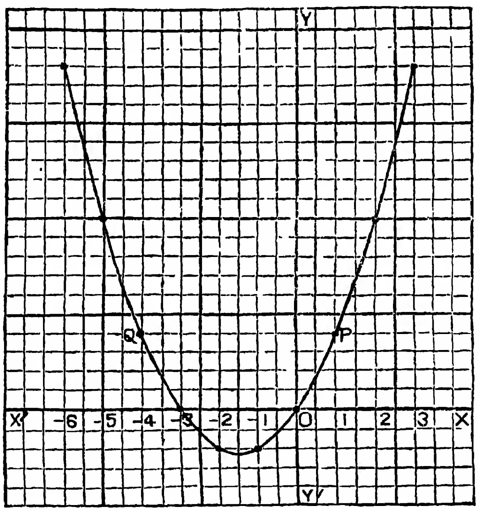


Fig. 19.

Now, we have to solve the equation $x^2+3x-4=0$.

Since $x^2+3x-4=0$, $\therefore$ $x^2+3x=4$.

Hence, we must find two points on the curve where $y=4$.

The two such points are P and Q.

At P, $x=1$ and at Q, $x=-4$.

Hence 1 and -4 are the required roots.

Plotting these points and joining them, we see that they all lie on a **circle**, whose centre is the origin and whose radius is 5, as shewn in Fig. 22, above.

474. Distance between two points. Let **P** be the point (x, y) and **Q** the point (a, b); draw **PN**, **QM** perpendicular to the axis of x and **QR** perp. to **PN**.

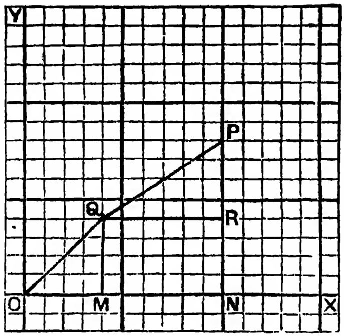


Fig. 23.

$$\text{PR}=\text{PN}-\text{NR}=y-b, \text{ and } \text{QR}=\text{ON}-\text{OM}=x-a.$$

Now, $\text{PQ}^2=\text{QR}^2+\text{PR}^2$ and $\therefore$ $\text{PQ}=\sqrt{(\text{QR}^2+\text{PR}^2)}$.

Thus $\text{PQ}=\sqrt{\{(x-a)^2+(y-b)^2\}}$.

Hence the distance between two points (x, y) and (a, b).

$$=\sqrt{\{(x-a)^2+(y-b)^2\}}.$$

$$\text{Also } \text{OQ}=\sqrt{(\text{OM}^2+\text{MQ}^2)}=\sqrt{(a^2+b^2)}.$$

475. Equation of a Circle. We have already found equation of a circle. It is easy to find the equation of *any* circle

Let the centre of the circle be at the point **A** (a, b) and let its radius be c; then if **P** (x, y) is any point on it,

we have, by Art. 474, $(x-a)^2+(y-b)^2=\text{AP}^2=c^2 \ldots\ldots (1)$ which is the required equation.

Ex. 1. Draw the graph of $x^2+y^2-8x-6y=0$.

The equation may be written $(x-4)^2+(y-3)^2=25$.

$\therefore$ $\sqrt{\{(x-4)^2+(y-3)^2\}}=5$.

Thus the graph is a circle, whose centre is at 4, 3 and whose radius is 5.

Ex. 2. Solve graphically the simultaneous equations

(i) $x^2+y^2=61$, (ii) $3y=4x-2$.

The graph of (i) is a circle. Since the equation is satisfied by $x=5, y=6$ (the point P), the graph may be drawn by describing a circle with centre O and radius OP.

The graph of (ii) is a straight line, which cuts the axes at the points $(0, -\frac{2}{3})$, $(\frac{1}{2}, 0)$.

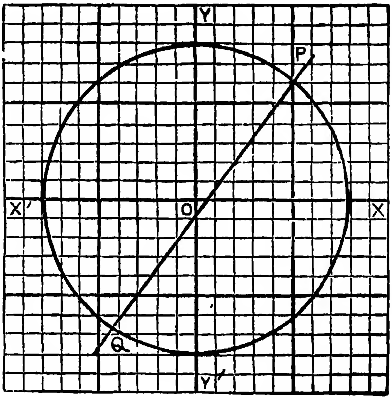


Fig. 24.

This line produced cuts the circle at P and Q. The co-ordinates of these points are $(5, 6)$ and $(-4{\cdot}36, -6{\cdot}48)$; thus the solution of the equations is given by

$$x=5,\ y=6, \text{ and } x=-4{\cdot}36,\ y=-6{\cdot}48.$$

Exercise CLXXVII.

Draw the graphs of the following :—

1. $x^2+y^2=36$. **2.** $x^2+y^2=64$. **3.** $x^2+y^2=81$. **4.** $x^2+y^2=0$.

5. $x^2+y^2-6x-8y=0$. **6.** $x^2+y^2=49$. **7.** $x^2+y^2=16$.

8. $x^2+y^2=12$. **9.** $x^2+y^2=8$. **10.** $x^2+y^2=20$. **11.** $x^2+y^2=35$.

Shew that the following equations represent circles and find their centres and radii :—

12. $x^2+y^2-4x+6y+7=0$. **13.** $x^2+y^2+6x+4y+4=0$.

14. $4x^2+4y^2-16x+8y+11=0$ **15.** $x^2+y^2-6x+4y+3=0$.

Find the equations of the following circles :—

16. Centre $(-2, 3)$, radius $=5$. 17. Centre $(-1\frac{1}{2}, -2\frac{1}{2})$, radius $=6$.

18. Shew that the equation (where a, b, c are constants)

$$x^2+y^2+ax+by+c=0$$

represents a circle and find its centre and radius.

19. Solve the following equations graphically :—

(i) $\left.\begin{aligned} x^2+y^2&=41 \\ 3x-2y&=2 \end{aligned}\right\}$ (ii) $\left.\begin{aligned} x^2+y^2&=34 \\ x+2y&=13 \end{aligned}\right\}$ (iii) $\left.\begin{aligned} x^2+y^2&=85 \\ 2x&=y+5 \end{aligned}\right\}$

20. Solve the following equations graphically :—

$x^2+y^2-4x-2y+1=0$, $2x-3y=3$.

VI. FRACTIONAL FUNCTIONS.

476. Infinity. Consider the fraction $\frac{a}{x}$ whose numerator has a *certain assigned value* but whose denominator x is a *variable quantity*; then it is plain that as x diminishes the value of the fraction $\frac{a}{x}$ increases.

For instance

$$\frac{a}{\cdot 1}=10a,\ \frac{a}{\cdot 001}=1000a,\ \frac{a}{\cdot 0000001}=10000000a, \text{ and so on.}$$

Thus we see that by giving to x a value small enough, we can make the fraction a/x larger than any assigned number, no matter how large that number may be; *i. e.* if x be made less than any assigned quantity, the value of the fraction a/x can be made larger than any quantity that could be named. This is expressed by saying that

when $x=0$, the value of $\frac{a}{x}$ is ∞,

for any number less than any assigned number is termed **zero** and is denoted by the symbol 0, and any quantity greater than a quantity that could be named is termed **infinity** and is denoted by the symbol ∞.

Similarly, if x is a quantity which gradually increases and finally becomes greater than any quantity that could be named, the fraction a/x becomes smaller than any assigned quantity. This is expressed by saying that

when $x=\infty$, the value of $\frac{a}{x}$ is 0.

Again, when the quotient a/x is positive, $a/0$ is said to be positively infinite $(+\infty)$; when a/x is negative, $a/0$ is said to be negatively infinite $(-\infty)$, and so on.

Ex. Draw the graph of $xy=8$. Shew that it consists of two infinite branches, one in the first quadrant and the other in the third.

The equation can be expressed in the form $y=\dfrac{8}{x}$.

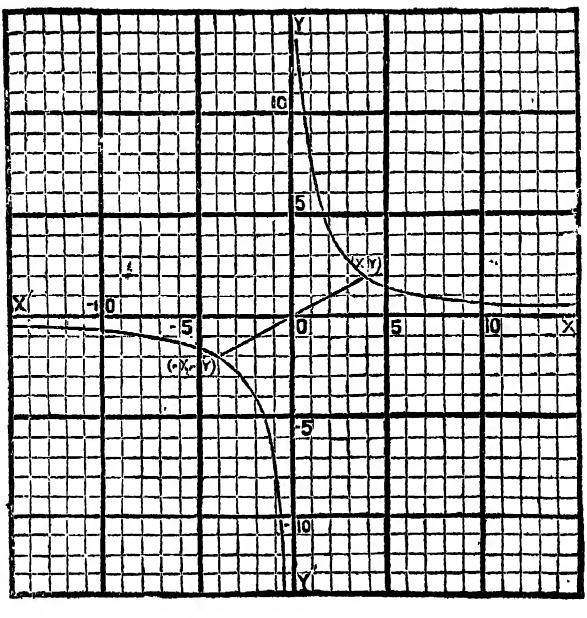
(i) It will be noticed that when $x=0, y=\infty$ and when $x=\infty, y=0$. Also y is positive when x is positive and negative when x is negative. Hence the graph must lie entirely in the first and third quadrants.

(ii) If (x, y) is any point on the curve, $(-x, -y)$ is also on the curve, for if $xy=8$, $(-x)(-y)=8$.

Hence the part of the curve in the third quadrant is exactly similar to the part in the first quadrant; or, any chord of the curve through the origin is bisected at the origin.

Tabulate the values of x and y as follows :—

x	± 0	± 1	± 2	± 4	± 6	± 8	± 12	± 16	...	$\pm \infty$
y	$\pm \infty$	± 8	± 4	± 2	$\pm 1\frac{1}{3}$	± 1	$\pm \frac{2}{3}$	$\pm \frac{1}{2}$	...	± 0



Plotting the above points, we see that the graph continually approaches the x-axis in such a way that for a sufficiently large positive value of x we obtain a point on the graph as near as we please to the x-axis but never actually reaching it until $x=\infty$. In the same way the curve approaches indefinitely near to the axis of y, but never meets it. Such a curve, having two branches, is called a **rectangular hyperbola.**

Such lines are called **asymptotes** of the curve.

Ex. Solve the following graphically :-

(i) $\left.\begin{aligned} x - y &= 3 \\ xy &= 28 \end{aligned}\right\}$ (ii) $\left.\begin{aligned} x^2+y^2 &= 65 \\ xy &= 28 \end{aligned}\right\}$

In each case we require the graph of $xy=28$. Proceeding as above, we find that the curve is a rectangular hyperbola lying the first and third quadrants.

In (i) $x-y=3$ is a straight line **QS** making intercepts 3 and -3 on the axes.

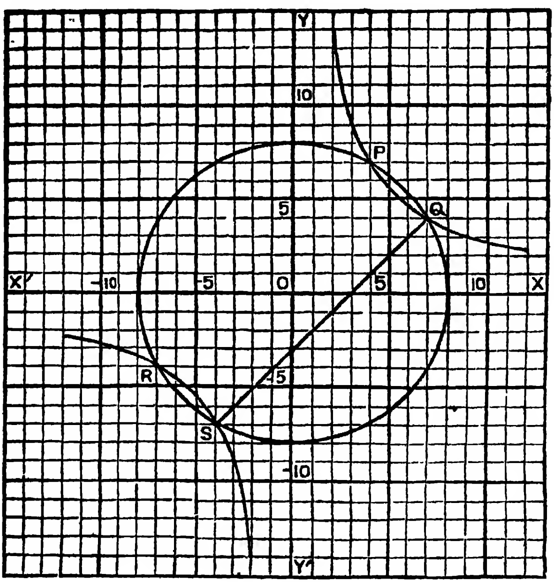


Fig. 26.

In (ii) $x^2+y^2=65$ is a circle. As the equation is satisfied by $x=4$, $y=7$, the graph can be drawn by finding the point **P** (4, 7) and describing a circle with centre **O** and radius **OP**.

The roots of (i) are the coordinates of **Q** and **S**; that is,

$x=7, y=4$; or $x=-4, y=-7$.

The roots of (ii) are the coordinates of **P**, **Q**, **R**, and **S**; that is,

$x=4, y=7$; $x=7, y=4$; $x=-7, y=-4$; $x=-4, y=-7$.

Exercise CLXXVIII.

1. Plot the graphs of the following.

(1) $xy=1$. (2) $xy=4$. (3) $xy=16$. (4) $xy=32$.

(5) $xy=12$. (6) $xy=35$. (7) $xy=36$. (8) $xy=40$.

2. Solve the following equations graphically :—

(1) $\left.\begin{array}{r} x \quad 15 \\ xy=36 \end{array}\right\}$ (2) $\left.\begin{array}{r} x-y=3 \\ xy=18 \end{array}\right\}$ (3) $\left.\begin{array}{r} x^2+y^2=25 \\ xy=12 \end{array}\right\}$

(4) $\left.\begin{array}{r} x+y=7 \\ xy=12 \end{array}\right\}$ (5) $\left.\begin{array}{r} x^2+y^2=20 \\ xy=8 \end{array}\right\}$ (6) $\left.\begin{array}{r} x-y=3 \\ xy=8 \end{array}\right\}$

(7) $\left.\begin{array}{r} x-2y=10 \\ xy=80 \end{array}\right\}$ (8) $\left.\begin{array}{r} x^2+y^2=16 \\ xy=6 \end{array}\right\}$ (9) $\left.\begin{array}{r} x+y=9 \\ xy=16 \end{array}\right\}$

VII. HARDER GRAPHICAL PROBLEMS.

477. The following are typical examples with their solutions.

Ex. 1. One tap will fill a cistern in 4 hours, and a second will fill it in 6 hours. How long will they take to fill the cistern running together?

Measure time horizontally along **OX** as shown in the diagram, and let **OP** (1 in.) drawn vertically along **OY** denote the capacity of the cistern.

Then **OA** is the graph of the first tap.

From **C**, the point where this meets the vertical 6 hour line, take **CD** = 1 in. upwards.

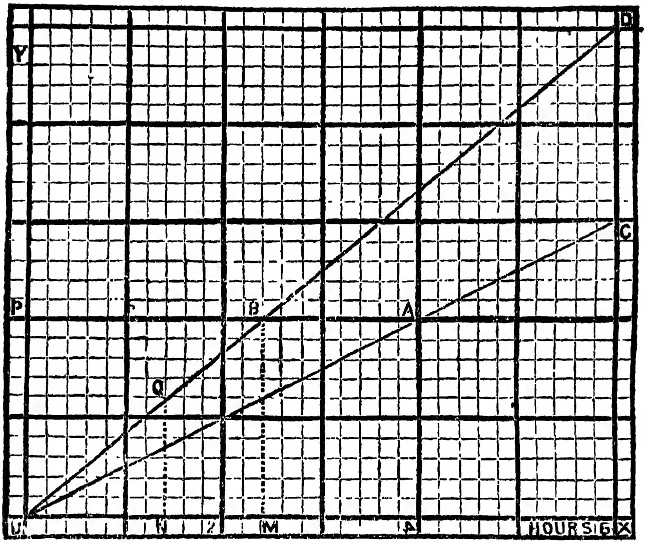


Fig. 27.

Then **OD** is the graph of the work done by the two taps. Now, examining any ordinate **QN**, we see that **QN** is the po on of the cistern filled by the two taps, running together, in time **ON**.

Since **BM** = **OP** ; ∴ **BM** is the reqd. time = 2·4 hours.

Ex. 2. A workman is engaged for 28 days at *Re.*1. 4*a.* pe day, but instead of receiving anything, is to pay 8*a.* a day on al days upon which he was idle ; he receives altogether *Rs.*26. 4*a.* ; for how many idle days did he pay ?

Let **OA** represent 28 days and **OB** represent 35 rupees.

Then **OC** is the graph of the money the man earns.

Draw **AD**, **OD** being 14, the graph of his fines, from the point **A**, instead of from **O**.

Now, examining any ordinate **PQN**, we see that

PN represents the money he earns in **ON** days,

and **QN**...............amount of his fines in **AN** days ;

Hence **PQ** represents the money he actually receives

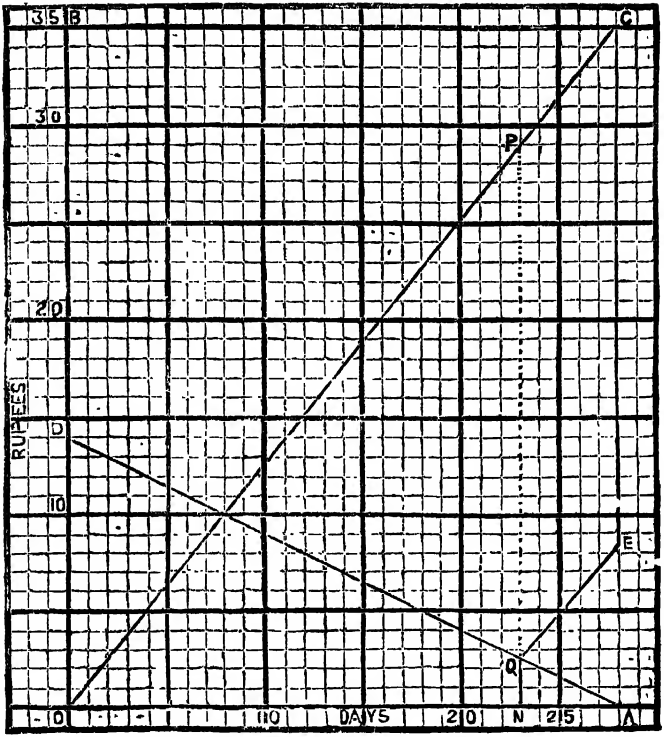


Fig. 28.

Thus, take $CE = 26\frac{1}{4}$ rupees, and draw EQ parallel to OC to meet AD at Q. Draw the ordinate PQN.

$$PQ = EC = 26\tfrac{1}{4}.$$

Hence ON represents the no. of days he is at work,

and AN the number of days he is absent.

From the diagram, $AN = 5$ days.

Ex. 3. A walks round a circular track one mile long in 20 minutes, and B motors round it in 5 minutes in the opposite direction,

but starting from the same point. Draw graphs to shew when and where they meet, distances to be measured in A's direction.

Measure time horizontally along OX as shewn in the diagram, and OE vertically along OY to represent one mile.

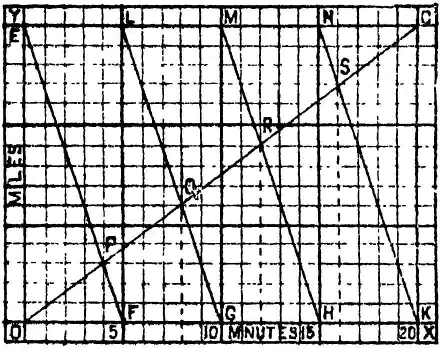


Fig. 29.

Then OC is A's graph for one mile.

Take E as B's starting point; then EF is his graph for the first 5 min.

Then B is at the starting point. Hence we take LG as his graph for the next 5 min.

Similarly, MH is his graph for the third 5 min. and NK is his graph for the fourth 5 min.

The points P, Q, R, S, where these lines meet, give us the times reqd., which are 4, 8, 12, 16 minutes from the start.

Also the distances are 352, 704, 1056, 1408 yds. from the start.

Ex. 4. A, B and C run a race of 180 yards. A and C start from the same point and at the same time, and A covers the distance in 40 seconds, beating C by 30 yards. B, with 12 yards' start, beats A by 6 seconds. Supposing the rates of running in each case to be uniform, find graphically the relative positions of the runners when B passes the winning post.

Measure time horizontally and distance vertically as shewn in the diagram.

Take O the starting point for A and C; then OE, representing 12 yards, on the vertical axis, denotes B's starting point.

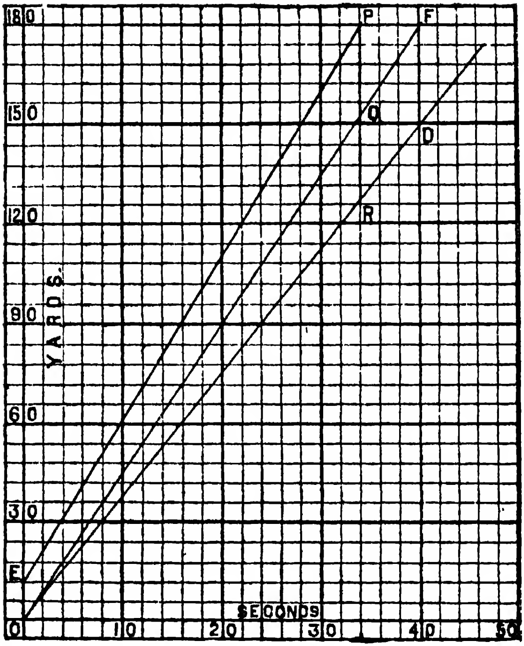


Fig. 30.

A's graph is drawn by joining O to the point F, which marks 40 seconds. From this point measure a vertical distance of 30 yards downwards to D. Then since FD represents 30 yards, D is C's position when A is at the winning post, and OD is C's graph.

Along the time-axis take a point P denoting 34 seconds ; then EP is B's graph.

Through P draw a vertical line to meet the graphs of A and C in Q and R respectively. Then Q and R mark the positions of A and C when B passes the winning post.

By inspection PQ and PR denote 28 and 54 yards respectively.

Thus B is 28 yards ahead of A and A is 54 yards ahead of C.

Ex. 5. A man walked a certain distance at the rate of 4 miles an hour, without delay ran part of the way back at the rate of 6 miles

an hour for half-an-hour, then waited for 12 minutes, and walked the remaining distance home in $\frac{1}{2}$ hour. The whole journey took him $2\frac{1}{2}$ hours. Find the distance.

Measure time horizontally and distance vertically as shewn in the diagram.

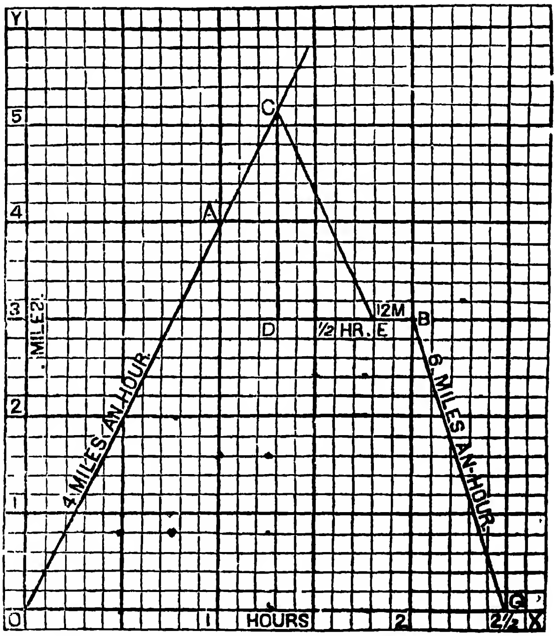


Fig. 31.

Now **OA** is the graph of 4 miles an hour, **BG** is the graph of 6 miles an hour (the distance run) and **BE** = 12 min. the waiting time.

Take **ED** = $\frac{1}{2}$ hour. Through **D** draw a vertical line to meet **OA** produced at **C**. Then **CE** is the graph of his walk at the end of the journey.

Hence the vertical distance of **C** from **OX** will give the required distance.

Thus, from the diagram, distance reqd. = 5·2 miles

Exercise CLXXIX.

1. Two taps, A and B, fill a cistern in 10 and 20 hours respectively. In what time will they fill it together?

2. Two pipes together can fill a cistern in 8 min., and one of them alone in 24 min. How long would the other alone take?

3. A man can do a piece of work in 5 days, and a boy in 10 days. What fraction of the work will they do working together for 2 days?

4. A alone can do a piece of work in 11 days, and B alone can do it in 17 days. How many complete days they would take to do it together?

5. A cistern has three pipes A, B, and C; A and B can fill it in 4 and 5 hours respectively and C can empty it in 2 hours. If these pipes are opened in order at noon, 1 o'clock, and 2 o'clock, find when the cistern will be empty.

6. A labourer is engaged for 30 days on condition that he receives 2*s*. 6*d*. for each day he works, and loses 1*s*. for each day he is idle: he receives £2. 7*s*. in all. How many days does he work, and how many days is he idle? (C. E. 1869).

7. A travels at the rate of 7 miles an hour, and B at 2 miles an hour round a circular track one mile long, starting at the same time from the same point in the same direction. Find the first three times when A passes B.

8. A man walked from A to B at the rate of 3 miles an hour, bicycled back without delay for 2 miles at the rate of 8 miles an hour, and walked the remaining distance home at the rate of 2 miles an hour, taking 4 hours over the journey. How far is it from A to B?

9. A walks at the rate of 2 miles an hour, B at the rate of 3 miles an hour, round a circular track 1 mile long, starting from the same point, and at the same time in opposite directions. Find the times of their first four meetings.

10. A man receives *Rs*.2 for every day that he works, but is fined 12*a*. for every day he is absent. After 25 days he receives *Rs*.28. How many days was he absent?

11. A man receives *Re*.1. 12*a*. for every day that he works, but is fined 8*a*. for every day he is absent. After 20 days he receives the same wages that he would have earned by working steadily for 11 days. How many days was he absent from work?

12. Two sums of money are put out at simple interest at different rates of interest. In the first case the amounts at the end of 6 and 15 years are *Rs.*260 and *Rs.*350 respectively. In the second case the amounts for 5 and 20 years are *Rs.*330 and *Rs.*420. Find graphically the year in which the principal with accrued interest will amount to the same in the two cases. Also from the graphs read off the value of each principal.

13. A can beat B by 20 yards in 120 and B can beat C by 10 yards in 50. Supposing their rates of running to be uniform, find graphically how much start A can give C in 120 yards so as to run a dead heat with him. If A, B and C start together, where are A and C when B has run 80 yards?

14. A man walked a certain distance at the rate of 4 miles an hour, and then ran part of the way back at the rate of 6 miles an hour, walking the remaining distance home in 15 minutes. The whole journey took him 1 hour 20 min. How far did he run, and what is the distance?

15. A tap which would fill a cistern in 3 hours, and a plug which would empty it in 7 hours, are both opened at the same instant, when the cistern is empty. How long will they take to fill the cistern?

CHAPTER XXIII.

PROGRESSIONS.

I. ARITHMETICAL PROGRESSION.

478. Quantities are said to be in **Arithmetical Progression,** when they proceed by a *common difference.*

Thus, each of the following series is in *Arithmetical Progression.* (A. P) :—

$$1, 3, 5, 7, \ldots\ldots$$
$$8, 4, 0, -4, \ldots$$
$$a, a+d, a+2d, a+3d, \ldots\ldots$$

the *common differences* being $2, -4, d,$ respectively.

479. The **common difference** of the terms of an arithmetical progression is found *by subtracting any term of the series from that which follows it.*

Thus, in the series $a, a+d, a+2d, a+3d, \ldots\ldots$
the common difference $=(a+d)-a=(a+2d)-(a+d)=\ldots\ldots=d$

480. *To find the* nth *term of an A. P.*

Let $a=$ 1st term, and $d=$ common difference.

Then the series will be $a, a+d, a+2d, a+3d, \ldots\ldots$

where the coefficient of d in any term is just *less by one* than the number of the term of the series.

Thus, the 5th term $=a+4d$;
19th term $=a+18d$;
30th term $=a+29d$;

and generally, the pth term $=a+(p-1)d$.

Hence, if n be the number of terms, and if l denote the last or nth term, we have

$$l=a+(n-1)d.$$

Ex. 1. Find the 10th term of the series $1, 5, 9, \ldots\ldots$

Here $a=1$, $d=5-1=4$, $n=10$.

$\therefore$ 10th term $=1+(10-1)\times 4=1+9\times 4=37$.

Ex. 2. Find the 13th term of the series $-48, \ -44, \ -40, \ldots\ldots$

Here $a=-48$, $d=-44-(-48)=4$, $n=13$.

$\therefore$ 13th term $=-48+(13-1)\times 4=-48+12\times 4=0$.

Ex. 3. Which term is $89\frac{1}{2}$ of the *A. P.* $10, \ 11\frac{1}{2}, \ 13, \ \ldots\ldots$?

Let $89\frac{1}{2}$ be the pth term of the series;

then, since the common difference $=11\frac{1}{2}-10=1\frac{1}{2}$.

$\therefore \ 89\frac{1}{2}=10+(p-1)\times 1\frac{1}{2}$; $\therefore \ 159=3(p-1)$;

$\therefore \ p-1=53$, and $\therefore \ p=54$.

Hence the required term is the 54th term.

Ex. 4. Is 576 a term of the series $11, \ 17, \ 23, \ldots$?

Here, the common difference $=17-11=6$.

Suppose, if possible, that 576 is the pth term of the series.

$\therefore \ 576=11+(p-1)\times 6$; $\quad \therefore \ p-1=\frac{565}{6}=94\frac{1}{6}$.

and $\therefore \ p=94\frac{1}{6}+1=95\frac{1}{6}$.

The value of p being fractional, is inadmissible.

Hence 576 is not a term of the given series.

481. Again, beginning at the end of the series, we may write down the terms by successively *subtracting d.*

Thus, if the last term is l, the term *before* it is $l-d$, the one before that is $l-2d$, the one before that is $l-3d$, and so on.

482. An Arithmetical Progression may completely be determined if *any two* of its terms are given, for then we shall have two equations, each of the first degree, which solved will give the first term and the common difference.

Ex. The 9th and 35th terms of an *A. P.* are respectively $\frac{1}{4}$ and $39\frac{1}{4}$; find the series and write down the 24th term.

Let a = first term and d = common difference.

Then the 9th term $=a+8d$ and the 35th term $=a+34d$.

Hence $a+8d=\frac{1}{4}$ and $a+34d=39\frac{1}{4}$, so that $a=-11\frac{3}{4}$, $d=\frac{3}{2}$.

Thus the series is $-11\frac{3}{4}, \ -10\frac{1}{4}, \ -8\frac{3}{4}, \ -7\frac{1}{4} \ldots\ldots$

Also the 24th term $=a+23d=-11\frac{3}{4}+23\times\frac{3}{2}=-11\frac{3}{4}+34\frac{1}{2}=22\frac{3}{4}$.

Exercise CLXXX.

1. Find the

(1) 5th and 31st term in the series 13, 10, 7,...

(2) 13th and 20th term in the series $-3, -2, -1, \ldots$

(3) 9th and 102nd term in the series $1, -\frac{1}{2}, -2, \ldots$

(4) 16th and 51st term in the series $-11, 4, 19, \ldots$

(5) 13th and 50th term in the series $-3\frac{1}{2}, -2\frac{3}{4}, -2, \ldots$

(6) 40th and 90th term in the series $-2{\cdot}8, 0, 2{\cdot}8, \ldots$

2. Find the last term in the following series :—

(1) $16, 15\frac{4}{5}, 15\frac{3}{5}, \ldots$ to 80 terms.   (2) $13\frac{1}{2}, 9, 4\frac{1}{2} \ldots$ to 15 terms.

(3) 11, 17, 23,...to 95 terms.   (4) $a, 2a, 3a, \ldots$ to 25 terms.

(5) $6a-3b, 4a-2b, 2a-b, \ldots\ldots\ldots$ to 25 terms.

(6) $5a+3b, 3a+2b, a+b, \ldots\ldots\ldots$ to 11 terms.

3. Find the nth term in each of the following series :—

(1) $9+\frac{25}{3}+7\frac{2}{3}+\ldots\ldots\ldots$ (C. F. A. 1884).

(2) $\frac{1}{n}+\frac{n+1}{n}+\frac{2n+1}{n}+\ldots\ldots\ldots$ (C. F. A. 1886).

(3) $(na-b)+(n-1)a+\{(n-2)a+b\}+\ldots\ldots\ldots$

(4) $\frac{a-n}{n}+\frac{a-2n}{n}+\frac{a-3n}{n}+\ldots$

(5) $\left(\frac{1}{a}-\frac{n}{x}\right)+\left(\frac{1}{a}-\frac{n-1}{x}\right)+\left(\frac{1}{a}-\frac{n-2}{x}\right)+\ldots$

4. Find the

(1) 30th and nth term of the series $-27-20-13-\ldots\ldots$

(2) nth and $(2n-1)$th term of the series $\frac{n-1}{n}+\frac{n-2}{n}+\frac{n-3}{n}+$

5. Which term of the series $\cdot 6, 1{\cdot}2, 1{\cdot}8, \ldots\ldots$ is $7{\cdot}2$?

6. Which term of the series $3\frac{1}{2}, 1, -1\frac{1}{2}, \ldots\ldots$ is $-21\frac{1}{2}$?

7. Is 544 a term of the series 64, 96, 128, ...

8. Is -389 a term of the series 9, 5, 1, ...

9. If the nth term of the two series $2+3\frac{7}{8}+5\frac{3}{4}+\ldots\ldots$ and $187+184\frac{1}{4}+181\frac{1}{2}+\ldots\ldots$ are the same, find n.

10 The 18th term of an A P. of which 1 is the first term is 35; find its 50th term.

11. The first term of an A. P is 3 and the 13th term 55; find the common difference.

12. The first term of an A P. is 17 and the 100th term -16; find the 30th term.

13. The 2nd and 31st terms of an A P. are $7\frac{1}{4}$ and $\frac{1}{2}$ respectively, find the 59th term

14 Find the first term and the common difference of an A. P. whose 7th and 17th terms are 1 and $-6\frac{1}{2}$

15 The 12th, 85th and last term of an A P are 38, 257 and 395 respectively. Find the first term and the number of terms

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483 Sum of the Series *To find the sum of a given number of terms in Arithmetical Progression, the first term and the common difference being given*

Let a denote the first term, d the common difference, and n the number of terms Also let l denote the last term and S the required sum. Then

$$S=a+(a+d)+(a+2d)+\ldots+(l-2d)+(l-d)+l\ ;\ldots(1)$$

Again, writing the series in the reverse order, we have

$$S=l+(l-d)+(l-2d)+\ldots+(a+2d)+(a+d)+a\ldots(2)$$

Now, adding together (1) and (2), we have

$$2S=(a+l)+(a+l)+(a+l)+\ldots\text{to } n \text{ terms}$$
$$=n(a+l)$$

$$\therefore\ S=\frac{n}{2}(a+l)\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Also $l=a+(n-1)d\ \ldots\ldots\ldots\ldots(2)$

$$\therefore\ S=\frac{n}{2}\{2a+(n-1)d\}\ldots\ldots\ldots(3)$$

Note. To find the sum of an A. P of a given number of terms, formula (1) should be used when the first and last terms are given, but (3) should be used when the first term and common difference are given.

Ex. 1 Find the sum of 20 terms of the series 14, 64, 114, ...

Here, $a=14$, $d=64-14=50$, $n=20$.

$\therefore\ S=\frac{20}{2}(2\times14+19\times50)=10\times978=9780.$

Ex. 2. Find the 9th term and the sum of 9 terms of $7, 5\frac{1}{2}, 4, \ldots$

Here, $a=7$, $d=5\frac{1}{2}-7=-\frac{3}{2}$, $n=9$.

$\therefore\ l=7+(9-1)\times-\frac{3}{2}=7-8\times\frac{3}{2}=7-12=-5.$

and $S=(7-5)\times\frac{9}{2}=2\times\frac{9}{2}=9.$

Ex. 3. Find the sum of 7 terms of the series $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}, \ldots$

Here, $a=\frac{1}{2}$, $d=\frac{1}{3}-\frac{1}{2}=-\frac{1}{6}$, $n=7$.

$\therefore\ S=\frac{7}{2}(2\times\frac{1}{2}+6\times-\frac{1}{6})=\frac{7}{2}(1-1)=\frac{7}{2}\times 0=0.$

In this case, the series, continued, is $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 0, -\frac{1}{6}, -\frac{1}{3}, -\frac{1}{2}$ ....., where the first 7 terms together amount to zero.

Exercise CLXXXI.

1. Find the last term and sum of the following series :

(1) $2, 5, 8, \ldots$ to 10 terms.

(2) $3, 9, 15, \ldots$ to 13 terms.

(3) $7, 5, 3, \ldots$ to 24 terms.

(4) $4, -3, -10, \ldots$ to 20 terms.

(5) $1, \frac{3}{2}, 2, \ldots$ to 12 terms.

(6) $5, 8, 11, \ldots$ to 17 terms.

(7) $\frac{1}{3}, \frac{7}{15}, \frac{9}{15} \ldots$ to 6 terms.

(8) $\frac{1}{3}, \frac{5}{6}, \frac{4}{3}, \ldots$ to n terms.

(9) $40, 38, 36, \ldots$ to 36 terms.

(10) $\frac{1}{2}, \frac{3}{4}, 1, \ldots$ to 100 terms

2. Find the sum of the following series :—

(1) $2, 4, 6, \ldots$ to 16 terms.

(2) $1, 3, 5, \ldots$ to 20 terms.

(3) $3, 9, 15, \ldots$ to 11 terms.

(4) $1, 8, 15, \ldots$ to 100 terms

(5) $\frac{2}{3}, \frac{7}{15}, \frac{4}{15}, \ldots$ to 21 terms.

(6) $\frac{1}{2}, -\frac{2}{3}, -\frac{11}{6}, \ldots$ to 13 terms.

(7) $-5, -3, -1, \ldots$ to 8 terms.

(8) $1, \frac{6}{7}, \frac{5}{7}, \ldots$ to 15 terms.

(9) $4, -3, -10, \ldots$ to 10 terms.

(10) $\frac{1}{2}, \frac{3}{4}, 1, \ldots$ to 10 terms.

(11) $1, 2\frac{2}{3}, 4\frac{1}{3}, \ldots$ to 20 terms.

(12) $\frac{3}{5}, -\frac{14}{15}, -\frac{51}{15}, \ldots$ to 10 terms.

(13) $12, 9\frac{1}{2}, 7, \ldots$ to 35 terms.

(14) $-17, -12, -7, \ldots$ to 11 terms.

(15) $\frac{1}{12}, \frac{1}{8}, \frac{1}{6}, \ldots$ to 8 terms and $3n$ terms.

(16) $a, 3a, 5a, \ldots$ to n terms.

(17) $2\frac{1}{3}, 3\frac{5}{12}, 4\frac{1}{2}, \ldots$ to n terms.

(18) $\frac{1}{2}(a-b), b, \frac{1}{2}(5b-a) \ldots$ to 19 terms.

(19) $a-3b, 2a-5b, 3a-7b, \ldots$ to 40 terms and n terms.

(20) $3x+4y, 5x+2y, 7x, \ldots$ to n terms and $(2n+1)$ terms

(21) $17\frac{1}{2}, 14\frac{1}{6}, 10\frac{5}{6}, \ldots$ to 24 terms. (B. P. E. 1883).

(22) $\frac{1}{a}, \frac{2}{a}, \frac{3}{a}, \ldots$ to 20 terms.

(23) $(x+y)^2+(x^2+y^2)+(x-y)^2+\dots\dots$ to n terms
(C E A 1880, P I E 1891)

(24) $3+4+8+9+13+14+18+19+\dots$ to 20 terms (C I A 1881)

(25) $n-\frac{1}{n},\ 3n-\frac{2}{n},\ 5n-\frac{3}{n},$ to n terms

3 Find the sum of n terms of the A P whose

(1) 1st and 100th terms are 17 and -16 respectively

(2) 12th and 50th terms are 5 and $9\frac{3}{4}$ respectively

4 The 8th term of an A P is double the 13 term, prove that the 4th term is double the 11th term

5 If there are 6 terms in an A P, the sum of the first and last is equal to the sum of the 3rd and 4th (P E I 1893)

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484 By means of the equations given in Art 483, when any three of the quantities a, d, l, n, s are given, we may find the others

Ex 1 What number of terms of the series 10, 8, 6, .. must be taken to make 30? and what number to make 28?

(i) $s=30$, $a=10$, $d=-2$

$$\therefore\ 30=\frac{n}{2}\{20-2(n-1)\},\text{ from (3)}$$

$$=n(11-n),\ \therefore\ n^2-11n+30=0$$

By solving this quadratic, we shall obtain $n=5$ or 6, either of which satisfies the question, since the *sixth* term of the series is zero

(ii) $s=28$, $a=10$, $d=-2$

$$\therefore\ 28=\frac{n}{2}\{20-2(n-1)\},\text{ from (3)}$$

$$=n(11-n),\ \therefore\ n^2-11n+28=0.$$

Now, from the above quadratic, we obtain $n=4$ or 7, either of which satisfies the question, since the 5th, 6th, and 7th terms of the series, *viz* 2, 0, -2 together = zero

Ex 2 How many terms of the series 3, 5, 7,..... make up 24?

Here, $s=24$, $a=3$, $d=2$

$$\therefore\ 24=\frac{n}{2}\{6+2(n-1)\}=n(n+2)\ ;\ \therefore\ n^2+2n-24=0$$

Solving this quadratic, we shall obtain $n=4$ or -6, of which the first only is admissible by the conditions of the question

Ex 3 The sum of 10 terms of an A P, whose first term is 3, is $41\frac{1}{4}$ find the common difference

Here, $s=41\frac{1}{4}$, so that the equation (3) of Art 480 gives

$$41\frac{1}{4}=\frac{10}{2}(6+9d) \text{ or } 45d=11\frac{1}{4}, \therefore d=\frac{1}{4}$$

Ex 4 The first term of a series in A P is 3, the last term 90 and the sum 1395 find the number of terms, and the common difference

If n be the number of terms, then from (we have

$$1395=\frac{n}{2}(3+90), \text{ whence } n=30$$

If d be the common difference, then we have

$$= \text{the 30th term}$$
$$-3+29d \quad \therefore d=3$$

Exercise CLXXXII.

1 The first term of a series in A P is n^2-n+1 and the common difference 2, find the sum of terms

2. The first term of an arithmetic series is 2, the common difference 7, and the last term 79 find the number of terms

3. The sum of 15 terms of an arithmetic series is 600 and the common difference is 5, find the first term

4. The first term of an A P is $13\frac{1}{7}$ the common difference $-\frac{1}{7}$, and the last term $\frac{2}{3}$, find the number of terms

5. The sum of 11 terms of an A P is $14\frac{4}{7}$, and the common difference is $\frac{1}{7}$, find the first term

6. Find how many terms of the series 13, $12\frac{1}{3}$, $11\frac{2}{3}$,......must be taken to make zero

7. The sum of a certain number of terms of the series $-7-5\frac{3}{4}-4\frac{1}{2}-$ ... is 133; find the number of terms

8. How many terms of the series 7, 6, 5,. ...must be taken to make $-24\cdot8$?

9 Find n in each of the following cases —

(1) $a=32,\ d=-3,\ s=-2075$ (2) $a=3,\ d=6,\ s=507.$

(3) $a=1,\ d=1\frac{1}{4},\ s=94\frac{1}{2}$ (4) $a=2,\ d=-7,\ s=-438.$

10. How many terms of $1\cdot2+2\cdot4+3\cdot6+$ &c amount to $163\cdot2$?

11 The first term in an A P is 1, the number of terms is 33; what must the common difference be, in order that the sum may be $149\frac{1}{2}$? (C F A 1864).

12. How many terms of the series $5+7+9+$&c., must be taken in order that the sum may be 480? (P. I. E. 1889).

13. The 5th term of an A. P. is -5, and the 11th term -23; find the 30th term and the sum of 30 terms.

14. The 11th term of an A. P. is 36 and the 20th term 27; find the first term and the sum of 25 terms.

15. The sum of the 8th and 4th terms of an A. P. is 24 and the sum of the 15th and 19th is 68. What is the series?

16. The sum of 24 terms of an A. P. is -18 and the 7th term is 2. Find the series and the sum to 48 terms.

17. Find the last term in the series 201, 204, 207,..., when the sum is 8217.

18. If the sum of n terms of an A. P. be n^2 and the common difference 2, find the first term.

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485. The following are ty[illegible] examples with their solutions.

Ex. 1. Find the $A.P.$ which is such that the sum of n terms is always $\frac{1}{2}n(3n+1)$ for all values of n.

1st Method. Putting $n=1$, the sum of one term, *i.e.*, 1st term $=2$.

Putting $n=2$, the sum of two terms $=\frac{1}{2}\times 2(3.2+1)=7$, *i.e.*, the 1st term + the 2nd term $=7$, *i.e.*

$$\text{the 2nd term} = 7 - \text{the 1st term} = 7-2=5.$$

The $A.P.$ is thus 2, 5, 8,....

Note. This method is open to objection, for it is based on the assumption that a series must be in $A.P.$, which may not sometimes happen.

2nd Method. Let S_r denote the sum to r terms of the series, so that S_{r-1} will denote the sum to $(r-1)$ terms. Then

$$\text{the } r\text{th term} = S_r - S_{r-1}.$$

Now $S_r = \frac{3}{2}r^2+\frac{1}{2}r$, and $S_{r-1}=\frac{3}{2}(r-1)^2+\frac{1}{2}(r-1)$.

$$\therefore\ S_r - S_{r-1} = \tfrac{3}{2}\{r^2-(r-1)^2\}+\tfrac{1}{2}\{r-(r-1)\}$$
$$=\tfrac{3}{2}(2r-1)+\tfrac{1}{2}=3r-1.$$

Then making $r=1, 2, 3,$ &c. the result follows.

Ex. 2. In an $A.P.$ shew that the sum of any two terms equidistant from the beginning and end is constant.

Let a = the first term, d = the common difference and l = the last term, so that the rth term from the end $= l-(r-1)\,d$. Art. 481.

Also the rth term from the beginning $= a+(r-1)d$.

Hence the sum of the rth term from the beginning and the rth term from the end

$$= a+(r-1)d+l-(r-1)d$$

$= a+l$, which is constant, being independent of r.

Ex. 3. Find the middle term or terms of a series in *A.P.* of n terms, whose first and last terms are a and l respectively.

First. Let n be *odd*, and of the form $2p+1$.

Then, obviously, there will be *one* middle term, *viz.* the $(p+1)$th, there being p terms before, and p terms after it.

Now $2p+1=n$; $\therefore p=\frac{1}{2}(n-1)$ and $p+1=\frac{1}{2}(n+1)$.

Hence the middle term or the $\frac{1}{2}(n+1)$th term

$$=a+\left(\frac{n+1}{2}-1\right)d=a+\frac{n-1}{2}d=\tfrac{1}{2}\{2a+(n-1)d\}=\tfrac{1}{2}(a+l),$$

(where d is the common difference).

Secondly. Let n be even, and of the form $2p$.

Then obviously, there will be two middle terms, *viz.* the pth and the $(p+1)$th, there being $p-1$ terms before and $p-1$ terms after them.

Now, $2p=n$; $\therefore p=\frac{1}{2}n$ and $p+1=\frac{1}{2}(n+2)$.

Hence the two middle terms, *i.e.*, the pth and $(p+1)$th terms are $a+\left(\frac{n}{2}-1\right)d$ and $a+\left(\frac{n+2}{2}-1\right)d$ respectively,

(where d is the common difference),

i.e. $\frac{1}{2}\{2a+(n-2)d\}$ and $\frac{1}{2}(2a+nd)$ respectively,

i.e. $\dfrac{na+(n-2)l}{2(n-1)}$ and $\dfrac{(n-2)a+nl}{2(n-1)}$, $\left(\text{for } d=\dfrac{l-a}{n-1}\right)$

Note. The sum of the two middle terms

$$=\tfrac{1}{2}\{4a+(2n-2)d\}=2a+(n-1)d=a+l.$$

Ex. 4. In the two series 2, 5, 8, &c., and 3, 7, 11, &c., each continued to 100 terms, find how many terms are identical.

Let the rth term of the first series be identical with the pth term of the second series.

Then, by question, $2+(r-1)\times 3=3+(p-1)\times 4$.

$$\therefore\ 3r-1=4p-1, \text{ or } 3r=4p;\ \therefore\ r=p+\tfrac{1}{3}p.$$

Let $\frac{1}{3}p=m$, then $p=3m$ and $\therefore\ r=4m$.

Now, since r and p must each be a positive integer not greater than 100, therefore m must not be greater than 25.

Hence 25 terms are identical.

Ex. 5. If the sum of n terms of one *A.P.* be to the sum of n terms of another as $1+2n:5+3n$, find the ratio of their 4th terms.

Let a = first term and b = com. diff. of the 1st *A.P.*

and A = ... and B = ...2nd... ..

Then $\dfrac{\frac{1}{2}n\{2a+(n-1)b\}}{\frac{1}{2}n\{2A+(n-1)B\}}=\dfrac{2a+(n-1)b}{2A+(n-1)B}=\dfrac{1+2n}{5+3n}$.(1)

Now, to obtain the ratio of the 4th terms, we must find the value of $\dfrac{a+3b}{A+3B}$, *i.e.*, of $\dfrac{2a+6b}{2A+6B}$.

Therefore, making $n=7$ in (1), we have

$$\frac{2a+6b}{2A+6B} \text{ or } \frac{a+3b}{A+3B}=\frac{1+14}{5+21}=\frac{15}{26}.$$

Ex. 6. Find the sum of the series in the nth group of

$$4+(6+8)+(10+12+14)+(16+18+20+22)+\ldots\ldots$$

The no. of terms of the series 4, 6, 8, 10,...in the first $(n-1)$ groups is evidently the sum of the series $1+2+3+\ldots$to $(n-1)$ terms $=\frac{1}{2}n(n-1)$. Similarly, the no. of terms of the above series in the first n groups $=\frac{1}{2}n(n+1)$.

Now, the sum of the series 4, 6, 8, 10...to $\frac{1}{2}n(n+1)$ terms

$$=\tfrac{1}{4}n(n+1)[2\times 4+\{\tfrac{1}{2}n(n+1)-1\}\times 2]=\tfrac{1}{4}n^2(n+1)^2+\tfrac{3}{2}n(n+1).$$

Similarly, the sum to $\frac{1}{2}n(n-1)$ terms

$$=\tfrac{1}{4}n^2(n-1)^2+\tfrac{3}{2}n(n-1), \text{ (writing } n-1 \text{ for } n).$$

Hence the sum of the series in the nth group

$$=\tfrac{1}{4}n^2\{(n+1)^2-(n-1)^2\}+\tfrac{3}{2}n\{(n+1)-(n-1)\}$$

$$=\tfrac{1}{4}n^2\times 4n+\tfrac{3}{2}n\times 2=n(n^2+3).$$

Exercise CLXXXIII.

1. If the first term of an arithmetic series be $=1$, and the common difference $=m$, the sum of n terms of the series is $\frac{1}{2}\{mn^2-(m-2)n\}$.

2. Prove that the sum of the rth and $(n-r)$th terms of an *A. P.* is equal to the sum of the first and $(n-1)$th terms.

3. If the sum of n terms of an *A. P.* is always equal to n^2, for all values of n, find the first term and the common difference.

4. The sum of the latter half of $2n$ terms of any Arithmetical series $=$ one-third of the sum of $3n$ terms of the same series. Prove it. (C. F. A 1876).

5. The sum of n terms of an *A. P.* is $2n^2$; find the first term and the common difference. (C. F A. 1878).

6. If the sum of n terms of an *A. P.* be pn^2+qn, for all values of n, prove that its nth term is always $(2n-1)p+q$.

7. The sum of n terms of two series in *A. P.* are as $1+3n : 17-2n$; compare their fifth terms and their rth terms.

8. If the mth term of an *A. P.* be n and the nth term m; of how many terms is the sum $\frac{1}{2}(m+n)(m+n-1)$ and what is the last term?

9. The pth term of an *A. P.* is a and the qth term is b. Show that the sum of the first $(p+q)$ terms is

$$\frac{p+q}{2}\left\{a+b+\frac{a-b}{p-q}\right\}. \quad \text{(M. T. A. 1887).}$$

10. If the sum of the first n terms of an *A. P.* be one-third of the sum of the next n terms, prove that the common difference is double of the first term.

11. If the $(p-q)$th and $(p+q)$th terms of an *A. P.* be m and n respectively, find the pth and qth terms.

12. The sum of the first ten terms of an *A. P.* is to the sum of the first five terms as $13 : 4$; find the ratio of the first term to the common difference.

13. The sum of n terms of one *A. P.* is to the sum of n terms of another as $2n+1 : 3n-1$; find the ratio of their 9th terms.

14. If the sum of n terms of an *A. P.* be $\frac{1}{12}n(5-3n)$, find the 7th term.

15. The sum of m terms of an $A.\ P.$ is n, and the sum of n terms is m. Find the sum of $(m+n)$ terms and also the sum of $(m-n)$ terms.

16. Sum to n terms the series whose rth term is $5r+4$.

17. The sums of n terms of two arithmetic series are as $3n+31 : 5n-3$; shew that their ninth terms are the same.

18. Find the sum of 11 terms of an $A.\ P.$, of which 121 is the middle term.

19. Find the first term and common difference of an $A.\ P.$ in which the sum of n terms is equal to $\frac{1}{7}n^2+5n$.

20. Find the sum of the series in the nth group of

(1) $2+(7+12)+(17+22+27)+(32+37+42+47)+\ldots\ldots\ldots\ldots$

(2) $1+(8+15)+(22+29+36)+(43+50+57+64)+\ldots\ldots\ldots\ldots$

(3) $(1+3)+(5+7+9+11)+(13+15+17+19+21+23)+\ldots\ldots$

21. The series $3+9+15+\ldots$and $2+7+12+\ldots$extend each to 50 terms; find how many terms are the same in both.

22. The series $3+8+13+\ldots$and $4+6+8+\ldots\ldots$extend each to 100 terms; find how many terms are the same in both.

23. Find the sum of 15 terms of an $A.P.$ of which the 8th is 6.

24. The sum of the first and fourth terms of an $A.P.$ is 19 and the sum of the third and sixth terms is 31. What is the first term?

25 The first and last of $(2n+1)$ terms of an $A.P.$ are a and h. Write down the sum and the middle term of the series.

26. Find the rth term of a series the sum of whose first n terms is $32n^2$ for all values of n.

27. If P, Q, R be the pth, qth, rth terms of an $A.P.$, shew that

$$(q-r)P+(r-p)Q+(p-q)R=0.$$

28. Find the sum of the first n numbers of the form $3r+1$.

29. The first two terms of an $A.P.$ are $1\frac{1}{2}$ and $2\frac{1}{3}$. How many terms must be taken that the sum may be 171?

30. If s be the sum of an Arithmetic series whose first term is a, and common difference is $2a$, find the number of terms and the last term.

486. Arithmetic Means When three quantities are in Arithmetical Progression, the middle one is called the **Arithmetic Mean** of the other two.

Thus, if a, x, b are in *A.P.*, x is called the *Arithmetic Mean* (*A. M.*) between a and b.

By definition of *A.P.* given in Art. 479, we have

$$x-a=b-x\ ;\ \therefore\ 2x=a+b \text{ and } \therefore\ x=\tfrac{1}{2}(a+b).$$

Thus, *the Arithmetic Mean of any two quantities is half their sum.*

487. When any number of quantities are in arithmetical progression, all the intermediate terms are called the **Arithmetic Means** between the first and last terms.

Thus, to insert any number of *arithmetic means* between two given quantities, is the same as to determine an *A.P.* whose first and last terms and also the number of terms are given.

488. *To insert* n *arithmetic means between two quantities* a *and* b.

Let d be the common difference of the required *A.P.*

Here, we have to find an *A.P.* of $(n+2)$ terms, of which a is the first, b is the last, so that b is the $(n+2)$th term of the *A.P.*

Now, the $(n+2)$th term of a series, whose first term is a and whose common difference is d, is

$$a+(\overline{n+2}-1)d, \text{ i.e. } a+(n+1)d. \quad \text{(Art. 480)}$$

$$\therefore\ a+(n+1)d=b, \text{ or } (n+1)d=b-a\ ;$$

$$\therefore\ d=\frac{b-a}{n+1}.$$

The means may now be easily determined; for they are

$$a+d,\ a+2d,\ a+3d, \ldots\ldots\ldots\ldots a+nd.$$

$$\text{The } p\text{th mean}=a+pb=a+p\left(\frac{b-a}{n+1}\right)=\frac{(n-p+1)a+pb}{n+1}.$$

Thus, on calculation, the means will be found to be

$$\frac{na+b}{n+1},\ \frac{(n-1)a+2b}{n+1},\ \frac{(n-2)a+3b}{n+1}, \ldots\ldots\ldots \frac{a+nb}{n+1}.$$

Ex. 1. Insert 3 arithmetic means between 6 and 26.

Here, we have to find *three* numbers between 6 and 26, so that the *five* may be in *A.P.* This case then reduces itself to finding d, when $a=6$, $b=26$ and $n=5$.

We have $26=6+4d$; whence $d=5$.

Thus the means required are 11, 16, 21.

Ex. 2. Find the number of Arithmetic means between 1 and 19, when the first mean is to the last as 1 to 4.

Let n be the number of means, and $d=$ the com. difference.

Then the 1st mean $=1+d$ and the last mean $=19-d$.

Hence, by the question,

$$1+d:19-d=1\,.\,4;\ \therefore\ 4+4d=19-d;\ \therefore\ d=3.$$

$$\text{But } d=\frac{b-a}{n+1}=\frac{18}{n+1};\ \therefore\ \frac{18}{n+1}=3,\text{ and }\therefore\ n=5.$$

Exercise CLXXXIV.

1 Write down the arithmetic mean of

(1) 7 and 13. (2) 9 and -9. (3) $x+y$ and $x-y$.

2. Insert 4 arithmetic means between 2 and 17.

3. Insert 9 $A.M$'s between 3 and 9 and 7 between -13 and 3.

4. Insert 4 $A.M.$'s between 2 and -18. and 8 between -3 and $-\frac{3}{4}$.

5. Insert 10 arithmetic means between -7 and 114.

6. Insert 9 $A.M.$'s between (i) $-2\frac{3}{4}$ and $4\frac{3}{4}$, and (ii) $-3\frac{2}{5}$ and $2\frac{3}{5}$.

7. Find 4 arithmetic means between 4 and 324. (C. F. A. 1890).

8 Insert 10 arithmetic means between $5a-6b$ and $5b-6a$.

9. Find the sum of n arithmetic means inserted between a and b.

10. There are n Arithmetic means between 3 and 17; and the 5th mean : last mean :: 1 : 2; find n.

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489. Natural Numbers. The numbers 1, 2, 3, 4,...are called the **natural numbers**.

Ex. 1. Find the sum of $1+2+3+4+\ldots\ldots$to n terms.

Here, the nth term$=n$. Hence $s=\frac{1}{2}n(n+1)$. (Art. 483).

Ex. 2. Find the sum of $1+3+5+7+\ldots\ldots$to n terms.

Here, the nth term$=1+(n-1)\times 2=2n-1$.

Hence, $s=\frac{1}{2}n(1+2n-1)=\frac{1}{2}n\times 2n=n^2$.

Thus, *the sum of* n *consecutive odd numbers beginning with unity is* n^2.

Ex. 3. Find the sum of all the even numbers which are greater than 150 and less than 350.

The first even number greater than 150 is 152 and the last less than 350 is 348, of which common difference is 2.

If n be the no. of terms, we have

$348=152+(n-1)\times 2$, which solved gives $n=99$.

Hence, $s=\frac{1}{2}\times 99\times(152+348)=\frac{1}{2}\times 99\times 500=24750$.

Exercise CLXXXV.

1. Find the sum of the first 40 odd numbers which are greater than 150.

2. Find the sum of all the odd numbers between 100 and 200.

3. Find the sum of all the even numbers which are between 101 and 999.

4. Shew that, if unity be added to the sum of any number of terms of the series 8, 16, 24, &c., the result will be the square of an odd number.

5. Find the sum of all the numbers between 100 and 500 which are divisible by 3.

6. Find the sum of all the numbers between 100 and 1000 which are divisible by 7.

7. If a, b, c, d are in $A.P.$, shew that $a+d=b+c$.

8. If a, b, c are in $A.P.$, prove that $a^2(b+c)$, $b^2(c+a)$ and $c^2(a+b)$ are in A.P.

9. If a^2, b^2 and c^2 be in $A.P.$, prove that $\frac{1}{b+c}$, $\frac{1}{c+a}$ and $\frac{1}{a+b}$ are in A.P.

10. If the roots of the equation $(b-c)x^2+(c-a)x+(a-b)=0$ are equal, prove that a, b, and c are in $A.P.$

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490. *To find the sum of the* **squares** *of the first* n *natural numbers.*

Let S be the required sum, so that we have

$$S=1^2+2^2+3^2+\ldots\ldots\ldots\ldots+n^2.$$

We have $x^3-(x-1)^3=3x^2-3x+1$; (*identically*).

Putting $n, n-1, n-2, \ldots\ldots 3, 2, 1$ for x in the above identity, we have

$$\begin{array}{lllll}
n^3-(n-1)^3 & =3n^2 & -3n & +1, \\
(n-1)^3-(n-2)^3 & =3(n-1)^2 & -3(n-1) & +1, \\
(n-2)^3-(n-3)^3 & =3(n-2)^2 & -3(n-2) & +1, \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots \\
3^3-2^3 & =3.3^2 & -3.3 & +1, \\
2^3-1^3 & =3.2^2 & -3.2 & +1, \\
\text{and } 1^3-0^3 & =3.1^2 & -3.1 & +1,
\end{array}$$

Also there are n of these equations.

Adding together the vertical columns, we obtain

$$n^3-0^3=3(1^2+2^2+3^2+\ \ldots\ldots\ +n^2)$$
$$-3(1+2+3+\ldots\ldots\ldots+n)+n\times 1,$$

i.e. $n^3=3S-3\times\dfrac{n(n+1)}{2}+n$; (Art. 489. **Ex. 1.**)

$$\therefore\ 3S=n^3+\frac{3n(n+1)}{2}-n=\frac{2n^3+3n^2+n}{2}.$$

$$=\frac{n(2n^2+3n+1)}{2}=\frac{n(n+1)(2n+1)}{2}.$$

Hence $S=\dfrac{n(n+1)(2n+1)}{6}$.

491. *To find the sum of the* **cubes** *of the first* n *natural numbers.*

Let S denote the required sum, so that we have

$$S=1^3+2^3+3^3+\ldots\ldots+n^3.$$

We have $x^4-(x-1)^4=4x^3-6x^2+4x-1$, (*identically*).

Putting $n, n-1, n-2, \ldots 3, 2, 1$ for x in the above identity, we have

$$\begin{array}{llllll}
n^4-(n-1)^4 & =4n^3 & -6n^2 & +4n & -1, \\
(n-1)^4-(n-2)^4 & =4(n-1)^3 & -6(n-1)^2 & +4(n-1) & -1, \\
(n-2)^4-(n-3)^4 & =4(n-2)^3 & -6(n-2)^2 & +4(n-2) & -1, \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots \\
3^4-2^4 & =4.3^3 & -6.3^2 & +4.3 & -1, \\
2^4-1^4 & =4.2^3 & -6.2^2 & +4.2 & -1, \\
1^4-0^4 & =4.1^3 & -6.1^2 & +4.1 & -1.
\end{array}$$

Also there are n of these equations.

Adding together the vertical columns, we obtain

$$n^4 - 0^4 = 4(1^3 + 2^3 + 3^3 + \ldots + n^3) - 6(1^2 + 2^2 + 3^2 + \ldots + n^2)$$
$$+ 4(1 + 2 + 3 + \ldots + n) - n \times 1,$$

$$i.e.\ n^4 = 4S - 6 \times \frac{n(n+1)(2n+1)}{6} + 4 \times \frac{n(n+1)}{2} - n,$$

(Arts. 490 and 489 Ex. 1)

$$= 4S - n(n+1)(2n+1) + 2n(n+1) - n\ ;$$

$$\therefore\ 4S = n^4 + n(n+1)(2n+1) - 2n(n+1) + n$$
$$= n(n^3+1) + n(n+1)\{(2n+1) - 2\}$$
$$= n(n+1)\{(n^2 - n + 1) + (2n - 1)\}$$
$$= n(n+1)(n^2 + n) = n^2(n+1)^2.$$

Hence $S = \dfrac{n^2(n+1)^2}{4} = \left\{\dfrac{n(n+1)}{2}\right\}^2.$

Cor. Thus we see that

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2 = (1 + 2 + 3 + \ldots + n)^2.$$

i. e. The sum of the cubes of the first n *natural numbers is equal to the square of the sum of these first* n *natural numbers.*

492. Problems. The methods employed in solving the following examples deserve special notice.

Ex. 1. The sum of three numbers in *A. P.* is 21, and the sum of their squares is 179; find them.

Let the *middle* number be x and the common difference of the numbers be y, so that they are $x-y$, x and $x+y$ (this being a convenient assumption in problems of this kind).

Then, we have $(x-y) + x + (x+y)$ or $3x = 21$ ....(1)
and $(x-y)^2 + x^2 + (x+y)^2$ or $3x^2 + 2y^2 = 179$...(2)

From (1) $x = 7$ and $2y^2 = 179 - 3x^2 = 179 - 147 = 32$; $\therefore y = \pm 4$.

Hence the reqd. numbers are 3, 7, and 11.

Ex. 2. Find five numbers in *A. P.* whose sum is 15 and the sum of whose squares is 55.

Let x be the middle number and y the common difference, so that the numbers are $x - 2y$, $x - y$, x, $x + y$ and $x + 2y$.

We have then $\left.\begin{array}{l}(x-2y)+(x-y)+x+(x+y)+(x+2y)=15\ldots(1)\\ \text{and } (x-2y)^2+(x-y)^2+x^2+(x+y)^2+(x+2y)^2=55\ldots(2)\end{array}\right\}$

From (1), $5x=15$ and $\therefore$ $x=3$.

From (2) $5x^2+10y^2=55$, or $x^2+2y^2=11$.

$\therefore$ $2y^2=11-x^2=11-9=2$; $\therefore$ $y=\pm 1$.

Hence the reqd. numbers are 1, 2, 3, 4, and 5.

Ex. 3. A man stands by a heap of 100 stones. How far must he walk, carrying one stone at a time to place the stones separately, at intervals of 10 yards apart, in a straight line having one end where the heap is?

To carry the 1st stone, the man shall have to walk 20 yds.
............2nd......,.......................... ....40 yds.
....3rd. ...,............... ....60 yds.
and so on, till he carries the 99th stone, for one stone should remain at the place where the heap is.

Thus, we have to sum the series

$20+40+60+\ldots$ to 99 terms.

Hence, distance travelled on the whole

$=\frac{99}{2}\{2\times 20+98\times 20\}$yds. $=(\frac{99}{2}\times 2000)$yds. $=99000$ yds.

Exercise CLXXXVI.

1. How many strokes a-day do the astronomical clocks make, which strike from one to twenty-four?

2. How many strokes does a common clock make in 12 hours? and how many, if it strikes also the half-hours?

3. Find the three numbers in *A. P.*, whose sum shall be 21, and the sum of the first and second $=\frac{3}{4}$ that of the second and third.

4. There are three numbers in *A. P.*, whose sum is 10, and the product of the second and third is $33\frac{1}{3}$; find them.

5. Find three numbers in *A. P.* whose sum is 21, and the sum of whose squares is 155.

6. Find three numbers in *A. P.* whose common difference is 1, such that the product of the second and third exceeds that of the first and second by $\frac{1}{2}$.

7. Find five numbers in *A. P.* whose sum is 40, and the sum of whose cubes is 4720.

8. Find five numbers in *A. P.* whose sum is 40 and the sum of whose squares is 410.

9. Find three numbers in *A. P.* whose sum is 21 and whose product is 315.

10. A debt can be discharged in a year by paying one shilling the first week, three the second, five the third, &c.: required the last payment and the amount of the debt.

11. Divide 25 into five parts which are in *A.P.*, and which are such that the sum of the squares of the least and greatest of them is one less than the sum of the squares of the other three.

12. A number of three digits is equal to 26 times the sum of the digits and the digits are in *A.P*: if 396 be added to the number, the digits are reversed: find the number.

13. One hundred stones being placed on the ground at the distance of a yard from one another, how far will a person travel, who shall bring them, one by one, to a basket, placed at the distance of a yard from the first stone?

14. A class consists of a number of boys whose ages are in *A.P.*, the common difference being four months. If the youngest boy is just eight years old, and the sum of the ages is 168 years, find the number of boys in the class. (C. F. A. 1872).

15. **A** sets out from a place and travels $2\frac{1}{2}$ miles an hour. **B** sets out 3 hours after **A**, and travels in the same direction, 3 miles the first hour, $3\frac{1}{2}$ miles the second, 4 miles the third, and so on. In how many hours will **B** overtake **A**?

16. A man saves each year *Rs.*10 more than he saved in the preceding year and he saves *Rs.*120 in the first year; in how many years will his savings, not including interest, be more than *Rs.*1000?

17. The sum of three numbers in *A.P.* is 18. The sum of the squares of the first and third exceeds twice the square of the middle one by 32. Find the numbers.

18. A person is employed to count *Rs.*12000. He counts at the rate of *Rs.*150 per minute for an hour, at the end of which time he begins to count at the rate of *Rs.*2 less every minute than he did the previous minute Find when he will finish his task, and explain the fact that two solutions occur. (M. F. A. 1886).

II. GEOMETRICAL PROGRESSION.

493. Quantities are said to be in **Geometrical Progression**, when they proceed by a *constant factor*.

Thus, each of the following series is in *Geometrical Progression* (*G. P.*) :—

$$1,\ 3,\ 9,\ 27, \ldots$$
$$4,\ 1,\ \tfrac{1}{4},\ \tfrac{1}{16}, \ldots$$
$$-\tfrac{1}{3},\ \tfrac{1}{9},\ -\tfrac{1}{27},\ \tfrac{1}{81}, \ldots$$
$$a,\ ar,\ ar^2,\ ar^3, \ldots$$

the *constant factors* being 3, $\frac{1}{4}$, $-\frac{1}{3}$, r, respectively.

494. The constant factor is called the **common ratio** of the series, which may be found *by dividing any term of the series by the term preceding*.

Thus, in the series $a, ar, ar^2, ar^3, \ldots$

$$\text{the common ratio} = \frac{ar}{a} = \frac{ar^2}{ar} = \frac{ar^3}{ar^2} = \ldots = r.$$

495. *To find the* nth *term of a G. P.*

Let a = 1st term, and r = common ratio.

Then the series will be $a, ar, ar^2, ar^3, \ldots$ where the index of r in any term is just *less by one* than the number of the term.

Thus, the 7th term $= ar^6$;
13th term $= ar^{12}$;
30th term $= ar^{29}$;
and generally, the pth term $= ar^{p-1}$.

Hence, if n be the number of terms, and l denote the last or nth term, we have

$$l = ar^{n-1}.$$

Ex. 1. Find the 8th term of the series $81, -27, 9, \ldots\ldots$

Here $a = 81$, $r = -\frac{27}{81} = -\frac{1}{3}$, $n = 8$;

Hence the 8th term $= 81 \times (-\frac{1}{3})^7 = 3^4 \times -\dfrac{1}{3^7} = -\dfrac{1}{3^3} = -\frac{1}{27}$.

496. When any two terms of a series in Geometrical Progression are given, the series can be completely determined, for

then we shall have two equations to determine the first term and the common ratio.

Ex. 2. Find the *G.P.* whose 7th term is 1 and whose 11th term is $\frac{1}{16}$.

Here, we have $ar^6 = 1$ and $ar^{10} = \frac{1}{16}$.

Thus, by division, $r^4 = \frac{1}{16}$, and so $r = \frac{1}{2}$.

Hence $a(\frac{1}{2})^6 = 1$, that is, $a = 2^6 = 64$.

Thus, the series is 64, 32, 16, 8,...

Exercise CLXXXVII.

1. Find the

(1) 5th and 14th terms of the series 9, 3, 1,.....

(2) 6th and 16th terms of the series $2, -3, \frac{9}{2}, \ldots$

(3) 10th and *n*th terms of the series $6, -2, \frac{2}{3},$

(4) 8th and 17th terms of the series ·6, ·03, ·0015,......

(5) 12th and *n*th terms of the series $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8},\ldots\ldots$

2. Find the *n*th term of the series $\sqrt{3} + \frac{1}{\sqrt{3}} + \frac{1}{3\sqrt{3}} + \ldots$ (C.F.A. 1886).

3. Write down the 12th term of the series $2, -2\sqrt{2}, 4, \ldots$

4. Find the last term of the series $3, -3^2, 3^3, \ldots$ to $2n$ terms.

5. Find the *p*th term of the series $a, a^3, a^5, \ldots\ldots$

6. What is the fifth term of the *G.P.* whose first term is 3 and whose third term is 4?

7. The 5th term of a *G.P.* is 20 and the 8th term -160, find the *n*th term.

8. The second term of a *G.P.* is 4 and the 5th term 256; find the series.

9 The fifth term of a geometric series is 8 times the second and the third term is 12; find the series.

10. The fifth term of a geometric series is 4 times the third, and the sum of the first two is -4; find the series.

11. Find a geometric series, whose first term is 2 and 7th term is $\frac{1}{32}$.

12. Given 6 the second term of a geometric series and 54 the fourth, find the first term.

13. The 3rd term of a $G.P.$ is 1, and the 6th term is $-\frac{1}{8}$; what is the 10th term.

14. Find the series in which the 5th term is $\frac{27}{16}$ and the 9th term is $\frac{1}{3}$.

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497. Sum of the Series. *To find the sum of a given number of terms in Geometrical Progression, the first term and the common ratio being given.*

Let a denote the first term, r the common ratio, n the number of terms, and S the sum of the terms. Then

$$S=a+ar+ar^2+ar^3+\ldots+ar^{n-1};\ldots\ldots\ (1)$$

Multiply by r; then

$$Sr=\quad ar+ar^2+ar^3+\ldots+ar^{n-1}+ar^n\ldots\ldots(2)$$

Hence, by subtraction of (1) from (2), we have

$$Sr-S=ar^n-a;$$

i.e. $S(r-1)=a(r^n-1)$;

$$\therefore\ S=\frac{ar^n-a}{r-1}=a\frac{r^n-1}{r-1}\ldots\ldots\ \ldots\ldots(1)$$

Changing the signs in numerator and denominator.

$$S=a\frac{1-r^n}{1-r}\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Note. The form (2) is the most convenient to use when r is *negative*.

If l denote the last term, we have

$$l=ar^{n-1},\ldots\ldots\ldots\ldots(3)$$

so that the formula (1) may be written

$$S=\frac{r.ar^{n-1}-a}{r-1}=\frac{rl-a}{r-1};\ldots\ldots\ldots\ldots(4)$$

a form which is sometimes convenient to use.

Ex. 1. Find the sum of $3-6+12-\ldots$to 6 terms.

Here, $a=3$, $r=-2$, $n=6$;

$$\therefore\ S=\frac{3\{(-2)^6-1\}}{-2-1}=\frac{3(64-1)}{-3}=-63.$$

Ex. 2. Find the sum of 4 terms of the series $1, -\frac{4}{3}, \frac{16}{9}, \ldots$

Here, $a=1$, $r=-\frac{4}{3}$, $n=4$;

$$\therefore S=\frac{\left(-\frac{4}{3}\right)^4-1}{-\frac{4}{3}-1}=\frac{\frac{4^4}{3^4}-1}{-\frac{4}{3}-1}=\frac{\frac{4^4-3^4}{3^4}}{-\frac{7}{3}}=-\frac{3}{7}\times\frac{256-81}{3^4}$$

$$=\frac{175}{7\cdot 3^3}=-\frac{25}{27}.$$

Ex. 3. Find the sum of $2\frac{1}{2}-1+\frac{2}{5}-$&c., to 5 terms.

Here, $a=\frac{5}{2}$, $r=-\frac{2}{5}$, $n=5$;

$$\therefore S=\frac{5}{2}\cdot\frac{\left\{\left(-\frac{2}{5}\right)^5-1\right\}}{-\frac{2}{5}-1}=\frac{5}{2}\cdot\frac{-\frac{2^5}{5^5}-1}{-\frac{7}{5}-1}=\frac{5}{2}\cdot\frac{\frac{2^5+5^5}{5^5}}{\frac{7}{5}}$$

$$=\frac{5}{2}\cdot\frac{5}{7}\cdot\frac{32+3125}{5^5}=\frac{3157}{14\cdot 5^3}=1\frac{201}{250}.$$

Ex. 4. Determine the nth term and the sum of n terms of $\frac{1}{5}-\frac{2}{15}+\frac{4}{45}-\frac{8}{135}+$&c.

Here, $a=\frac{1}{5}$, $r=-\frac{2}{15}\div\frac{1}{5}=-\frac{2}{3}$, $n=n$;

$\therefore$ nth term $=\frac{1}{5}(-\frac{2}{3})^{n-1}=\frac{1}{5}(\frac{2}{3})^{n-1}(-1)^{n-1}$,

and $S=\frac{1}{5}\cdot\frac{1-(-\frac{2}{3})^n}{1-(-\frac{2}{3})}=\frac{1}{5}\cdot\frac{1-(-\frac{2}{3})^n}{\frac{5}{3}}=\frac{3}{25}\{1-(\frac{2}{3})^n(-1)^n\}$.

Exercise CLXXXVIII.

1. Find the last term and the sum of

(1) $1+4+16+$&c. to 4 terms. (2) $5+20+80+$&c. to 5 terms.

(3) $3+6+12+$&c. to 6 terms. (4) $2-4+8-$&c. to 8 terms.

(5) $1-4+16-$&c. to 7 terms. (6) $1-2+2^2-$&c. to 10 terms.

(7) $1+2+4+$&c. to 6 terms. (8) $81-27+9-$&c. to 8 terms.

2. Find the sum of

(1) $\frac{1}{3}+\frac{1}{6}+\frac{1}{12}+$&c. to 8 terms. (2) $\frac{1}{2}+\frac{1}{3}+\frac{2}{9}+$&c. to 6 terms.

(3) $\frac{2}{3}+1+\frac{3}{2}+$&c. to 6 terms. (4) $3-\frac{1}{2}+\frac{1}{12}-$&c. to 5 terms.

(5) $9-6+4-$&c. to 9 terms. (6) $100-40+16-$&c. to 5 terms.

3. Find the last term and sum of the following series :—

(1) $1+2+4+8+\ldots\ldots$ to n terms.

(2) $8+20+50+125+\ldots\ldots$ to n terms.

(3) $1+3+9+27+\ldots\ldots$ to 9 terms and to n terms.

(4) $1-2+2^2-2^3+\ldots\ldots$ to 10 terms and to n terms.

(5) $3\frac{3}{8}+2\frac{1}{4}+1\frac{1}{2}+\ldots\ldots$ to 8 terms and to n terms.

(6) $\frac{1}{3}+\frac{1}{2}+\frac{3}{4}+\ldots$ to n terms. (7) $\frac{1}{2}-\frac{1}{3}+\frac{2}{9}-\ldots\ldots$ to n terms.

(8) $\cdot2+\cdot02+\cdot002+\ldots\ldots$ to n terms.

(9) $(1\frac{1}{2})^{-1}+2^{-1}+(2\frac{1}{3})^{-1}+\ldots\ldots$ to n terms.

(10) $\frac{1}{\sqrt{2}}+\frac{1}{2}+\frac{1}{2\sqrt{2}}+\ldots\ldots$ to 10 terms.

4. Sum the following series :—

(1) $1-\frac{2}{3}+\frac{4}{9}-$ &c. to 6 terms. (P. I. E. 1888).

(2) $\frac{2}{3}-(\frac{2}{3})^{\frac{1}{2}}+1-$ &c. to n terms. (C. E. A. 1865).

(3) $\frac{1}{\sqrt{3}}+1+\frac{3}{\sqrt{3}}+$ &c.... to 18 terms. (P. I. E. 1891).

(4) $2+\sqrt{2}+1+$ &c. to n terms. (P. I. E. 1890).

(5) $\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}+\frac{1}{2\sqrt{3}}+$ &c. to 10 terms.

(6) $3\sqrt{x}+\frac{1}{\sqrt{x}}+3\sqrt{x}+\frac{1}{x}+\sqrt{x}+\frac{1}{x\sqrt{x}}+$ &c. to $2n$ terms.

5. Find the sum of n terms of the following series :—

(1) $1+\frac{3}{5}+(\frac{3}{5})^2+\ldots\ldots$ (2) $1-\frac{3}{2}+\frac{9}{4}-\frac{27}{8}+\ldots\ldots$

(3) $\frac{3}{10}-\frac{5}{6}+\frac{125}{54}-\ldots\ldots$ (4) $\frac{1}{\sqrt{2}}-\sqrt{\frac{3}{2}}+\frac{3}{\sqrt{2}}-\ldots\ldots$

(5) $\frac{a}{x}+\frac{b}{x^2}+\frac{b^2}{ax^3}+\ldots\ldots$ (6) $x-y+\frac{y^2}{x}-\frac{y^3}{x^2}+\ldots\ldots$

(7) $\frac{\sqrt{3}+1}{\sqrt{3}-1}+1+\frac{\sqrt{3}-1}{\sqrt{3}+1}+\ldots$ (8) $\frac{\sqrt{2}+1}{\sqrt{2}-1}+2+\frac{2(\sqrt{2}-1)}{\sqrt{2}+1}+\ldots$

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498. The following are illustrative examples.

Ex. 1. In a *G.P.*, shew that the product of any two terms equidistant respectively from the first and the last terms is constant.

Suppose a is the first term and r the common ratio.

Let the two terms be the pth term from the beginning and the pth term from the end, both the terms inclusive.

The pth term from the beginning $= ar^{p-1}$.

Also, the pth term from the end is the $(n-p+1)$th term from the beginning, and hence it is

$$= ar^{n-p+1-1} = ar^{n-p},$$

Hence their product $= ar^{p-1} \times ar^{n-p} = a.\ ar^{n-1}$

$$= al = \text{a constant.}$$

Ex. 2 Find the middle term or terms of a series of n terms in *G. P.*

Let a, ar, ar^2,...denote the series to n terms.

First. Let n be *odd.*

Then there will be *one* middle term, *viz.* the $\left(\frac{n+1}{2}\right)$th term, and it is $= ar^{\frac{n+1}{2}-1} = ar^{\frac{n-1}{2}} = \sqrt{a(ar^{n-1})} = \sqrt{al}.$

Secondly. Let n be *even.*

Then there will be two middle terms, *viz.* the $\frac{n}{2}$th and $\frac{n+2}{2}$th terms, and they are

$$ar^{\frac{n}{2}-1} \text{ and } ar^{\frac{n+2}{2}-1} \text{ or } ar^{\frac{n-2}{2}} \text{ and } ar^{\frac{n}{2}}.$$

Note. It should be noticed here, that the product of the two middle terms

$$= ar^{\frac{n-2}{2}} \times ar^{\frac{n}{2}} = a^2r^{n-1} = a.ar^{n-1} = al.$$

Ex. 3. Sum to n terms the series whose pth term is $(-1)^p a^{4p}$.

When $p=1$, the first term $=(-1)a^4 = -a^4$,

When $p=2$, the second term $=(-1)^2a^8 = a^8$,

When $p=3$, the third term $=(-1)^3a^{12} = -a^{12}$, and so on.

Hence, the first term $= -a^4$ and com. ratio $= -a^4$.

$$\therefore S = \frac{-a^4\{(-a^4)^n - 1\}}{-a^4 - 1} = \frac{a^4}{a^4+1}\{(-1)^n a^{4n} - 1\}.$$

Ex. 4. The sum of the first 10 terms of a certain *G. P.* is equal to 33 times the sum of the first 5 terms. What is the common ratio?

Let a = the first term and r = the common ratio.

Then, by the question, we have

$$\frac{a(1-r^{10})}{1-r}=33\cdot\frac{a(1-r^5)}{1-r}\ ;\ \therefore\ 1+r^5=33,\ \text{or}\ r^5=32\ ;\ \therefore\ r=2.$$

Ex. 5. If S be the sum of an odd number of terms in $G.\ P.$ and S' the sum of the series when the signs of the even terms are changed, prove that the sum of the squares of the terms will be SS'.

Let a=the first term, r=the common ratio and $2n+1$=no. of terms. Then

$$S=a+ar+ar^2+\ldots\ldots+ar^{2n}=\frac{a(1-r^{2n+1})}{1-r}.$$

$$\text{and } S'=a-ar+ar^2-\ldots+ar^{2n}=\frac{a(1+r^{2n+1})}{1+r}.$$

$$\therefore\ SS'=\frac{a^2\{1-(r^{2n+1})^2\}}{1-r^2}=\frac{a^2\{1-(r^2)^{2n+1}\}}{1-r^2}.$$

$$\text{Also } a^2+(ar)^2+(ar^2)^2+\ldots+(ar^{2n})^2=\frac{a^2\{1-(r^2)^{2n+1}\}}{1-r^2}.$$

Hence the result.

Exercise CLXXXIX.

1. If all the terms of a $G.P.$ be multiplied or divided by the same quantity, the resulting terms will form another G. P. with the same common ratio.

2. If every alternate term of a G. P. be taken away, the remaining terms will be in $G.P.$

3. Shew that the reciprocals of the terms of a $G.P.$, are in $G.P.$

4. If a, b, c, d be in $G.P.$, shew that $ad=bc$.

5. Shew that the product of any two terms of a $G.P.$, which are respectively equally distant from the first and the last terms, is equal to the product of the first and last terms.

6. From three given numbers which are in $G.\ P.$, three other numbers in $G.P.$ are subtracted, and the remainders are found to be also in $G.\ P.$, prove that the three series have the same common ratio. (B. P. E. 1890).

7. If an odd number of quantities be in $G.P.$, prove that the first, the middle, and the last of them are in $G.P.$

8. If ab be the $(2m+1)$th term of a $G.P.$, whose first term is a, find the middle term.

9. The sum of three terms in *G.P.* is 63, and the difference of the first and third terms is 45 : find the terms.

10. When the number of terms of a series in *G. P.*, is even, shew that the product of two terms equidistant from the beginning and end is equal to the product of the two middle terms. (P.I.E. 1890).

11. Show that the sum of n terms of a *G.P.* beginning with the pth term is r^{p-q} times the sum of an equal number of terms of the same series beginning with the qth term. (M. F. A. 1884).

12. A *G. P.* has $2n$ terms. The sum of the n odd terms is equal to a and that of the n even terms is b. Find the G. P.

13. If there are six terms in *G. P.*, prove that the product of first and last is equal to the product of third and fourth. (P.I.E. 1893).

14. In a *G.P.* if the $(p+q)$th term $=m$, and the $(p-q)$th term $=n$, find the pth and the qth terms. (B. P. E. 1888).

15. The sum of the first four terms of a *G.P.* is 40, and the sum of the first eight terms is 3280 ; find the series.

16. Shew that the product of all the terms of a *G.P.* is equal to the nth power of the middle term, when n is odd, and is equal to the $\frac{1}{2}n$th power of the product of the two middle terms, when n is even (n being the number of the terms in each case).

17. If a, b, c, d be in *G.P.*, prove that

(1) $a^2b^2c^2(a^{-3}+b^{-3}+c^{-3})=a^3+b^3+c^3$.

(2) a^2+b^2, b^2+c^2 and c^2+d^2 are also in *G.P.*

(3) $(a+b+c+d)^2=(a+b)^2+(c+d)^2+2(b+c)^2$. (C. F. A. 1900).

(4) $(a-d)^2=(b-c)^2+(c-a)^2+(d-b)^2$. (C. F. A. 1900).

(5) $(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$.

18. If a, b, and c are in *A. P.* and x, y, z in *G. P.*, prove that $x^by^cz^a=x^cy^az^b$.

19. If P, Q and R be the pth, qth and rth terms of *G. P.* prove $P^{q-r}.Q^{r-p}.R^{p-q}=1$. (B. P. E. 1889).

20. Show that the $2n$th term of a geometrical series is the mean proportional between the nth and $3n$th terms. (C.F.A. 1877).

21. If a be the first term and ar^{n-1} be the last term of a geometric series, and ar^{n-1} be the first and a the last term of the same series reversed, and if the terms of the first series be divided by equidistant terms from the beginning of the second, then the sum of the resulting series will be $\dfrac{r^n-r^{-n}}{r-r^{-1}}$. (C. F. A. 1868).

22. If S be the sum, P the product and R the sum of the reciprocals of the series a, ar, ar^2, &c. to n terms, prove that $P^2=\left(\frac{S}{R}\right)^n$.
(C. F. A. 1883.)

23. The common ratio of a series in Geometrical Progression is 3; the sum of the first and third terms is equal to the squares of the first and second terms; find the sum of n terms. If $n=6$, shew that the sum is 364. (C. F. A. 1866).

24. If a, b, c, x be real quantities, and if

$$(a^2+b^2)x^2-2b(a+c)x+b^2+c^2=0,$$

prove that a, b, c are in $G. P.$ and x is their common ratio.

25. If S denote the sum of $a+ar+ar^2+\ldots\ldots$, and R that of $a+ar^{-1}+ar^{-2}+\ldots\ldots$, each to n terms, prove that $aS=lR$, when l is the last term of the first series.

26. If S_1, S_2, S_3 be the sums to n, $2n$, $3n$ terms of a $G. P.$ respectively, prove that

(i) $S_1^2+S_2^2=S_1(S_2+S_3)$.

(ii) $S_1(S_3-S_2)=(S_2-S_1)^2$. (B. P. E. 1882).

27. Prove that in the product

$$(1+x+x^2+\ldots\ldots\quad+x^{2n})(1-x+x^2\ldots\ldots\ldots+x^{2n}),$$

the coefficients of odd powers of x is zero, and of even powers unity. (B. P. E. 1893).

28. If n terms be in $G. P.$, whose common ratio is r, and S_m denote the sum of the first m terms, prove that the sum of the products of every two terms $=\frac{r}{r+1}S_mS_{m-1}$.

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499. **Sum to Infinity.** If r be a *proper* fraction, *i. e.*, if r be <1, its powers, r^2, r^3, &c., r^n will, *a fortiori*, be also <1, and therefore, ar^n will be $<a$: hence, instead of writing $S=\frac{ar^n-a}{r-1}$, in which fraction both numerator and denominator are *negative*, we may write, in this case,

$$S=\frac{a-ar^n}{1-r}=\frac{a}{1-r}-\frac{ar^n}{1-r}.$$

Now the *greater* we take the value of n, (*i.e.* the more terms we take of the series) the less will be the value of ar^n; and, by taking n sufficiently great, we may get ar^n as small as we please, only never so small as actually to *vanish*. If ar^n vanished, we should have the

sum of the series $= \frac{a}{1-r}$; but since, however small may be the value of ar^n, the second fraction will never actually become *zero*, it follows that the sum of the series will never actually reach the above value, though, by increasing n, *i.e.* taking more terms of the series, it may be made to approach it as nearly as we please.

On this account, the *Limit* of the sum of the series, $a+ar+ar^2+\ldots$, or sometimes (but less correctly) the sum of the series *ad infinitum* or the sum of an *infinite number* of terms, in which r is *numerically less than unity*, is $\frac{a}{1-r}$.

Note. It is usual to denote the **Limit** of such a sum by Σ.

Ex. 1. Find the Limit of the sum of the series $1+\frac{1}{2}+\frac{1}{4}+\ldots\ldots$

Here, $a=1$, $r=\frac{1}{2}$; $\therefore \Sigma = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$;

i.e. the more terms we take of this series, the more nearly will their sum $=2$, but will never actually reach it.

Ex. 2. Sum $2\frac{1}{2}-\frac{1}{2}+\frac{1}{10}-\ldots\ldots$ to *infinity*.

Hence, $a=2\frac{1}{2}$, $r=-\frac{1}{5}$.

$\therefore$ reqd. sum $= \frac{\frac{5}{2}}{1-(-\frac{1}{5})} = \frac{\frac{5}{2}}{1+\frac{1}{5}} = \frac{\frac{5}{2}}{\frac{6}{5}} = \frac{5}{2}\times\frac{5}{6} = \frac{25}{12} = 2\frac{1}{12}$.

500. Recurring Decimals are examples of infinite geometrical series. Thus, for example

$$\cdot 9282828\ldots\ldots \text{denotes } \frac{9}{10}+\frac{28}{10^3}+\frac{28}{10^5}+\frac{28}{10^7}+\ldots\ldots$$

Here the terms after $\frac{9}{10}$ form a *G. P.*, of which the first term is $\frac{28}{10^3}$ and the common ratio is $\frac{1}{10^2}$

Hence the sum of an infinite number of terms of this series is $\frac{28}{10^3} \div \left(1-\frac{1}{10^2}\right)$ that is $\frac{28}{990}$.

Therefore the value of the recurring decimal is

$$\frac{9}{10}+\frac{28}{990} = \frac{891+28}{990} = \frac{919}{990}.$$

Note. The value of a recurring decimal may be found practically thus:—

Let $s = \cdot 9282828\ldots\ldots$
then $10s = 9\cdot 282828\ldots\ldots$
and $1000s = 928\cdot 282828\ldots\ldots$

Hence by subtraction, $(1000-10)s = 928-9$;

so that $990s = 919$, $\therefore s = \frac{919}{990}$.

Exercise CLXL.

1. Sum to infinity the following series :—

(1) $4+2+1+\ldots\ldots$ (2) $\frac{1}{2}+\frac{1}{3}+\frac{2}{9}+\ldots\ldots$ (3) $\frac{1}{4}-\frac{1}{16}+\frac{1}{64}-\ldots$

(4) $\frac{3}{2}-1+\frac{2}{3}-\ldots\ldots$ (5) $1-\frac{1}{2}+\frac{1}{4}-\ldots\ldots$ (6) $1-\frac{2}{5}+\frac{4}{25}-\ldots\ldots$

(7) $\frac{5}{8}+\frac{1}{2}+\frac{2}{5}+\frac{8}{25}+\ldots\ldots$ (C. F. A. 1888). (8) $\frac{1}{5}+\frac{1}{15}+\frac{1}{45}+\ldots\ldots$

(9) $432+324+243+\ldots\ldots$ (C. F. A. 1894). (10) $\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\ldots\ldots$

(11) $3-1+\frac{1}{3}-\frac{1}{9}+\ldots\ldots$ (C. F. A. 1876 ; B. P. E. 1886).

(12) $2-\frac{3}{4}+\frac{9}{32}-\ldots\ldots$ (13) $2-1\frac{1}{3}+\frac{8}{9}-\ldots\ldots$ (14) $\frac{2}{3}+\frac{2}{5}+\frac{6}{25}+\ldots\ldots$

(15) $9+6+4+\ldots\ldots$ (16) $2+\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\ldots$ (17) $3\frac{3}{8}+2\frac{1}{4}+1\frac{1}{2}+\ldots\ldots$

(18) $-3\frac{1}{5}+1\frac{3}{5}-\frac{4}{5}+\ldots$ (19) $(1\frac{1}{2})^{-1}+2^{-1}+(2\frac{2}{3})^{-1}+\ldots$ (20) $6-2+\frac{2}{3}-\ldots$

(21) $6-3+\frac{3}{2}-\frac{3}{4}+\ldots\ldots$ (also to n terms). (C. F. A. 1878).

(22) $48-36+27-20\frac{1}{4}+\ldots\ldots$ (also to n terms). (C. F. A. 1879).

(23) $1-\frac{1}{10}+\frac{1}{10^2}-\frac{1}{10^3}+\ldots$. (24) $\frac{1}{2}-\frac{1}{2\cdot2^3}+\frac{1}{2\cdot2^6}-\ldots\ldots$

(25) $1-x+x^2-x^3+\ldots\ldots(x<1)$.

2. Sum to infinity :—

(1) $\sqrt{3}+\frac{1}{\sqrt{3}}+\frac{1}{3\sqrt{3}}+\ldots\ldots$ (C. F. A. 1886).

(2) $(\sqrt{2}+1)+1+(\sqrt{2}-1)+\ldots\ldots$ (C. F. A. 1887).

(3) $(2+\sqrt{3})+1+(2-\sqrt{3})+\ldots\ldots$ (C. F. A. 1891).

(4) $\sqrt{\frac{3}{2}}+\sqrt{\frac{2}{3}}+\frac{2}{3}\sqrt{\frac{2}{3}}+\ldots\ldots$ (5) $1+(\sqrt{2}-1)+(3-2\sqrt{2})+\ldots\ldots$

(6) $\frac{5+2\sqrt{2}}{5-2\sqrt{2}}+1+\frac{5-2\sqrt{2}}{5+2\sqrt{2}}+\ldots\ldots$ (7) $\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}-\ldots\ldots(x>1)$.

(8) $\sqrt{\frac{a^3}{x^2}}+\sqrt{\frac{a}{x}}+\sqrt{\frac{1}{a}}+\ldots\ldots$ (9) $(a^2-x^2)+(a-x)+\frac{a-x}{a+x}+\ldots\ldots$

(10) $\frac{a}{(1+x)^n}+\frac{ax}{(1+x)^{n+1}}+\frac{ax^2}{(1+x)^{n+2}}+\ldots\ldots$

3. Find the sum of an infinite number of terms of the series

$$a+\frac{b}{r}+\frac{c}{r^2}+\frac{a}{r^3}+\frac{b}{r^4}+\frac{c}{r^5}+\frac{a}{r^6}+\frac{b}{r^7}+\frac{c}{r^8}+\ldots\ldots$$

4. Find the values of the following recurring decimals :—

(1) ·2343434...... (2) ·43285285...... (3) ·75363636......

5. If $2\frac{1}{4}$ and 1 be the first and third terms of a *G. P.*, find the sum of the series *ad infinitum*.

6. In a *G. P.* continued to infinity, the common ratio being less than unity, each term bears a constant ratio to the sum of all the terms which follow it.

7. Shew that $\dfrac{a-ar+ar^2-ar^3+\ldots\ldots\text{to } inf.}{a+ar+ar^2+ar^3+\ \ldots\text{to } inf.}=\dfrac{1-r}{1+r}$.

8. Given a and b the first two terms of a decreasing geometric series, find the sum to *infinity*; and the sum of the same series to *inf.* commencing after the nth term.

9. The first term of a Geometric series continued to infinity is 1, and any term is equal to the sum of all the succeeding terms. Find the series. (M. F. A. 1881).

10. In an infinite *G. P.* whose terms are all positive, the common ratio being less than unity, prove that any term is $>$, $=$ or $<$ the sum of all the succeeding terms, according as the common ratio is $<$, $=$ or $>$ $\frac{1}{2}$. (B. P. E. 1887).

11. If S_1, S_2, $S_3, \ldots S_p$ are the sums of infinite Geometric series,
se

$\ldots \dfrac{1}{p+1}$, respectively, prove that

$$S_1+S_2+S_3+\ldots+S_p=\tfrac{1}{2}p(p+3). \quad \text{(B. P. E. 188}$$

12. If S_1, S_2, S_3 are the sums to n terms, $2n$ terms and to infinity of a *G. P.*, shew that

$$S_1(S_1-S_3)=S_3(S_1-S_2). \quad \text{(C. P. A. 1877).}$$

501. By means of the equations of a *G. P.* given in Art. 482, we may find any one of the four quantities a, r, n, and s, when the other three are given. It is not, however, generally easy to find n, when the other quantities are given, because this quantity occurs in the form of an index The student should be able to guess at its value in the simple instances we shall here give; but in other cases, it could only be found by the aid of logarithms.

Ex. 1. In a *G. P.* the first term is 4, and the sum of the first 8 terms is $7\frac{31}{32}$. Find the common ratio.

Here, $a=4$, $n=8$, and $s=7\frac{31}{32}=\frac{255}{32}$, so that $\frac{255}{32}=4.\dfrac{r^8-1}{r-1}$,

which simplified gives $128r^8-255r+127=0$;

or $256r^8-510r+254=0$; $\therefore$ $(2r)^8-1-255(2r-1)=0$.

Hence, $2r-1$ (being a com. factor)$=0$, and $\therefore$ $r=\frac{1}{2}$.

Ex. 2. How many terms of the series $6, -2, \frac{2}{3}$...must be taken that the sum may be $4\frac{364}{729}$?

Here, $a=6$, $r=-\frac{1}{3}$ and $s=4\frac{364}{729}=\frac{3280}{729}$;

so that, if n be the number of terms, we have

$$\frac{3280}{729}=6.\frac{1-(-\frac{1}{3})^n}{1-(-\frac{1}{3})}=\frac{9}{2}\{1-(-\frac{1}{3})^n\}.$$

Hence $(-\frac{1}{3})^n=1-\frac{3280}{729}\times\frac{2}{9}=\frac{1}{6561}=\frac{1}{3^8}=(-\frac{1}{3})^8$; $\therefore$ $n=8$.

502. In the case of an infinite Geometric series, the cases considered in the preceding Article are much simplified.

Ex. 1. The sum of an infinite *G. P.* is 4 and the second term is $\frac{3}{4}$, find the series.

Let a be the first term and r the common ratio.

Hence, $\frac{a}{1-r}=4$...(1), and $ar=\frac{3}{4}$...(2)

By division, we have $r(1-r)=\frac{3}{4}\div 4=\frac{3}{16}$.

$\therefore$ $r^2-r+\frac{3}{16}=0$, whence $r=\frac{1}{4}$ or $\frac{3}{4}$.

As both the values of r are less than unity, both are admissible. Also from (2) $a=\frac{3}{4}\div r=3$ or 1.

Hence the series is either $3, \frac{3}{4}$, &c. or $1, \frac{3}{4}$, &c.

Ex. 2. The sum of an infinite Geometric series is 3, and the sum of its first two terms is $2\frac{2}{3}$. Find the series.

Let a be the first term and r the common ratio.

Hence, $\frac{a}{1-r}=3$...(1) and $a+ar=2\frac{2}{3}$...(2).

By division, we have $\frac{1}{1-r^2}=3\times\frac{3}{8}=\frac{9}{8}$.

$\therefore$ $1-r^2=\frac{8}{9}$ and $\therefore$ $r^2=1-\frac{8}{9}=\frac{1}{9}$; and $\therefore$ $r=\pm\frac{1}{3}$.

Hence, from (1) $a=3(1-r)=3(1\mp\frac{1}{3})=2$ or 4.

Thus, the series is either $2, \frac{2}{3}$, &c...or $4, -\frac{4}{3}$, &c.

Exercise CLXLI.

1. How many terms of the series $2-3+\frac{9}{2}-$&c. must be taken that the sum may be $-8\frac{5}{16}$?

2. The first term of a *G. P.* is 12 and the sum to 6 terms is $39\frac{117}{256}$. Find the common ratio.

3. The first term of a *G. P.* exceeds the second by 2, and the sum to infinity is 50. Find the series. (C. F. A. 1892).

4. The sum of an infinite Geometric series is 2, and the second term is $-\frac{8}{9}$, find the series.

5. If the sum of a *G. P.* continued to infinity be n times the first term, find the common ratio.

6. Given the first term 3, the last term 768, and the number of terms 9, to find the common ratio.

7. The sum of a *G.P.* whose common ratio is -3 is -1092 and the last term is -1458. Find the first term.

8. Given $a=5$, $r=4$, $l=327680$; find s and n.

9. Find the *G. P.* whose second term is $-\frac{1}{2}$ and whose sum to infinity is $4\frac{6}{11}$.

10. The sum of an infinite *G. P.* is 10, and the sum of the first two terms is $6\frac{2}{5}$ Find the series.

11. The first term of an *A. P.* is the same as that of a *G. P.*, and the common difference of the one and the common ratio of the other are both 2; and the sum of 5 terms of each series is the same. Find the 5th term of each series. (C. F. A. 1873.)

12. Find the *G. P.* whose sum to infinity is 9, and whose second term is -4.

503. Geometric Mean. When three quantities are in Geometrical Progression, the middle one is called the **Geometric Mean** of the other two.

Thus, when a, x, b are in *G. P.*, x is called the *Geometric Mean* (*G. M.*) between a and b.

By definition of *G. P.*, we have

$$\frac{x}{a}=\frac{b}{x};\ \therefore\ x^2=ab;\ \text{and}\ \therefore\ x=\pm\sqrt{ab}.$$

Thus, *the geometric mean of any two quantities is the square root of their product.*

Note. It is worthy of notice here that quantities which are in *G. P.* are in continued proportion (Art. 409), and that the geometric mean of two quantities is the same as their mean proportional.

504. When any number of quantities are in geometrical progression, all the intermediate terms are called the **Geometric Means** of the two extremes.

Thus to insert any number of *geometric means* between two given quantities, is the same as to determine a *G.P.*, whose first and last terms and also the number of terms are given.

505. *To insert* n *geometric means between* a *and* b.

Let r be the common ratio of the required *G. P.*

Here, we have to find a *G. P.* of $(n+2)$ terms, of which a is the first term, b the last, so that b is the $(n+2)$th term of the *G. P.*

Now, the $(n+2)$th term of a series, whose first term is a, and whose common ratio is r, is

$$ar^{(n+2)-1}, \text{ i. e., } ar^{n+1}. \quad \text{(Art. 481).}$$

$$\therefore ar^{n+1}=b, \text{ so that } r^{n+1}=b/a\,;$$

$$\text{and } \therefore r=\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

The means may now be easily determined ; for they are

$$ar,\ ar^2,\ ar^3,\ \ldots\ \ldots ar^n.$$

$$\text{The } n\text{th mean}=ar^n=a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}=(ab^n)^{\frac{1}{n+1}}.$$

Ex. 1. Insert 3 geometric means between 2 and $10\frac{1}{8}$.

Here, $a=2$, $b=10\frac{1}{8}$; $\therefore r=(10\frac{1}{8}\div 2)^{\frac{1}{4}}=(\frac{81}{16})^{\frac{1}{4}}=\frac{3}{2}$.

Hence the means are 3, $4\frac{1}{2}$ and $6\frac{3}{4}$.

Exercise CLXLII.

1. Insert

(1) 3 *G. M.*'s between 1 and 16, and also between $\frac{1}{2}$ and 128.

(2) 4 *G. M.*'s between $-\frac{1}{10}$ and $3\frac{1}{5}$, and also between $\frac{2}{3}$ and $-5\frac{1}{16}$.

(3) 3 *G. M.*'s between 4 and 324. (C. F. A. 1890).

(4) 5 *G. M.*'s between $\frac{32}{81}$ and $4\frac{1}{2}$.

(5) 3 *G. M.*'s between 2 and 32, and also between 37 and 2997.

2. Insert 5 *mean proportionals* between 8 and 27.

3. Insert 2 *G. M.*'s between $\sqrt{2}$ and $\sqrt{3}$.

4. If n geometric means be inserted between a and b, prove their product is $(ab)^{\frac{n}{2}}$.

5. The arithmetic mean of the first and third terms of a *G. P.* is five times the second term. Find the common ratio.

6. Insert two numbers between 6 and 16 such that the first three may be in *A. P.* and the last three in *G. P.*

7. Find the ratio of $a : b$ when their arithmetic mean is to their geometric mean as $13 : 5$.

8. If a, b, c be in *G. P.*, and x and y be the *A. M.*'s between a, b and b, c respectively, prove that

$$\frac{2}{b}=\frac{1}{x}+\frac{1}{y} \text{ and } 2=\frac{a}{x}+\frac{c}{y}. \quad \text{(P. I. E. 1892).}$$

9. Find the geometric mean of $9x^2-24x+16$ and $4x^2+20x+25$.

10. The *G. P.* between a and b is to their *A. M.* as m is to n; shew that $a : b=n+\sqrt{(n^2-m^2)} : n-\sqrt{(n^2-m^2)}$. (A. I. E. 1889).

506. Problems. The following are illustrative examples.

Ex. 1. The sum of three numbers in *G.P.* is 21 and the sum of their squares is 189; find the numbers.

Let x, xy and xy^2 be the numbers, so that we have

$$x(1+y+y^2)=21 \ldots(1) \text{ and } x^2(1+y^2+y^4)=189 \ldots.(2)$$

Dividing the square of (1) by (2), we get

$$\frac{1+y+y^2}{1-y+y^2}=\frac{21\times 21}{189}=\frac{7}{3}, \text{ which solved gives } y=2 \text{ or } \tfrac{1}{2}.$$

Hence from (1) $x=3$ or 12, so that the numbers are 3, 6, 12.

Ex. 2. The sum of three numbers in *G. P.* is 21, and their product is 216. Find the numbers.

Let x/y, x, and xy be the numbers, so that

$$\frac{x}{y}+x+xy=21 \ldots(1) \text{ and } \frac{x}{y}.x.xy=216 \ldots(2)$$

From (2), $x^3=216$ and $\therefore x=6$.

" (1) $\frac{1}{y}+1+y=\frac{21}{6}=\frac{7}{2}$, or $y^2-\frac{5}{2}y+1=0$;

which solved gives $y=2$ or $\frac{1}{2}$.

Hence the numbers are 3, 6 and 12.

Exercise CLXIII.

1. The difference between the first and second of four numbers in *G. P.* is 12, and the difference between the third and fourth is 300; find them.

2. The sum of three quantities in *G. P.* is $24\frac{4}{5}$ and their product is 64; find them. (A. I. E. 1891).

3. The continued product of three numbers in $G.P.$ is 216, and the sum of their products, taken in pairs, is 156. Find the numbers.

4. The sum of three numbers in $G. P.$ is 38 and the sum of their squares is 532; find them.

5. A man saves each year half as much again as he did in the previous year. If he saved *Rs.*400 in the first year, in how many years will he have saved *Rs.*8312. 8*a.*?

6. Suppose a body moves eternally in this manner, *viz.* 20 miles the first minute, 19 miles the second, $18\frac{1}{20}$ miles the third, and so on in Geometrical Progression; required the utmost distance it can reach (C. F. A. 1864).

7. Find three numbers in $G. P.$, such that their sum is 19, and their continued product is 216.

8. The population of a country increases annually in $G.P.$, and in 4 years was raised from 10000 to 14641 souls; by what part of itself was it annually increased?

III HARMONICAL PROGRESSION.

507. Any number of quantities are in **Harmonical Progression** when the difference between the first and second of any three consecutive of them is to the difference between the second and the third as the first is to the third.

Thus, a, b, c, d, e &c., are in Harmonical Progression ($H. P.$), if $a-b : b-c :: a : c$,

$b-c : c-d :: b : d$, and so on.

508. *The reciprocals of quantities in Harmonical Progression are in Arithmetical Progression.*

Let a, b, c be three quantities in $H. P.$, then

$a-b : b-c :: a : c$; $\therefore$ $c(a-b)=a(b-c)$,

or $ac-bc=ab-ac$. Divide by abc; then

$$\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}; \therefore \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \text{ are in } A. P.$$

Thus, since 1, 3, 5, &c, $\frac{1}{4}$, $-\frac{1}{4}$, $-\frac{3}{4}$, &c., are in $A. P.$, their reciprocals 1, $\frac{1}{3}$, $\frac{1}{5}$, &c., 4, -4, $-\frac{4}{3}$, &c., are in $H.P.$

509. We cannot find the sum of *any* number of terms of an Harmonic series; but many problems with respect to such series may be solved by *inverting* the terms, and treating their reciprocals as in $A. P.$

Ex. 1. The 15th term of a *H. P.* is $\frac{1}{25}$, and the 23rd term is $\frac{1}{41}$: find the series.

Let a be the first term and d the common difference of the corresponding *A. P.* ; then

$25=$ the 15th term $=a+14d$;

and $41=$ the 23rd term $=a+22d$;

whence $d=2$ and $a=-3$.

Hence the *A. P.* is $-3,\ -1,\ 1,\ 3,\ 5\ldots\ldots$;

and the *H. P.* is $-\frac{1}{3},\ -1,\ 1,\ \frac{1}{3},\ \frac{1}{5},\ldots\ldots$

Ex. 2. Continue to 3 terms each way the series 2, 3, 6.

Since $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$ are in *A. P.* with the common difference $-\frac{1}{6}$,

$\therefore$ the Arithmetic series continued each way is

$$1,\ \frac{5}{6},\ \frac{2}{3},\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{6},\ 0,\ -\frac{1}{6},\ -\frac{1}{3};$$

and the Harmonic series is

$$1,\ \frac{6}{5},\ \frac{3}{2},\ 2,\ 3,\ 6,\ \infty,\ -6,\ -3.$$

Exercise CLXLIV.

1. Find a *H. P* in which

(1) the 3rd term is $1\frac{2}{13}$, and the 21st term is $\frac{3}{14}$.

(2) the 2nd term is $1\frac{1}{2}$, and the 15th term is $\frac{1}{5}$.

2. Find the nth term of the series

$$4+4\tfrac{2}{7}+4\tfrac{8}{13}+5+\&c. \quad \text{(C. F. A. 1886).}$$

3. Find the series in *H. P.*, in which the 39th term is $\frac{1}{11}$ and the 54th term is $\frac{1}{26}$.

4. Continue the *H. P.* to 3 terms each way :—

(1) $2,\ \frac{4}{3},\ 1.$ (2) $1\frac{1}{2},\ 2\frac{1}{7},\ 3\frac{3}{4}.$ (3) $1,\ 1\frac{1}{6},\ 1\frac{2}{5}.$

5. The 1st and 5th terms of a *H. P.* are 3 and 7 ; find the 20th term.

6. Find the mth term of a *H. P.*, whose first term is a, whose last term is c, and whose number of terms is n. (M. F. A. 1884).

7. In a *H. P.*, if the pth term $=qr$ and the qth term $=pr$, prove that the rth term $=pq$. (A. I. E. 1892).

8. If the mth term of a *H. P.* be n, and the nth term m, find the $(m+n)$th term.

9. If P, Q, R be the pth, qth and rth terms of a *H. P.*, shew that $(q-r)QR+(r-p)RP+(p-q)PQ=0$. (B. P. E. 1887).

10. An $A. P.$ and a $H. P.$ have the same first term, the same last term and the same number of terms; prove that the product of the rth term from the beginning in one series and the rth term from the end in the other, is independent of r. (B. P. E. 1890).

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510. Harmonic Mean. When three quantities are in Harmonical Progression, the middle one is called the **Harmonic Mean** of the other two.

Thus, if a, x and b are in $H. P.$, then x is the *Harmonic Mean* between a and b.

Hence, by Art. 493, we have $\frac{1}{a}, \frac{1}{x}, \frac{1}{b}$ in $A. P.$

$$\therefore \frac{1}{x}-\frac{1}{a}=\frac{1}{b}-\frac{1}{x}; \quad \therefore \frac{2}{x}=\frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab}.$$

$$\therefore x=\frac{2ab}{a+b}.$$

Thus, *the harmonic mean of two quantities is equal to twice their product divided by their sum.*

511. When any number of quantities are in harmonical progression, all the intermediate terms are called the **Harmonic Means** of the two extremes.

We find the harmonic means between two given quantities by first finding the arithmetic means between the reciprocals of the two given quantities

512. *To insert* n *harmonic means between* a *and* b.

Insert n arithmetic means between $1/a$ and $1/b$.

From Art. 474, we see that they are

$$\frac{n.\frac{1}{a}+\frac{1}{b}}{n+1}, \quad \frac{(n-1).\frac{1}{a}+\frac{2}{b}}{n+1}, \quad \frac{(n-2).\frac{1}{a}+\frac{3}{b}}{n+1}, \ldots\ldots \frac{\frac{1}{a}+\frac{n}{b}}{n+1},$$

$$i.\ e., \quad \frac{nb+a}{(n+1)ab}, \quad \frac{(n-1)b+2a}{(n+1)ab}, \quad \frac{(n-2)b+3a}{(n+1)ab}, \ldots\ldots\ldots \frac{b+na}{(n+1)ab},$$

Hence the required harmonic means are

$$\frac{(n+1)ab}{a+nb}, \quad \frac{(n+1)ab}{2a+(n-1)b}, \quad \frac{(n+1)ab}{3a+(n-2)b}, \ldots\ldots\ldots\ldots \frac{(n+1)ab}{4a+b}.$$

Thus, the pth mean $=\dfrac{(n+1)ab}{pa+(n-p+1)b}$.

Ex. Insert 4 harmonic means between 2 and 12.

Here, we have to insert 4 *arithmetic* means between $\frac{1}{2}$ and $\frac{1}{12}$.

Hence, by Art. 488, we have

$$\tfrac{1}{12}=\tfrac{1}{2}+5d, \text{ so that } 5d=-\tfrac{5}{12} \text{ and } \therefore d=-\tfrac{1}{12}.$$

Hence, the Arith. means are $\frac{5}{12}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$;

and $\therefore$ the Harm. means are $2\frac{2}{5}$, 3, 4, 6.

513. If **A**, **G**, **H** be the arithmetic, geometric, and harmonic means between a and b, we have proved that

$$A=\frac{a+b}{2}\ldots\ldots(1)\ ;\quad G=\sqrt{ab}\ldots\ldots.(2)\ ;\quad H=\frac{2ab}{a+b}\ldots\ldots(3).$$

514. To prove that G is the Geometric mean between A and H; and that A, G, H, are in order of magnitude, A being the greatest.

Since $A=\frac{a+b}{2}$, and $H=\frac{2ab}{a+b}$, $\therefore A\times H=\frac{a+b}{2}\times\frac{2ab}{a+b}=ab=G^2$;

$\therefore G=\sqrt{AH}$, or G is the Geom. mean between A and H.

$$\text{Also, } A-H=\frac{a+b}{2}-\frac{2ab}{a+b}=\frac{(a+b)^2-4ab}{2(a+b)}=\frac{(a-b)^2}{2(a+b)}$$

= a positive quantity for all positive values of a and b.

Hence A is $>H$, and of course, $>G$, whose value (being the Geom. mean between them) lies between those of A and H.

Thus, A, G, and H form a descending $G.\ P.$

515. Three quantities a, b, c, are in Arith., Geom., or Harm. Progression, according as

$$\frac{a-b}{b-c}=\frac{a}{a},\ \text{ or }=\frac{a}{b},\ \text{ or }=\frac{a}{c}.$$

(i) $\frac{a-b}{b-c}=\frac{a}{a}=1$; $\therefore a-b=b-c$, and a, b, c, are in $A.P.$

(ii) $\frac{a-b}{b-c}=\frac{a}{b}$; $\therefore ab-b^2=ab-ac$, or $b^2=ac$;

$\therefore b/a=c/b$, and a, b, c, are in $G.P.$

(iii) $\frac{a-b}{b-c}=\frac{a}{c}$; $\therefore ac-bc=ab-ac$, or (dividing each by abc),

$\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}$; whence $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ are in $A.P.$,

and $\therefore a$, b, c, are in $H.P.$

Exercise CLXLV.

1. Insert

 (1) 2 harmonic means between 2 and 4.

 (2) 6 harmonic means between 3 and $\frac{6}{23}$.

 (3) 5 harmonic means between $\frac{2}{3}$ and $\frac{6}{15}$.

2. Find the Arith., Geom. and Harm. means between

 (1) 2 and $4\frac{1}{2}$. (2) $3\frac{3}{8}$ and $1\frac{1}{2}$.

3. Find a fourth harmonic proportional to 6, 8, and 12.

4. Insert 3 *H. M.*'s between 4 and 2. (C. F. A. 1867).

5. Insert 4 *H. M.*'s between 1 and 30. (A. I. E. 1892).

6. If a, b and c are in *H.P.*, prove that

$$\frac{1}{a}+\frac{1}{b+c},\ \frac{1}{b}+\frac{1}{c+a} \text{ and } \frac{1}{c}+\frac{1}{a+b} \text{ are in } H.P.$$

7. If a, b and c be in *A.P.*, prove that

$$\frac{bc}{a(b+c)},\ \frac{ca}{b(c+a)} \text{ and } \frac{ab}{c(a+b)} \text{ are in } H.P. \text{ (B. P. E. 1891).}$$

8. Prove that $(x^2+y^2)(x^2+xy+y^2)$, $x^4+x^2y^2+y^4$ and $(x^2+y^2)\times(x^2-xy+y^2)$ are in *H.P.*

9. If a, b and c are in *H.P.*, prove that

$$\frac{a}{b+c},\ \frac{b}{c+a} \text{ and } \frac{c}{a+b} \text{ are also in } H.P.$$

10. The *A.M.* of two numbers exceeds the *G.M.* by $\frac{3}{2}$, and the *G.M.* exceeds the *H.M.* by $\frac{6}{5}$. Find the numbers. (C. F. A. 1870).

11. The sum and difference of the Arith. and Geom. means between two numbers are 9 and 1 respectively; find them.

12. The Harm. mean between two numbers is $\frac{24}{25}$ of the Arith. mean, and one of the numbers is 4; find the other.

13. The difference of the *A.M.* and *H.M.* between two numbers is $1\frac{4}{5}$; find the numbers, one being four times the other.

14. Find two numbers whose difference is 8, and the *H.M.* between them is $1\frac{4}{5}$.

15. The square of the *A.M.* between two numbers exceeds that of the *G.M.* by 400, and square of the *G.M.* exceeds that of the *H.M.* by 144. What are the numbers? (C. F. A. 1874).

16. Find two numbers such that the sum of their *A.M.*, *G.M.*, and *H.M.* is $9\frac{4}{5}$, and the product of these means is 27.

17. If a, b and c be in $A.P.$, and $b, c,$ and d in $H.P.$, prove that a, b, c and d are proportionals.

18. If a_1, a_2 be the $A.M.$'s, h_1, h_2 the $H.M.$'s and g_1, g_2 the $G.M.$'s between a and b, shew that

$$a_1 h_2 = a_2 h_1 = g_1 g_2 = ab. \quad \text{(M. F. A. 1891).}$$

19. If a, b and c be in $H.P.$, prove that

(1) $a^2+c^2>2b^2$, if a, b and c be positive.

(2) $a : c :: 2a^2+bc : 2c^2+ab$.

20. If $a, b,$ and c be in $A.P.$, and a^2, b^2 and c^2 in $H.P.$, prove that either $-\frac{1}{2}a$, b and c are in $G.P.$, or else $a, b,$ and c are equal. (C. F. A. 1904).

IV. OTHER SIMPLE SERIES.

516 Besides the Progressions, there are some other simple, but important, series the successive terms of which are formed according to simple laws. We shall now consider the summation of some such series, which depend on the rules laid down in the preceding Articles.

Ex. 1. Sum the series $1.2+2.3+3.4+\ldots\ldots$ to n terms.

Here, the rth term of $1+2+3+\ldots$ is r,

and the rth term of $2+3+4+\ldots\ldots$ is $r+1$.

Hence the rth term of the given series $r(r+1)=r^2+r$.

Making $r=1, 2, 3, \ldots\ldots n$, in succession, we have

the series $=(1^2+2^2+3^2+\ldots\ldots+n^2)+(1+2+3+\ldots+n)$.

$$\text{Hence sum reqd.} = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}, \text{ (Arts. 490 \& 489).}$$

$$= \frac{n(n+1)}{2}\left\{\frac{2n+1}{3}+1\right\} = \frac{n(n+1)(n+2)}{3}.$$

Ex. 2. Sum the series $1^2+3^2+5^2+\ldots\ldots$ to n terms. (M.F.A. 1889).

Here, the rth term of $1+3+5+\ldots\ldots$ is $2r-1$.

Hence, the rth term of the given series is $(2r-1)^2=4r^2-4r+1$.

Making $r=1, 2, 3, \ldots n$, in succession, we have

the series $=4(1^2+2^2+3^2+\ldots\ldots+n^2)-4(1+2+3+\ldots\ldots+n)$
$+(1+1+\ldots\ldots$ to n terms.)

Hence, sum reqd $=4\times\frac{n(n+1)(2n+1)}{6}-4\times\frac{n(n+1)}{2}+1\times n$

$$=2n(n+1)\left\{\frac{2n+1}{3}-1\right\}+n=\frac{4n(n^2-1)}{3}+n$$

$$=\frac{n(4n^2-1)}{3}.$$

Ex. 3. Sum the series $1+3+6+10+15+\ldots\ldots$to n terms.

Let S denote the reqd. sum, and t_n the nth term.

Then $S=1+3+6+10+15+\ \ldots\ldots\ +t_n$,

also $S=\quad 1+3+\ 6+10+\ \ldots\ldots+t_{n-1}+t_n$.

Hence, by subtraction, we have

$$0=(1+2+3+4+5+\ldots\ldots\text{to } n \text{ terms})-t_n.$$

$$\therefore\ t_n=1+2+3+4+5+\ldots\ldots\text{to } n \text{ terms}=\tfrac{1}{2}n(n+1).$$

$$=\tfrac{1}{2}(n^2+n).$$

Making $n=1, 2, 3,\ldots$successively, we get

the series $=\frac{1}{2}\{(1^2+2^2+3^2+\ldots\ +n^2)+(1+2+3+\ldots+n)\}$.

Hence sum reqd. $=\frac{1}{2}\left\{\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}\right\}$

$$=\tfrac{1}{2}\times\frac{n(n+1)}{2}\left(\frac{2n+1}{3}+1\right)=\frac{n(n+1)(n+2)}{6}.$$

Ex. 4. Sum to n terms the series

$5+55+555+5555+\ldots\ldots\ldots\ldots$

The given series $=\frac{5}{9}(9+99+999+9999+\ldots\ldots\ldots\ldots\ldots\ldots)$

$$=\tfrac{5}{9}\{(10-1)+(10^2-1)+(10^3-1)+\ldots\ldots\}$$

$$=\tfrac{5}{9}\{(10+10^2+10^3+\ldots\ldots\text{ to } n \text{ terms})$$

$$-(1+1+1+\ldots\ \text{to } n \text{ terms})\}.$$

Hence sum reqd. $=\frac{5}{9}\left\{\frac{10(10^n-1)}{10-1}-n\right\}=\frac{50}{81}(10^n-1)-\frac{5}{9}n.$

Ex. 5. Sum to n terms the series

$1+3+7+15+31+\ldots\ldots\ldots$(C. F. A. 1876).

Let S denote the reqd. sum and t_n the nth term.

Then $S=1+3+7+15+31+\ldots\ldots\ldots+t_n$,

also $S=\quad 1+3+\ 7+15+\ldots\ldots\ldots+t_{n-1}+t_n$.

Hence by subtraction, we have

$$0=(1+2+4+8+16+\dots\dots\text{to } n \text{ terms})-t_n;$$

$$\therefore\ t_n=1+2+4+8+16+\dots\dots\text{to } n \text{ terms}=\frac{2^n-1}{2-1}=2^n-1.$$

Making $n=1, 2, 3, \dots\dots$ successively, we get

the series $=(2+2^2+2^3+\dots\text{to } n \text{ terms})-(1+1+1\dots\text{to } n \text{ terms})$

$$=\frac{2(2^n-1)}{2-1}-n=2^{n+1}-(n+2).$$

Ex. 6. Sum the series $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\dots$ to n terms.

Here, the rth term $=\frac{1}{r(r+1)}=\frac{1}{r}-\frac{1}{r+1}$.

Making $r=1, 2, 3, \dots n$, in succession, we obtain

the series $=(\frac{1}{1}-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+\dots+\left(\frac{1}{n-1}-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{n+1}\right)$

$$=1-\frac{1}{n+1}=\frac{n}{n+1},$$

(since all the terms except the first and last destroy one another).

Note. When n is infinite, $\frac{1}{n+1}$ is zero. Hence the sum of an infinite number of terms of the given series $=1$.

517 Mixed Series. The rth term of such a series consists of a pair of factors, one of these factors forming an *A.P.* and the other a *G. P.*

Ex. Sum to n terms the series $1+3x+5x^2+7x^3+\dots$

Here, the factors $1, 3, 5, \dots$ form an *A.P.*, whose first term is 1 and whose common difference is 2, so that the nth term $=1+2(n-1)=2n-1$.

The other factors $1, x, x^2, x^3\dots$ form a *G.P.*, whose nth term $=x^{n-1}$.

Hence the nth term of the given series $=(2n-1)x^{n-1}$.

Let S denote the required sum; then

$$S=1+3x+5x^2+7x^3+\dots\dots+(2n-1)x^{n-1}\dots\dots\ (1)$$

Multiply both sides of the equation by x,

$$\therefore\ Sx=x+3x^2+5x^3+\dots+(2n-3)x^{n-1}+(2n-1)x^n\dots\dots(2)$$

Subtracting (2) from (1), we have

$$S-Sx=1+2x+2x^2+2x^3+\ldots\ldots+2x^{n-1}-(2n-1)x^n,$$

$$\text{or } S(1-x)=1+2x(1+x+x^2+\ldots\ldots+x^{n-2})-(2n-1)x^n \ldots(3)$$

But $1+x+x^2+\ldots+x^{n-2}$ is a *G. P.* of $(n-1)$ terms, having 1 as the first term and x as the common ratio, and thus the sum $=\dfrac{1-x^{n-1}}{1-x}$.

Hence, from (3), we obtain

$$S(1-x)=1+2x.\frac{1-x^{n-1}}{1-x}-(2n-1)x^n=1+\frac{2x}{1-x}-\frac{2x^n}{1-x}-(2n-1)x^n,$$

$$=\frac{1+x}{1-x}-\frac{2x^n+(2n-1)x^n(1-x)}{1-x}=\frac{1+x}{1-x}-\frac{(2n+1)x^n-(2n-1)x^{n+1}}{1-x},$$

$$\therefore S=\frac{1+x}{(1-x)^2}-\frac{(2n+1)x^n-(2n-1)x^{n+1}}{(1-x)^2}.$$

Note. If x be < 1, and n be infinitely great, then x^n and x^{n+1} are both too small, and thus the sum to infinity $=\dfrac{1+x}{(1-x)^2}$.

Exercise CLXLVI.

1. Find the sum of the series to n terms :—

(1) $2+5+10+17+\ldots\ldots\ldots$ (C. F. A. 1877 ; B. P. E. 1885).

(2) $2+7+14+23+34+\ldots\ldots$ (C. F. A. 1878 ; B. P. E. 1885).

(3) $5^2+7^2+9^2+\ldots+25^2$. (C. F. A. 1888).

(4) $1.3+2.4+3.5+\ldots\ldots\ldots$

(5) $1.2.3+2.3.4+3.4.5+\ldots\ldots$

(6) $2^2+5^2+8^2+11^2+\ldots\ldots$

(7) $3.5+5.7+7.9+9.11+\ldots\ldots$

(8) $1.3.5+3.5.7+5.7.9\ldots\ldots$

(9) $1.2.4+2.3.5+3.4.6+\ldots\ldots$

(10) $1^3+3^3+5^3\ldots\ldots\ldots\ldots$

(11) $2.1^2+3.2^2+4.3^2\ldots$ (C. F. A. 1887).

2. Sum the series

$$n.1+(n-1).2+(n-2).3+(n-3).4+\ldots+1.n. \quad \text{(C.F.A 1889)}$$

3. Shew that

$$1+2^2+3+4^2+5+6^2+\ldots\ldots\ldots \text{ to } n \text{ terms}$$
$$=\tfrac{1}{12}(n+1)(2n^2+n+3) \text{ or } \tfrac{1}{12}n(n+4)(2n+1).$$

according as n is odd or even. (B. P. E. 1892).

4. Sum to n terms $a^2+(a+1)^2+(a+2)^2+\ldots\ldots$

5. Sum $1+(1+a)r+(1+a+a^2)r^2+\ldots\ldots\ldots$ to infinity, r and a being <1.

6. Sum the following series to n terms

$$a+b+(a^2+2ab)+(a^3+3a^2b)+\ldots\ldots\ldots \quad \text{(C. F. A. 1891).}$$

7. Shew that the sum of the products, taken two and two together, of the natural numbers from 1 to n is

$$\tfrac{1}{24}(n-1)n(n+1)(3n+2).$$

8. Sum to n terms :—

(1) $1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+\ldots$(C. F. A. 1884). (2) $\frac{1}{3}+\frac{3}{9}+\frac{5}{27}+\ldots$(C. F. A. 1880).

(3) $1+\frac{2}{5}+\frac{3}{5^2}+\frac{4}{5^3}+\ldots\ldots$(C. F. A. 1889).

(4) $a(a+b)+(a+b)(a+2b)+(a+2b)(a+3b)+\ldots\ldots$

9. Sum to infinity :—

(1) $\frac{5}{7}+\frac{7}{21}+\frac{9}{63}+\frac{11}{189}+\ldots\ldots\ldots$ (B. P. E. 1883).

(2) $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\ldots$, and also to n terms.

(3) $a+(a+b)r+(a+2b)r^2+(a+3b)r^3+\ldots(r<1)$ and also to n terms. (B. P. E. 1889).

10. Sum to n terms the series whose rth term is $(2r+1))3^r$.

11. Sum $2+22+222+2222+\ldots$ to n terms.

12. Sum to infinity (r and br being each <1)

$$ar+(a+ab)r^2+(a+ab+ab^2)r^3+\ldots\ldots$$

13. If S_n denote the sum of n terms of a given series in *G. P.*, find the value of $S_1+S_2+S_3+\ldots+S_n$. (C. F. A. 1861).

14. Sum to n terms :—

(1) $(a+b)+(a^2+2b)+(a^3+3b)+\ldots\ldots\ldots$

(2) $a+b+3a+2b+5a+4b+\ldots$, and to 10 terms. (C. F. A. 1868).

(3) $(2a-\frac{1}{3})+(3a+\frac{1}{12})+(4a-\frac{1}{48})+\ldots\ldots\ldots$

(4) $1+5+13+29+61+\ldots\ldots\ldots$ (5) $9+99+999+\ldots\ldots$

(6) $1+\frac{3}{2}+\frac{7}{4}+\frac{15}{8}+\ldots\ldots$(C. F. A. 1880).

(7) $3+6+11+20+\ldots\ldots$(B. P. E. 1886).

(8) $\cdot 55+\cdot 555+\cdot 5555+\ldots\ldots$

APPENDIX.

SIMULTANEOUS QUADRATIC EQUATIONS.

I. TWO UNKNOWNS.

1. We shall now consider a few Examples of Simultaneous Quadratic Equations involving **two** unknowns. The solution of these is generally more difficult; but there are certain cases of frequent occurrence, for which the following observations will be useful.

2 Elimination by Substitution. When one equation is linear and the other quadratic, find the value of one of the unknown quantities from the linear equation in terms of the other and then substitute in the quadratic equation. The resulting equation in very many cases will be a quadratic, which may be solved by the ordinary rules.

Ex. 1. Solve $x+2y=4 \quad \ldots(1), \quad 2xy-y^2=3 \ldots\ldots (2)$

From (1) express x in terms of y; thus

$$x=4-2y.$$

Substitute this value of x in (2) and we get

$$2(4-2y)y-y^2=3, \text{ or } 5y^2-8y+3=0;$$

$$\therefore (y-1)(5y-3)=0; \therefore y=1 \text{ or } \tfrac{3}{5}.$$

Hence, from (1) $x=2$ or $\tfrac{14}{5}$.

Ex. 2. Solve $2x-y+1=0 \ldots (1), \quad 13x^2-2y^2+8x=18 \ldots\ldots(2)$.

From (1), we have $y=2x+1$.

Substituting this value of y in (2), we get

$$13x^2-2(2x+1)^2+8x=18, \text{ or } 5x^2=20;$$

$$\therefore x^2=4 \text{ and } \therefore x=\pm 2.$$

Hence from (1) $y=5$ or -3.

Exercise I.

Solve the following equations :—

1. $\left.\begin{array}{l} x+2y=6 \\ 3x^2-xy=20 \end{array}\right\}$ 2. $\left.\begin{array}{l} 3y-2x=1 \\ 5x^2-2xy=3 \end{array}\right\}$ 3. $\left.\begin{array}{l} 5x+2y=7 \\ 7x^2-8xy=159 \end{array}\right\}$

4. $\left.\begin{array}{l} x^2+y^2=25 \\ 3x+4y=24 \end{array}\right\}$ 5. $\left.\begin{array}{l} 2x+3y=8 \\ x^2+xy+y^2=7 \end{array}\right\}$ 6. $\left.\begin{array}{l} 3x+1=2x+y \\ 2(x+y)=4x^2-xy \end{array}\right\}$

7. $\left.\begin{array}{r} x-2y=10 \\ x^2+y^2=25 \end{array}\right\}$ 8. $\left.\begin{array}{r} x-y=2 \\ 15(x^2-y^2)=16xy \end{array}\right\}$ 9. $\left.\begin{array}{r} x^2+3xy+y^2=45 \\ x+y=9 \end{array}\right\}$

10. $\left.\begin{array}{r} \frac{1}{10}(3x+5y)+\frac{1}{6}(4x-3y)=6\frac{23}{30} \\ 3x^2+2y^2=179 \end{array}\right\}$ 11. $\left.\begin{array}{r} xy=(x-\frac{3}{7})(y+\frac{2}{3}) \\ x^2y^2=(x^2+3)(y^2-4) \end{array}\right\}$

12. $\left.\begin{array}{l} 4x-5y=1 \\ 2x^2-xy+3y^2+3x-4y=47 \end{array}\right\}$ 13. $\left.\begin{array}{l} 2x+3y=17 \\ 3x^2-4xy+8y^2=183 \end{array}\right\}$

14. $\left.\begin{array}{r} 2x+3y=8 \\ 9x^2-y^2=5 \end{array}\right\}$ 15. $\left.\begin{array}{r} 5x+2y=12 \\ 2x^2+3xy+y^2=15 \end{array}\right\}$ (C. F A. 1888.)

3. Equations which can be reduced to such linear equations as $x+y=a$ and $x-y=b$.

Then x and y can be found by addition and subtraction.

Ex. 1. Solve $x+y=10 \ldots(1)$, $xy=24 \ldots(2)$

We have $(x-y)^2=(x+y)^2-4xy=100-96=4$.

$\left.\begin{array}{l} \therefore\ x-y=\pm 2 \\ \text{and } x+y=10 \end{array}\right\}$ Hence by addition and subtraction, $2x=12$ or 8 and $2y=8$ or 12.

$\therefore\ x=6$ or 4 and $y=4$ or 6.

Ex. 2. Solve $x^2+y^2=65 \ldots(1)$, $x+y=11 \ldots(2)$

We have $(x-y)^2=2(x^2+y^2)-(x+y)^2=130-121=9$.

$\left.\begin{array}{l} \therefore\ x-y=\pm 3 \\ \text{and } x+y=11 \end{array}\right\}$ Hence by addition and subtraction, $2x=14$ or 8 and $2y=8$ or 14.

$\therefore\ x=7$ or 4 and $y=4$ or 7.

Ex. 3. Solve $3x-2y=7 \ldots(1)$, $xy=20 \ldots(2)$

We have $(3x+2y)^2=(3x-2y)^2+24xy=49+480=529$.

$\left.\begin{array}{l} \therefore\ 3x+2y=\pm 23 \\ \text{and } 3x-2y=\quad 7 \end{array}\right\}$ Hence by addition and subtraction, $6x=30$ or -16 and $4y=16$ or -30.

$\therefore\ x=5$ or $-2\frac{2}{3}$ and $y=4$ or $-7\frac{1}{2}$.

Exercise II.

Solve the following equations :—

1. $\left.\begin{array}{r} x^2+y^2=25 \\ x+y=1 \end{array}\right\}$ 2. $\left.\begin{array}{r} 2(x-y)=11 \\ xy=20 \end{array}\right\}$ 3. $\left.\begin{array}{r} x^2+y^2=25 \\ xy=12 \end{array}\right\}$

4. $\left.\begin{array}{r} x+y=30 \\ xy=224 \end{array}\right\}$ 5. $\left.\begin{array}{r} x^2-y^2=16 \\ x+y=8 \end{array}\right\}$ 6. $\left.\begin{array}{r} x^2+y^2=85 \\ xy=42 \end{array}\right\}$

7. $\left.\begin{array}{r} x^2+y^2=89 \\ xy=40 \end{array}\right\}$ 8. $\left.\begin{array}{r} 2x+y=7 \\ xy=3 \end{array}\right\}$ 9. $\left.\begin{array}{r} 2x+3y=23 \\ xy=20 \end{array}\right\}$

10. $\left.\begin{array}{r} 3y-5x=1 \\ xy=2 \end{array}\right\}$ 11. $\left.\begin{array}{r} x-2y=2 \\ xy-12 \end{array}\right\}$ 12. $\left.\begin{array}{r} x^2+y^2=a^2 \\ x+y=b \end{array}\right\}$

13. $\left.\begin{array}{r} x-y=10 \\ x^2+y^2=178 \end{array}\right\}$ 14. $\left.\begin{array}{r} x-y=10 \\ xy=39 \end{array}\right\}$ 15. $\left.\begin{array}{r} xy=a^2 \\ x-y=b \end{array}\right\}$

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4. The solution of many other equations of a more difficult nature can be made to depend on the solution of such Examples as are in the previous Articles.

Ex. 1. Solve $x^3+y^3=341\ldots(1)$, $x+y=11\ldots(2)$

Dividing (1) by (2) we get $x^2-xy+y^2=31\ldots(3)$

Squaring (2) $\qquad x^2+2xy+y^2=121,$

Subtracting, $\qquad -3xy=-90$; $\therefore\ xy=30\ldots\ldots(4)$

Now, using (2) and (4) we get the required solutions, as in Art. 3.

Thus $x=5$ or 6 and $y=6$ or 5.

Otherwise thus :—

Cubing (2), we have $x^3+y^3+3xy(x+y)=1331\ldots(3)$

Substituting (1) and (2) in (3), we have

$3xy\times 11=1331-341=990$; $\therefore\ xy=30\ldots\ldots(4)$

Now, proceed as above.

Ex. 2. Solve $x^4+x^2y^2+y^4=651\ldots(1)$, $x^2-xy+y^2=21\ldots(2)$

Dividing (1) by (2), we have $\left.\begin{array}{r} x^2+xy+y^2=31 \\ \text{and } x^2-xy+y^2=21 \end{array}\right\}$

Hence by addition and subtraction, we have

$2(x^2+y^2)=52$ and $2xy=10$; $\therefore\ x^2+y^2=26$ and $xy=5$.

Now, proceed as in Art. 3.

Thus we get $x=5$ or 1, and $y=1$ or 5.

Ex. 3. Solve $4xy=96-x^2y^2\ldots(1)$, $x+y=6\ldots(2)$.

From (1), we get $x^2y^2+4xy+4=96+4=100$.

Taking the sq. root, $xy+2=\pm 10$,

$\therefore\ xy=\pm 10-2=8$ or $-12\ldots(3)$

Now, using (2) and (3), we get the required solutions.

Thus, $x=4$ or 2 and $y=2$ or 4.

Ex. 4. Solve $\left.\begin{array}{r} x^2+3x+y=73-2xy \ldots\ldots(1) \\ y^2+3y+x=44 \ldots\ldots(2) \end{array}\right\}$

Adding (1) and (2), we get

$(x+y)^2+4(x+y)=117$;

$\therefore\ (x+y)^2+4(x+y)+4=117+4=121.$

Taking the sq. root, $(x+y)+2=\pm 11$;

$\therefore\ x+y=-2\pm 11=9$ or -13.

Taking the first value of $x+y$, we have $x=9-y$.

Substituting the above in (2), we get

$$y^2+3y+9-y=44, \text{ or } y^2+2y+1=36.$$

Taking the sq. root, $y+1=\pm 6$; $\therefore\ y=-1\pm 6=5$ or -7.

Hence $x=4$ or 16.

5. The method of solving an equation will not be altered if instead of x and y, their **reciprocals** $1/x$ and $1/y$ occur throughout the equations. Thus,

Ex. 1. Solve $\frac{1}{x}+\frac{1}{y}=\frac{5}{6}\ldots\ldots(1)$, $\frac{1}{x^2}+\frac{1}{y^2}=\frac{13}{36}\ldots(2)$

Let $\frac{1}{x}=A$ and $\frac{1}{y}=B$, then the equations become

$$A+B=\tfrac{5}{6}\ldots(3), \text{ and } A^2+B^2=\tfrac{13}{36}\ldots(4).$$

Now, using (3) and (4), we shall get

$$A=\tfrac{1}{2} \text{ or } \tfrac{1}{3} \text{ and } B=\tfrac{1}{3} \text{ or } \tfrac{1}{2}.$$

Hence $x=\frac{1}{A}=2$ or 3 and $y=\frac{1}{B}=3$ or 2.

Exercise III.

Solve the following equations :—

1. $\left.\begin{array}{r} x^3+y^3=1331 \\ x+y=11 \end{array}\right\}$ **2.** $\left.\begin{array}{r} x+y=12 \\ x^3+y^3=2196 \end{array}\right\}$ **3.** $\left.\begin{array}{r} x^3-y^3=973 \\ x-y=7 \end{array}\right\}$

4. $\left.\begin{array}{r} x+y=11 \\ x^3+y^3=1001 \end{array}\right\}$ **5.** $\left.\begin{array}{r} x+y=6 \\ x^3+y^3=72 \end{array}\right\}$ **6.** $\left.\begin{array}{r} x-y=1 \\ x^3-y^3=19 \end{array}\right\}$

7. $\left.\begin{array}{r} x^3+y^3=189 \\ x^2y+xy^2=180 \end{array}\right\}$ **8.** $\left.\begin{array}{r} x+y=9 \\ \sqrt[3]{x}+\sqrt[3]{y}=3 \end{array}\right\}$ **9.** $\left.\begin{array}{r} x^2+xy=a^2 \\ y^2+xy=b^2 \end{array}\right\}$

10. $\left.\begin{array}{r} x^4+x^2y^2+y^4=133 \\ x^2-xy+y^2=\quad 7 \end{array}\right\}$ 11. $\left.\begin{array}{r} x^3-y^3=9 \\ x^2+xy+y^2=3 \end{array}\right\}$ 12. $\left.\begin{array}{r} x+y=\qquad 2a \\ x^3+y^3=2a^3+6a \end{array}\right\}$

13. $\left.\begin{array}{r} {}^3+y^3=407 \\ x+y=\quad 11 \end{array}\right\}$ 14. $\left.\begin{array}{r} x+y+x^2+y^2=18 \\ xy=\quad 6 \end{array}\right\}$ 15. $\left.\begin{array}{r} x^2+y^2=13 \\ 2xy-x-y=\quad 7 \end{array}\right\}$

16. $\begin{array}{r} x^4+x^2y^2+y^4=481 \\ x^2+xy+y^2=\quad 37 \end{array}$ 7. $\left.\begin{array}{r} x^4+x^2y^2+y^4=931 \\ x^2+xy+y^2=\quad 49 \end{array}\right\}$ 18. $\left.\begin{array}{r} x^3-y^3=124 \\ x-y=\quad 4 \end{array}\right\}$

19. $\frac{1}{x}+\frac{1}{y}=\frac{1}{6},\ \frac{1}{x^2}+\frac{1}{y^2}$ 20. $\frac{x^2}{y^2}=\frac{85}{9}-\frac{4x}{y},\ x-y=2.$

21. $\left.\begin{array}{l} 6(x+y)=5xy \\ \frac{1}{x^3}+\frac{1}{y^3}=\frac{35}{216} \end{array}\right|$ 22. $\begin{array}{l} \frac{1}{x} \\ \frac{1}{x^2}-\frac{1}{y^2} \end{array}$ 23. $\left.\begin{array}{r} \frac{1}{x^2}+\frac{1}{y^2}=\frac{41}{400} \\ xy=\quad 20 \end{array}\right\}$

24. $\frac{1}{x}+\frac{1}{y}=\frac{5}{6},\ \frac{1}{x^3}+\frac{1}{y^3}=\frac{[illegible]}{216}.$ 25. $\frac{1}{x^2}+\frac{1}{y^2}=\frac{[illegible]}{64},\ \frac{1}{x}-\frac{1}{y}=\frac{1}{4}.$

6. **Homogeneous Equations.** When both the equations are of the same degree and homogeneous with respect to x and y, in all those terms of it which involve x and y, put $y=vx$, by means of which we may generally without difficulty obtain an equation involving v only, which being determined, x and y may then be found.

Ex. 1. Solve $x^2+3xy+4y^2=14 \ldots\ldots(1)$
$3x^2+4xy+5y^2=25 \ldots\ldots(2)$

Assume $y=vx$.

Now substitute this value of y in both (1) and (2).

From (1), we have $x^2(1+3v+4v^2)=14 \ldots\ldots (3)$
" (2) " $x^2(3+4v+5v^2)=25 \ldots\ldots(4)$

Dividing (3) by (4), $\frac{1+3v+4v^2}{3+4v+5v^2}=\frac{14}{25}.$

Multiplying across and transposing,

$$30v^2+19v-17=0, \text{ or } (2v-1)(15v+17)=0.$$

$\therefore\ v=\frac{1}{2}$ or $-\frac{17}{15}$ and $\therefore\ y=\frac{1}{2}x$ or $-\frac{17}{15}x$

Using the value $y=\frac{1}{2}x$ in (1),	Using the value $y=-\frac{17}{15}x$ in (1),
we have $x^2+\frac{3}{2}x^2+x^2=14$	we have $x^2-\frac{17}{5}x^2+\frac{1156}{225}x^2=14$
$\therefore\ \frac{7}{2}x^2=14$, $\therefore\ x=\pm 2$.	$\therefore\ \frac{616}{225}x^2=14$; $\therefore\ x=\pm\dfrac{15}{2\sqrt{(11)}}$
Hence $y=\pm 1$.	Hence $y=\mp\dfrac{17}{2\sqrt{(11)}}$.

Otherwise thus :—

Dividing (1) by (2), $\dfrac{x^2+3xy+4y^2}{3x^2+4xy+5y^2}=\dfrac{14}{25}$.

Multiplying across, $25x^2+75xy+100y^2=42x^2+56xy+70y^2$,

$\therefore\ 30y^2+19xy-17x^2=0$ or $(2y-x)(15y+17x)=0$.

$\therefore\ 2y=x$ and $15y=-17x$, or $y=\frac{1}{2}x$ or $-\frac{17}{15}x$.

The work is now the same as given above.

7. Some *Homogeneous Equations* may easily be solved by other artifices than those illustrated above.

Ex. 1. Solve $x^2+3xy=22\ldots(1)$, $xy+4y^2=42\ldots(2)$

Adding (1) and (2), we have $x^2+4xy+4y^2=64$.

Taking the sq. root $x+2y=\pm 8\ldots(3)$

(i) Taking the upper sign, $x=8-2y$; substituting this in (1),
we have $(8-2y)^2+3(8-2y)y=22$, or $64-8y-2y^2=22$.

$\therefore\ y^2+4y-21=0$, or $(y-3)(y+7)=0$.

$\therefore\ y=3$ or -7.

Hence $x=8-2y=2$ or 22.

(ii) Taking the lower sign, $x=-(8+2y)$; substituting this in (1),
we have $(8+2y)^2-3(8+2y)y=22$, or $64+8y-2y^2=22$.

$\therefore\ y^2-4y-21=0$, or $(y-7)(y+3)=0$.

$\therefore\ y=7$ or -3.

Hence $x=-(8+2y)=-22$ or -2.

Thus, we have the four solutions :—

$x=\pm 2$ or ± 22, $y=\pm 3$ or ∓ 7.

Exercise IV.

Solve the following equations :—

1. $\left.\begin{array}{l} x^2+xy=66 \\ x^2-y^2=11 \end{array}\right\}$ **2.** $\left.\begin{array}{l} x^2+y^2=34 \\ x^2-xy=10 \end{array}\right\}$. **3.** $\left.\begin{array}{l} 3x^2+xy=68 \\ 4y^2+3xy=160 \end{array}\right\}$

4. $\left.\begin{aligned}2x^2+3xy&=26\\3y^2+2xy&=39\end{aligned}\right\}$ **5.** $\left.\begin{aligned}x^2-xy&=6\\5xy+4y^2&=19\end{aligned}\right\}$ **6.** $\left.\begin{aligned}x^2+xy&=80\\3xy+y^2&=52\end{aligned}\right\}$

7. $\left.\begin{aligned}3x^2-\tfrac{1}{3}xy+y^2&=361\\x^2-y^2+\tfrac{2}{3}xy&=134\end{aligned}\right\}$ **8.** $\left.\begin{aligned}3x^2-4xy+y^2&=20\\2x^2-7y^2&=4\end{aligned}\right\}$

9. $\left.\begin{aligned}x^2+xy&=80\\3xy+y^2&=52\end{aligned}\right\}$ **10.** $\left.\begin{aligned}x^2-xy&=15\\y^2-2xy&=16\end{aligned}\right\}$ **11.** $\left.\begin{aligned}7x^2-3y^2&=109\\x^2+xy&=20\end{aligned}\right\}$

12. $\left.\begin{aligned}x^2+3xy+3y^2&=19\\4x^2+xy+y^2&=10\end{aligned}\right\}$ **13.** $\left.\begin{aligned}x^2+4xy&=133\\xy+4y^2&=57\end{aligned}\right\}$ **14.** $\left.\begin{aligned}4x^2-xy&=2\\xy+11y^2&=1\end{aligned}\right\}$

15. $\left.\begin{aligned}5x^2+2y^2&=13\\x^2-xy+y^2&=3\end{aligned}\right\}$ **16.** $\left.\begin{aligned}3x^2-4xy+5y^2&=33\\4x^2-xy&=10\end{aligned}\right\}$ (C. F. A 1878)

17. $\left.\begin{aligned}2x^2+3xy+y^2&=20\\5x^2+4y^2&=41\end{aligned}\right\}$ (C. F. A. 1892). **18.** $\left.\begin{aligned}3x^2-4xy&=7\\3xy-4y^2&=5\end{aligned}\right\}$

19. $\left.\begin{aligned}4xy-x^2&=15\\39y^2-xy&=150\end{aligned}\right\}$ (C. F. A. 1890) **20.** $\left.\begin{aligned}7xy-5x^2&=36\\4xy-3y^2&=105\end{aligned}\right\}$

8. An expression is said to be **symmetrical** with respect to x and y, when these quantities are similarly involved in it. Thus

$$x^3+x^2y^2+y^3,\ 5xy+7x+7y+1 \text{ and } 3x^4-5x^2y-5xy^2+3y^4$$

are *symmetrical* with respect to x and y.

9. Symmetric Equations. When each of the two equations is *symmetrical* with respect to x and y, put $u+v$ for x and $u-v$ for y.

Ex. 1. Solve $x^3+y^3=18xy\ldots(1)$, $x+y=12\ldots(2)$

Put $u+v$ for x, and $u-v$ for y;

then (1) becomes $(u+v)^3+(u-v)^3=18(u+v)(u-v)$,

$$\text{or } u^3+3uv^2=9(u^2-v^2)\ldots\ldots(3)$$

and (2) becomes $(u+v)+(u-v)=12$, whence $u=6$.

Putting this for u in (3), we have

$216+18v^2=9(36-v^2)$, whence $v=\pm 2$.

Hence $x=u+v=6\pm 2=8$ or 4, and $y=u-v=6\mp 2=4$ or 8.

Ex. 2. Solve $x+y=3\ldots(1)$, $x^4+y^4=17\ldots(2)$

Assume $x=u+v$ and $y=u-v$; then from (1), we have

$$x+y=2u=3 \text{ and } \therefore\ u=\tfrac{3}{2}\ldots\ldots(3)$$

From (2) $(u+v)^4+(u-v)^4=17$ or $2(u^4+6u^2v^2+v^4)=17$.

Substituting (3) in this and simplifying, we get

$$16v^4+216v^2-55=0, \text{ or } (4v^2-1)(4v^2+55)=0;$$

$$\therefore\ v^2=\tfrac{1}{4} \text{ or } -\tfrac{55}{4}, \text{ and } \quad v=\pm\tfrac{1}{2} \text{ or } \pm\tfrac{1}{2}\sqrt{(-55)}.$$

Hence $x=\frac{3}{2}\pm\frac{1}{2}=2$ or 1 and $y=\frac{3}{2}\mp\frac{1}{2}=1$ or 2; &c.

10. Some equations though not *symmetrical* with respect to x and y may yet be solved by the preceding method.

Ex. 3. Solve $x-y=2\ldots(1)$, $x^4+y^4=706\ldots(2)$.

Assume $x=u+v$ and $y=u-v$; then from (1), we have

$$x-y=2v=2 \text{ and } \therefore\ v=1 \ \ldots\ldots\ldots\ldots(3)$$

From (2) $(u+v)^4+(u-v)^4=706$, or $2(u^4+6u^2v^2+v^4)=706$.

Substituting v in this and dividing by 2, we have

$$u^4+6u^2-352=0, \text{ or } (u^2-16)(u^2+22)=0;$$

$$\therefore\ u^2=16 \text{ or } -22 \text{ and } \therefore\ u=\pm4 \text{ or } \pm\sqrt{(-22)}.$$

Hence $x=\pm4+1=5$ or -3 and $y=\pm4-1=3$ or -5.

Exercise V.

Solve the following equations:

1. $\left.\begin{array}{l} x+y=12 \\ x^3+y^3=18xy \end{array}\right\}$
2. $\left.\begin{array}{l} x+y=6 \\ x^4+y^4=14x^2y^2 \end{array}\right\}$
3. $\left.\begin{array}{l} x+y=12 \\ x^3+y^3=2196 \end{array}\right\}$
4. $\left.\begin{array}{l} x-y=4 \\ x^3-y^3=124 \end{array}\right\}$
5. $\left.\begin{array}{l} x+y=5 \\ x^2+y^2=8xy \end{array}\right\}$
6. $\left.\begin{array}{l} x+y=8 \\ x^4+y^4=1312 \end{array}\right\}$

II. THREE UNKNOWNS.

11 The method of solution adopted in the following system of equations is deserving of special notice.

Ex. Solve $yz=a^2$ (1), $zx=b^2\ldots(2)$, $xy=c^2\ldots(3)$.

Multiplying together, $x^2y^2z^2=a^2b^2c^2$;

Taking the sq. root, $xyz=\pm abc\ldots\ldots(4)$

Dividing (4) by (1), we have $x=\pm\dfrac{bc}{a}$.

Similarly, $y=\pm\dfrac{ca}{b}$ and $z=\pm\dfrac{ab}{c}$.

These results are very useful and should be committed to memory. Certain types of equations may be reduced to this system by suitable transformations, as the following examples will shew.

Ex. 1. Solve $x(y+z)=80$, $y(z+x)=72$, $z(x+y)=56$.

The equations may be written thus :—

$$xy+xz=80,\ldots(1),\quad yz+xy=72\ldots(2),\quad zx+yz=56\ldots(3)$$

Adding and dividing by 2, we have

$$yz+zx+xy=104\ldots\ldots\ldots(4)$$

Subtracting (1), (2) and (3) separately from (4), we get

$$yz=24\ ;\quad zx=32 \text{ and } xy=48.$$

Hence, using the method of the preceding example,

we get $x=\pm 8$, $y=\pm 6$ and $z=\pm 4$.

Ex. 2. Solve
$$\left.\begin{aligned}(x+y)(x+z)&=56\ldots\ldots\ldots\ldots(1)\\(y+z)(y+x)&=77\ldots\ldots\ldots\ldots(2)\\(z+x)(z+y)&=88\ldots\ldots\ldots\ldots(3)\end{aligned}\right\}$$

Putting u, v and w for $y+z$, $z+x$ and $x+y$ respectively, the equations become

$$vw=56,\ uw=77 \text{ and } uv=88.$$

Therefore $u=\pm 11$, $v=\pm 8$ and $w=\pm 7$.

Hence $y+z=\pm 11$, $z+x=\pm 8$ and $x+y=\pm 7\ldots\ldots\ldots\ldots(4)$

Now adding and dividing by 2, we have

$$x+y+z=\pm 13\ \ldots\ldots\ldots\ldots(5)$$

Subtracting (4) separately from (5), we have

$$x=\pm 2,\ y=\pm 5 \text{ and } z=\pm 6.$$

Ex. 3. Solve
$$\left.\begin{aligned}xy+2(x+y)&=20\ldots\ldots\ldots(1)\\zx+2(z+x)&=24\ldots\ldots\ldots(2)\\yz+2(y+z)&=38\ldots\ldots\ldots(3)\end{aligned}\right\}$$

Adding 4 to each of the given equations, we get

$$(x+2)(y+2)=24,\ (z+2)(x+2)=28,\ (y+2)(z+2)=42.$$

Now, putting u, v and w for $x+2$, $y+2$ and $z+2$ respectively, the equations become

$$uv=24,\ wu=28 \text{ and } vw=42.$$

Therefore $u=\pm 4$, $v=\pm 6$ and $w=\pm 7$

Hence $x=2$ or -6, $y=4$ or -8, $z=5$ or -9.

Ex. 4. Solve $xyz=\frac{24}{5}(x+y)=4(x+z)=\frac{24}{7}(y+z)$.

Here, $\dfrac{x+y}{xyz}=\dfrac{5}{24}$ or $\dfrac{1}{yz}+\dfrac{1}{xz}=\dfrac{5}{24}$.

Similarly, $\frac{1}{yz}+\frac{1}{xy}=\frac{1}{4}$ and $\frac{1}{xz}+\frac{1}{xy}=\frac{7}{24}$.

Adding and dividing by 2, we have

$$\frac{1}{yz}+\frac{1}{zx}+\frac{1}{xy}=\frac{3}{8}.$$

$\therefore\ \frac{1}{yz}=\frac{1}{12},\ \frac{1}{zx}=\frac{1}{8}$ and $\frac{1}{xy}=\frac{1}{6}$.

$\therefore\ yz=12,\ zx=8$ and $xy=6$.

Hence $x=\pm 2,\ y=\pm 3$ and $z=\pm 4$.

12. Next, consider the following system of equations

$$x+y+z=a,\ yz+zx+xy=b,\ xyz=c.$$

We have $(x-x)(x-y)(x-z)=0$, (*identically*)

$$\text{or } x^3-(x+y+z)x^2+(yz+zx+xy)x-xyz=0.$$

Now substituting in the above from the given equations

$$x^3-ax^2+bx-c=0.$$

This equation may be solved by resolving the left side into factors, as the following example will shew.

Ex. Solve

$$\left.\begin{aligned} x+y+z&=9 \ldots\ldots\ldots\ldots(1)\\ yz+zx+xy&=26 \ldots\ldots\ldots\ldots(2)\\ xyz&=24 \ldots\ldots\ldots\ldots(3)\end{aligned}\right\}$$

Proceeding as in the above Art., we get

$$x^3-9x^2+26x-24=0$$

$$\text{or } (x-2)(x-3)(x-4)=0\ ;\ \therefore\ x=2,\ 3, \text{ or } 4.$$

Now from the nature of the question, each letter x, y, z may have any one of the three values, provided that the other two values are given to the other two letters ;

$\therefore\ x=2,\ 3$ or 4 ; $y=4,\ 2$ or 3 ; $z=3,\ 4$ or 2.

Exercise VI.

Solve the following equations :—

1. $x^2yz=a^3,\ y^2zx=b^3,\ z^2xy=c^3$.

2. $\frac{a^3x}{y^2z^2}=\frac{b^3y}{x^2z^2}=\frac{c^3z}{x^2y^2}=1$. **3.** $\frac{x}{y^2z^2}=\frac{y}{z^2x^2}=\frac{z}{x^2y^2}=1$.

4. $xy+x+y=19,\ xz+x+z=23,\ yz+y+z=29$.

5. $xy+10(x+y)=21,\ yz+10(y+z)=32,\ xz+10(x+z)=32$.

6. $xy+a(x+y)=xz+a(x+z)=yz+a(y+z)=3a^2$.

7. $xyz=2x+2y+z=\frac{1}{2}(3x+3y+4z)=\frac{1}{4}(6x+15y+5z)$.

8. $\left.\begin{array}{r} 5x+4y+3z=48xyz \\ 3x+6y+5z=46xyz \\ x+2y+3z=18xyz \end{array}\right\}$ 9. $\left.\begin{array}{r} 3x-4y+7z=0 \\ 2x-y-2z=0 \\ 3x^3-y^3+z^3=18 \end{array}\right\}$

10. $x(y+z)=6$, $y(z+x)=12$, $z(x+y)=10$.

11. $x+y+z=xyz=6$, $yz+zx+xy=11$.

12. $x+y+z=1/x+1/y+1/z=\frac{7}{2}$, $xyz=1$.

13. $x+y+z=23$, $yz+zx+xy=170$, $xyz=400$.

14. $xyz=\frac{1}{3}(x+y)=\frac{1}{4}(y+z)=\frac{1}{5}(z+x)$.

15. $x(y+z)=26$, $y(z+x)=50$, $z(x+y)=56$.

16. $x^2+xy+xz=18$, $y^2+xy+yz=27$, $z^2+xz+yz=36$.

III. PROBLEMS PRODUCING SIMULTANEOUS QUADRATIC EQUATIONS.

13. The following are illustrative examples.

Ex. 1. The sum of two numbers is 14 and the sum of their cubes is 854. Find the numbers.

Let x and y denote the numbers.

By the question, $x+y=14$ ......(1), $x^3+y^3=854$... ...(2).

Dividing (2) by (1), $x^2-xy+y^2=61$.

But $x^2-xy+y^2=(x+y)^2-3xy$; $\therefore$ $61=14\times14-3xy$, or $xy=45$.

Now, $(x-y)^2=(x+y)^2-4xy=14\times14-4\times45=16$;

$\left.\begin{array}{l} \therefore\ x-y=\pm4 \\ \text{and } x+y=14 \end{array}\right\}$ Hence by addt. and subtr., we get $x=9$ or 5 and $y=5$ or 9.

Thus the numbers are 9 and 5.

Ex. 2. A dealer sold 60 bullocks and 80 sheep for *Rs.*1060; but he sold 42 more sheep for *Rs.*90 than he did bullocks for *Rs.*45. Find the price of each.

Let x be the price of a bullock in rupees,

and y ....a sheep ....

Now, for *Rs* 90, he sells $\frac{90}{y}$ sheep, and for *Rs.*45, $\frac{45}{x}$ bullocks.

By the question, $\left.\begin{array}{r} 60x+80y=1060 \text{............(1)} \\ \dfrac{90}{y}-42=\dfrac{45}{x} \text{ ....(2)} \end{array}\right\}$

From (1), $3x+4y=53$ or $x=\frac{1}{3}(53-4y)$.

From (2), $30x-14xy=15y$......................(3)

Substituting the value of x in (3), we have

$$10(53-4y)-\tfrac{14}{3}y(53-4y)=15y,$$

which reduces to $56y^2-907y+1590=0$;

whence $y=2$ or $\frac{795}{56}$ and $\therefore$ $x=15$ or &c.

Hence the price of **a** sheep is *Rs.*2, and of a bullock is *Rs.*15.

Ex. 3. Find two numbers such, that their sum, product, and difference of their squares may be all equal.

Let $x+y$ and $x-y$ be the two numbers.

Then their sum $=2x$, their product $=x^2-y^2$,

and the difference of their squares $=(x+y)^2-(x-y)^2=4xy$.

By the question, $2x=4xy$......(1), $2x=x^2-y^2$.. .. ...(2).

From (1) $y=\frac{1}{2}$; and from (2) $2x=x^2-\frac{1}{4}$; or $4x^2-8x-1=0$.

$$\therefore x=\tfrac{1}{2}(2\pm\sqrt{5}).$$

Hence $x+y=\frac{1}{2}(3\pm\sqrt{5})$ and $x-y=\frac{1}{2}(1\pm\sqrt{5})$ are the numbers required.

Note. The above step in assuming the numbers should be noticed, as it simplifies much the solution of problems of this kind.

Ex. 4. In going a quarter of a mile along a straight road the hind wheel of a bicycle turns 11 times more than the front wheel. Had the front wheel been 3 inches longer in circumference than it actually is, the hind wheel would have turned 16 times more than the front wheel. Find the circumference of each wheel.

Let x be the circumference of the front wheel in feet,

and y .... hind wheel... ......

Now, on the first supposition, the hind wheel makes $\frac{1320}{y}$ turns and the front wheel $\frac{1320}{x}$ turns, in going over $\frac{1}{4}$ of a mile or 1320 ft.

By the question, $\frac{1320}{y}-11=\frac{1320}{x}$. ....(1).

On the second supposition, the front wheel makes $\frac{1320}{x+\frac{1}{4}}$ turns in going over $\frac{1}{4}$ of a mile.

By the question, $\frac{1320}{y}-16=\frac{1320}{x+\frac{1}{4}}$. ....(2)

Subtracting (2) from (1), we have

$$5=\frac{1320}{x}-\frac{1320}{x+\frac{1}{4}}, \text{ or } \frac{1}{x}-\frac{1}{x+\frac{1}{4}}=\frac{1}{264}.$$

$\therefore\ 4x^2+x-264=0$, or $(x-8)(4x+33)=0$.

$\therefore\ x=8$ or $-\frac{33}{4}$ and $\therefore\ y=7\frac{1}{2}$ or &c.

Hence the circumferences are 8 ft. and $7\frac{1}{2}$ ft.

Exercise VII.

1. The product of two numbers is 63 and the difference of their squares 32. Find the numbers.

2. The difference of two numbers is 4, and the difference of their cubes is 316. Find the numbers.

3 The sum of two numbers is 10 and the sum of their cubes is 370. Find the numbers.

4. Find two numbers whose sum is 32 and the sum of whose squares is 544.

5. A rectangular enclosure is half an acre in area, and its perimeter is 201 yards. Find the lengths of its sides.

6. A labourer undertakes to carry a load a certain distance, agreeing to take 8*a.* for each maund moved one mile. He earns *Rs.*40. 8*a.* and the distance in miles exceeds the number of maunds carried by 4 05 Find the load and the distance.

7. In a mixed number the integer is 98 times the fraction. The numerator of the fraction being unity, and its denominator less by 7 than the integer, find the mixed number.

8. A man being asked his age, answered, 'If you multiply my two digits together, the number formed will be my age 22 years ago, and if you add all the digits of the two ages you will have one-third of my present age'. How old is he?

9. The sum of two numbers is six times their difference, and their product exceeds twice their sum by 11. Find the numbers.

10. **A** and **B** gained by trading *Rs.*100. Half of **A**'s stock was less than **B**'s by *Rs.*100; and **A**'s gain was $\frac{3}{20}$ths of **B**'s stock. What did each put into the stock, and what are the respective shares of the gain?

11. What fraction will be increased by $\frac{1}{38}$ when unity is added to both numerator and denominator, and diminished by $\frac{8}{57}$ when 4 is subtracted from each of them?

12. The sum of the three sides of a right-angled triangle is 12 inches and their product is 60 cubic inches. Find the lengths of the sides.

13. If one year be added to the tenth part of the sum of the squares of the ages of two brothers, the total will be seven times the difference between their ages; and next year the elder will be half as old again as the younger. What are their ages? (C. F. A. 1881).

14. The fore-wheel of a carriage makes 20 more turns than the hind-wheel in 600 yards; but if the circumference of each were increased by 5 yards, then the fore-wheel would make only 15 turns more than the hind-wheel in 900 yards. Find the circumference of each wheel.

15. The figures which express the pounds and the pence in a certain sum of money will change places if £2. 19s 9d. be added to it, and those which express the shillings and the pence would be interchanged by subtracting 2s. 9d. What alteration would be produced in the sum of money by interchanging the figures which express the pounds and shillings?

IV. GRAPHS OF ELLIPSES AND HYPERBOLAS.

14. The graph of $\frac{b}{a}\sqrt{(a^2-x^2)}$ is an **Ellipse** and that of $\frac{b}{a}\times\sqrt{(x^2-a^2)}$ is a **Hyperbola.** (See Note).

GRAPHS OF ELLIPSES.

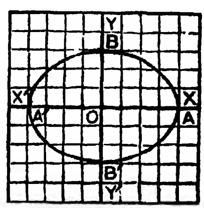
Ex. 1. Draw the graph of $\frac{1}{4}\sqrt{(144-9x^2)}$.

Let $y=\frac{1}{4}\sqrt{(144-9x^2)}$.

For each value of x, there are two equal and opposite values of y. The values of x and y may be tabulated thus:—

x	0	± 1	± 2	± 3	± 4
y	± 3	$\pm\frac{\sqrt{135}}{4}$	$\pm\frac{\sqrt{108}}{4}$	$\pm\frac{\sqrt{63}}{4}$	0

The curve is symmetrical with respect to the axes of x and y, (as shewn in the annexed Fig.)

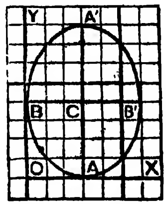


Here, $OA=4$, $OA'=-4$,
$OB=3$ and $OB'=-3$.

Ex. 2. Draw the graph of $16x^2+9y^2-96x-72y+144=0.$

The equation may be written thus : $16(x-3)^2+9(y-4)^2=144.$

The values of x and y can be easily tabulated and it can be easily seen that for real values of y, x must lie between 0 and 6 and for real values of x, y must lie between 0 and 8. The curve touches both the axes of x and y (as shewn in the annexed Fig.)



Here, AA$'=8$ and BB$'=6$.

It is an ellipse whose major axis is vertical and minor axis horizontal.

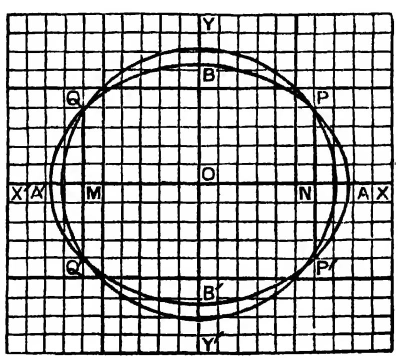
Ex. 3. Solve graphically $2x^2+3y^2=30$...(1), $x^2+y^2=13$...(2)

The graph of (1) is an ellipse whose centre is the origin, semi-axis major $\sqrt{(15)}$ and semi-axis minor $\sqrt{(10)}$, the axis major being horizontal. The graph of (2) is a circle whose centre is the origin and radius $\sqrt{(13)}$.

If the graphs be accurately drawn, it will be found (as in the Fig. below) that the values of x are ± 3 and of y are ± 2.

Here, ON $=3$, OM $=-3$, PN $=$ QM $=2$, P$'$N $=$ Q$'$M $=-2$.

The four points of intersection are P, P$'$, Q and Q$'$.



N.B.—The unit of length here is equal to twice the length of a side of the square.

Exercise VIII.

1. Draw the graphs of

(1) $\frac{1}{12}\sqrt{(3600-25x^2)}$. (2) $\sqrt{(64-3x^2)}$. (3) $\frac{3}{4}\sqrt{(16-x^2)}$.

(4) $\frac{5}{6}\sqrt{(125-x^2)}$. (5) $25x^2+144y^2-50x-576y-2999=0$,

and determine graphically the limits between which x must lie in order that y may be real and *vice versa*.

2. Solve graphically the following equations :—

(1) $\left.\begin{matrix}3x^2+4y^2=19\\4x^2+5y^2=24\end{matrix}\right\}$ (2) $\left.\begin{matrix}16x^2+25y^2=400\\x^2+y^2=25\end{matrix}\right\}$ (3) $\left.\begin{matrix}x^2+5y^2=120\\2x^2+y^2=204\end{matrix}\right\}$

(4) $\left.\begin{matrix}2x+3y=12\\3x^2+4y^2=43\end{matrix}\right\}$ (5) $\left.\begin{matrix}y=3x+4\\y^2=25-2x^2\end{matrix}\right\}$ (6) $\left.\begin{matrix}y=2x+3\\x^2+y^2-6x-8y=0\end{matrix}\right\}$

(7) $\left.\begin{matrix}2y=x+5\\2x^2+3y^2-10x=19\end{matrix}\right\}$ (8) $\left.\begin{matrix}x^2+y^2=36\\3x^2+4y^2-8y=96\end{matrix}\right\}$ (9) $\left.\begin{matrix}y=x^2\\2x^2+3y^2=35\end{matrix}\right\}$

(10) $x^2-y=2$, $4x^2+y^2=20$.

GRAPHS OF HYPERBOLAS.

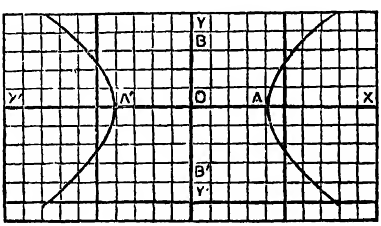
Ex. 1. Draw the graph of $\frac{3}{4}\sqrt{(x^2-16)}$.

Let $y=\frac{3}{4}\sqrt{(x^2-16)}$.

For each value of x, there are two equal and opposite values of y; x can have no value between 4 and -4 for real values of y. The curve is symmetrical with respect to the axes of x and y.

The values of x and y may be tabulated thus :—

x	± 4	± 5	± 6	± 8	...
y	0	$\pm\frac{9}{4}$	$\pm\frac{3}{2}\sqrt{5}$	$\pm 3\sqrt{3}$	...



As x increases, y increases. When x is infinitely large, y is

infinitely large. The curve extends to infinity on both sides of the axis of y, (as shewn in the Fig. above.)

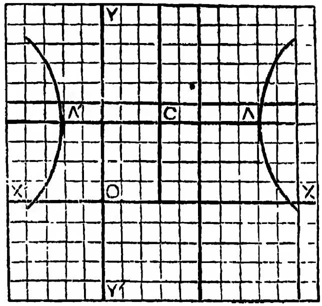
It is a hyperbola whose transverse semi-axis is 4 and conjugate semi-axis 3.

Ex. 2. Draw the graph of $x^2-y^2-6x+8y=32$.

The equation may be written thus :—

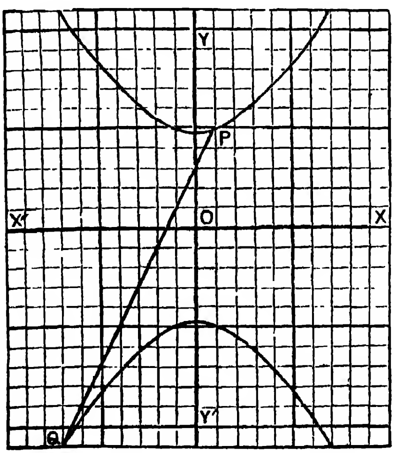
$$(x-3)^2-(y-4)^2=25.$$

The curve is a rectangular hyperbola whose centre is the point (3, 4); x can have no value between 8 and -2 for real values of y, (as shewn in the annexed Fig.)



Ex. 3. Solve graphically $y=2x+3$...(1), $y^2-2x^2=23$....(2).

The graph of (1) is a straight line and that of (2) is a hyperbola of which the transverse axis is vertical. To solve the two equations



graphically is to find practically the co-ordinates of the points of intersection of these two curves. The Fig. above gives the graphical solution.

P and Q are the points of intersection. It will be found that the abscissa of P is 1 and of Q is -7, and the ordinate of P is 5 and that of Q -11.

Exercise IX.

1. Draw the graphs of

(1) $\frac{2}{5}\sqrt{(x^2-36)}$. (2) $x^2-y^2=36$. (3) $y^2-5x^2=16$.

(4) $x^2-y^2-12x+16y=0$. (5) $y^2-2x^2+4y=0$.

2. Solve the following equations graphically :—

(1) $\left.\begin{array}{r} y=2x \\ x^2-y^2=36 \end{array}\right\}$ (2) $\left.\begin{array}{r} 2y=x+5 \\ y^2-2x^2=64 \end{array}\right\}$ (3) $\left.\begin{array}{r} x^2+y^2=41 \\ 2x^2-3y^2=2 \end{array}\right\}$

(4) $\left.\begin{array}{r} 2x^2+y^2=57 \\ 3x^2-4y^2=-52 \end{array}\right\}$ (5) $\left.\begin{array}{r} y=2x+3 \\ 2y^2-3x^2+6x=0 \end{array}\right\}$ (6) $\left.\begin{array}{r} y+1=2x \\ y^2-3x^2=1 \end{array}\right\}$

V. MISCELLANEOUS GRAPHS OF THE SECOND DEGREE.

15. The following are illustrative examples.

Ex. 1. Draw the graph of $\frac{1}{x+1}$.

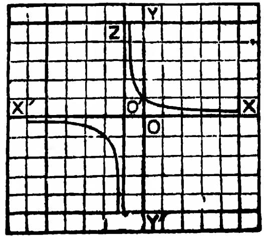
Let $y=\frac{1}{x+1}$; then $y(x+1)=1$.

If the origin be transferred to the point $(-1, 0)$, the equation becomes $xy=1$, which is a rectangular hyperbola, having the new axes for asymptotes.

The values of x and y may be tabulated for the original equation thus :—

x	∞	$-\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{7}{8}$	-1	0	1	2	3	4	...
y	0	2	4	8	∞	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	...

The annexed Fig. represents the graph. Here, the lines **O′X** and **O Y′** are the asymptotes, *i.e.* the straight lines which touch the curve at infinity.



Ex. 2. Draw the graph of $\sqrt{x}+\sqrt{y}=1$.

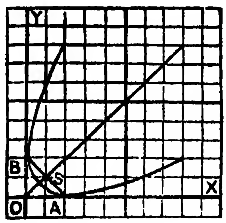
The equation may be written in the form

$(x+y-1)^2=4xy$, which shews that when $x=0$, $y=1, 1$;

and when $y=0$, $x=1, 1$. The curve therefore touches both the axes at the points $(1,0)$ and $(0,1)$. It is a parabola, because the terms of the second degree form a perfect square.

By tabulating the values of x and y, it will be found that the annexed Fig. represents the graph.

Here, **OA** $=1$, **OB** $=1$. The line drawn through **O** bisecting the angle **AOB** is the axis of the parabola. The line **AB** is the latus rectum.



Note. The student who knows the properties of the parabola will easily understand these. The unit of length here is equal to twice the side of a square.

Exercise X.

1. Draw the graphs of

(1) $\dfrac{1}{x+2}$. (2) $\dfrac{1}{3-x}$. (3) $\dfrac{1}{x+3}+2$ (4) $\dfrac{x+2}{x+3}$.

(5) $\dfrac{x^2+2}{x+1}$. (6) $y=\dfrac{x-1}{x-2}$. (7) $x^2-2xy+y^2-6(x+y)+9=0$.

2. Solve the following equations graphically :—

(1) $\left.\begin{array}{l} y=x+1 \\ y(2x+1)=1 \end{array}\right\}$ (2) $\left.\begin{array}{l} y=2x \\ x^2-2xy+y^2-8(x+y)+16=0 \end{array}\right\}$

(3) $x-y=2, \frac{1}{x}+\frac{1}{y}=\frac{3}{4}$. (4) $x+y=5, \frac{1}{x}-\frac{1}{y}=\frac{3}{4}$.

(5) $y(x-2)=1,\ x^2+y^2=10$. (6) $xy=1,\ 2x^2+3y^2=5$.

3. Trace graphically the changes in the sign and magnitude of the following expressions as x increases from *minus* to *plus* infinity :—

(1) $2x^2-3x+5$. (2) $3x-4x^2+10$. (3) $x^2/(x-1)$.

(4) $x+\frac{4}{x}$. (5) $\frac{25x^2+20x-86}{30x-87}$. (6) $\frac{(x-2)(x-4)}{x-3}$.

4. Plot the graphs of

(1) $\frac{1}{x^2}$. (2) $\frac{1}{x^3}$. (3) $\frac{1}{x^4}$. (4) $\frac{1}{x^5}$.

(5) $\frac{1}{x-1}$. (6) $1-\frac{1}{x}$. (7) $1-\frac{2}{x}$. (8) $\frac{x}{1-x}$.

5. Draw the graphs of the circle $x^2+y^2=9$ and of the straight lines $3x+4y=12$, $3x+4y=15$, and $3x+4y=18$; and shew that the circle meets the first in two real points, the third in no real points, and that it touches the second.

ANSWERS

Ex. I. (p. 4).

1. (1) 80. (2) 80. (3) 280. (4) 41. (5) 18. (6) 14.
(7) 6. (8) 31.

2. (1) 10. (2) 44. (3) 58. (4) 12. (5) 5. (6) 7.
(7) 33. (8) 14.

3. (1) 4. (2) 30. (3) 2. (4) 4. (5) $\frac{3}{4}$. (6) 3.
(7) 2. (8) 6. (9) 16.

4. (1) -8. (2) 1. (3) 106. (4) -1. (5) -178. (6) 192.

5. (1) 18. (2) 42. (3) 8. (4) 41. (5) -25.
(6) 78. (7) 8. (8) 6. **6.** 1.

Ex. II. (pp. 6-7).

1. (1) 2880. (2) 17496. (3) 4800. (4) 4032
(5) 238. (6) 41328. (7) 486.

2. (1) 94. (2) 89. (3) -64. (4) 16. (5) 1. (6) 7.
(7) 3. (8) $38\frac{3}{4}$. (9) 49. (10) $\frac{13}{31}$.

3. 264. **4.** $3\frac{6}{7}$. **5.** (1) -73. (2) 16. (3) 5. (4) $11\frac{1}{3}$

Ex. III. (p. 8).

1. (1) 9. (2) 1. (3) 4. (4) 6. (5) 12.
(6) 40. (7) 24. (8) 6. (9) 2.

2. (1) 30. (2) 1312. (3) 2040. (4) 17424
(5) 225. (6) 397. (7) 1120. (8) 750.

3. (1) 9. (2) 11. (3) 20. **4.** 4676.

5. (1) 46. (2) 24. (3) 7200.

Ex. IV. (pp. 9-10).

1. (1) 30. (2) 80. (3) 600. (4) 9. (5) 25.
(6) 11. (7) 4.

2. (1) 21. (2) 22. (3) 7. (4) 13. (5) 15. (6) 4.

3. (1) 35. (2) 10. **4.** 6. **5.** (1) 14. (2) 25.

Ex. V. (pp. 12-14).

1. 9 ; 27 ; 81 ; 81 ; 6 ; 0 ; 1. **2.** 12 ; 0 ; 144 ; 30 ; 72.

3. 486 ; 0 ; 2916.

4. (1) $6\frac{3}{4}$. (2) $4\frac{1}{3}$. (3) $20\frac{11}{12}$. (4) $6\frac{1}{6}$. (5) $23\frac{79}{280}$.

5. 88. **6.** 128. **7.** (1) $230\frac{2}{5}$. (2) $53\frac{4}{15}$. (3) $2\frac{1}{16}$. (4) $142\frac{13}{16}$.

8. (1) 3. (2) $6\frac{2}{3}$ (3) 146. (4) 55. (5) $\frac{1}{30}$. (6) 6,
(7) 5. (8) $\frac{2}{5}$. (9) 1. (10) $\frac{5}{8}$. (11) 2. (12) $\frac{2}{3}$.
9. (1) 8. (2) 3. (3) 27. (4) 125. (5) 64. (6) 3.
(7) 2. (8) 1.
10. (1) 8. (2) 120. (3) 4. (4) 384. **11.** 0
12. 0. **13.** 0. **14.** 3 *exponent* and 4 *coefficient.* **15.** 0.

Ex. VI. (pp. 15-16).

1. $x-y$. **2.** $x-5$. **3.** $64x+4y+z$. **4.** $x+1$; $x-1$.
5. $x-y$. **6.** $a+b$. **7.** (i) $16a$. (ii) $192a$. (iii) $64a$.
(iv) $4a$. (v) $8a$. **8.** (i) $3b$. (ii) $36b$.
9. (i) $b/12$. (ii) $b/36$. **10.** ax. **11.** $x-13$. **12.** $13-x$.
13. $2x$; $128x$. **14.** $5/a$; $100/a$. **15.** $192a+12b$.
16. xy ; a/y. **17.** x/y. **18.** (i) $3a$. (ii) [illegible]a. (iii) ax.
19. $144a$. **20.** $ax+by$. **21.** (1) $(x+y)/z$. (2) $5z^2(x-y)$.
(3) $(6a^3-x^4)c^2$. (4) $(a^2+b^3)/(a+b-d)$. (5) $(a+b)^2(c-d)$.
22. $4b$; $3b^2c$; $\frac{3}{4}ab$; ax^2 ; a^2. **23.** The sum of ab and ac
24. $a+a+a$; $a.a.a$. **25.** (i) 7. (ii) 9. (iii) 5.

Ex. VII. (pp. 18-19).

1. -12. **2.** -12 ; 12 ; 8. **3.** -120. **4.** (i) -15. (ii) 15.
5. (i) 15 miles. (ii) -15 miles. **6.** 4·45. **7.** 35 seers.
8. 50 years ; 30 years. **9.** 9 inches. **10.** -10.

Ex. VIII. (p. 20).

1. 9. **2.** -15. **3.** $-7a$. **4.** $-2x$. **5.** $5a-4c$.
6. $5a^2+4ab-b^2$. **7.** $9a^2-7a$. **8.** $-11x^2+6$.

Ex. IX. (pp. 21-22).

1. -8 ; -58. **2.** -202 ; -780. **3.** -304.
4. (1) -31. (2) -8. (3) $-5\frac{1}{3}$. **5.** -33. **6.** (1) 24. (2) -11.
7. 6. **8.** 6, 2, 0, 0, 2, 6. **9.** -10, -15, -13, -8, 11, 111, 286.
12. (1) 1640. (2) $\frac{1}{4}$. (3) -1560.

Ex. X. (p. 23).

1. $24a$. **2.** $27x$. **3.** $23ab$. **4.** $23a^2b$. **5.** $-24x$
6. $-35a$. **7.** $3ab$. **8.** 0. **9.** $10a$. **10.** $-2ab$
11. $10abc$. **12.** $-6a^2b^2$. **13.** $7a^2b^4$. **14.** $14ab^2$. **15.** 0.
16. $-7a^2b^2$. **17.** 0. **18.** $4ax$. **19.** $-3a^2$. **20.** 0.

Ex. XI. (p. 24).

$a-3b+5c+6d-7x.$ **2.** $9x^2y^2-3x^2y+5xy-6xy^2.$

$5a^2b-3ab-5b^2+4b^3.$ **4.** $2a-3a^2+2a^3-5a^4.$

$-5a^5+29a^4b+4a^3b^2-7a^2b^3-3ab^4-7b^5.$

Ex. XII. (pp. 25-26).

1. $18x+15y.$ **2.** $-25a+12b.$ **3.** $0.$ **4.** $a+b+c.$

5. $-a+15b-8c.$ **6** $15a+3b-6c+6d.$ **7.** $6by-7cz.$

8. $23a^2-26ab+14b^2.$ **9.** $14x-9y+10z-12.$

10. $44ab-cd-6c^2.$ **11.** $5x^3+50x^2y-14xy^2+4y^3.$

12. $22x-5xy+3x^2y^2.$ **13.** $21ax+7ax^2.$

14. $2x^2+2y^2+2z^2.$ **15.** $6ax^2-43a^2x.$

16. $-12a^2b^2+14abcd.$ **17.** $6abc.$ **18.** $28x^3.$

Ex. XIII. (pp. 26-27).

1. $-(a-b).$ **2.** $5(a-b)x^2.$ **3.** $9(x^2+y^2)-5ab(x^2-y^2).$

4. $11a.$ **5.** $\frac{11}{18}x^2.$ **6.** $-\frac{8}{3}xy.$ **7.** $-8a^2b.$

8. $a+3.$ **9.** $x^2+\frac{1}{2}x^3.$ **10.** $\frac{1}{6}a^2b^2+ab.$

Ex. XIV. (p. 28).

1. $5a.$ **2.** $-4ab.$ **3.** $13ab.$ **4.** $7xy.$ **5.** $20b.$

6. $-7a^2b.$ **7.** $4a.$ **8.** $-2a-5b.$ **9.** $18ax^2.$

10. $-2a.$ **11.** $-x.$ **12.** $a-b.$ **13.** $2a-3b.$

14. $2bcd-abc.$ **15.** $3a^2-3b^2.$ **16.** $a-b-c.$

17. $3ax+a.$ **18.** $3ax+2a.$ **19.** $x-y-a-b.$

20. $2b-c.$ **21.** $-3y.$ **22.** $-a^2+4b^2+2c^2.$

23. $4x^2-2ax-b.$ **24.** $-2b-c.$ **25.** $-x^2+14x-12.$

Ex. XV. (p. 29).

1. $a-3b+3c.$ **2.** $2a^2-2a-4.$ **3.** $-2x^2-7xy+3y^2.$

4. $4ax-9by+2cz.$ **5.** $5x^2-5x+5.$ **6.** $-2a^2+ab-2b^2.$

7. $a^2-3b^2+19c^2-1.$ **8.** $x^3-6x^2y+11xy^2-2y^3.$

9. $-12a^3+11a^2b+3b^3.$ **10.** $2a^3-11a^2b+14ab^2-4b^3+2$

11. $7a^2-3a+4b^2-7ab+2c^2-6bc$; $8a^2b+2b^2+xy.$

12. $-x^3-6x^2y-2y^2+6-3x^2-4y^3$; $-2x^2-9xy+4y^2.$

13. $3x^2+13xy-y^2-16xz-13yz$; $2x^2+12xz-5z^2.$

14. $3a^4-a^3-14a+14.$ **15.** $a-11b-3c-2d+4e.$

16. $-5p^2-19q^2+25pq.$ **17.** $-3p^2q^3.$ **18.** $x^2+xy+y^2.$

19. $-8x+9x^2-3x^3.$ **20.** $3a^4-4a^3b-4ab^3+2b^4.$

21. $3q^3-6pq+q^2.$ **22.** $a^2+a^2b+8a^2b^2-2.$

23. $5a+b-6c.$ **24.** $-2x^2.$ **25.** $a+b+c.$

Ex. XVI. (p. 30).

1. $2a(x-y)$. 2. $-2(a-b)(x-y)+2(x+y)b^2$.
3. $-2(a+b)^2+x(a^2+b^2)+2x^2(a+b)$. 4. $-\frac{5}{8}abc+\frac{19}{18}a^2b-\frac{3}{2}a^3$.
5. $\frac{1}{30}a^2+\frac{1}{12}a^2b^2+\frac{1}{20}a^2b^2$. 6. $4a^2b^2(a-b)-16x^2y^2(a^2+b^2)+9ab(a^3-b^3)$.
7. $\frac{5}{8}x^2y^2+\frac{7}{10}x^2yz+\frac{1}{2}xy^2z+\frac{13}{24}$.
8. $\frac{1}{24}a^2b^2-abc+\frac{13}{8}x^2y^2z^2+\frac{5}{4}(a-b)$. 9. $-a+b-c$.

Ex. XVII. (p. 32).

1. $5a-4b$. 2. $x+4y$. 3. $2x-y$. 4. $14-5x$.
5. $-2a+11b$. 6. $2a-2b+3c$. 7. $-2x+5y-2z$.
8. $-x-6y$. 9. $4a-4x$. 10. $4a^3-4a^2c$.
11. $x^2-3y^2-3z^2$. 12. $2ax^2+2by^2+2cz^2$. 13. $18x^2-y^2$.
14. $-9b+14c$. 15. $5a^3-11a^2b+b^3$.

Ex. XVIII. (p. 33).

1. $3a-3b$. 2. $2a+3c$. 3. $6a-3b$. 4. $6x-y$.
5. $a^2-3b^2+3c^2$. 6. $2ab+4b^2$. 7. 0. 8. $-3x-y+4z$.
9. $3a-2b$. 10. $b-c$. 11. $x+y-z$. 12. a.
13. a. 14. $-4c+4d$. 15. $11a-15b$. 16. $-x-y-m-n$.
17. $2a$. 18. $-3b+3c$. 19. $a+10b$. 20. $65x-33y$.
21. $8x-4y-z$. 22. $2x-y$; 0.

Ex. XIX. (pp. 34-35).

1. $(5b-4c-3a)$; $-(3a-5b+4c)$.
2. $(2\quad a+b-c)$; $-(a-b+c-2)$.
3. $(2a+3b-4c-5)$; $-(5-2a-3b+4c)$.
4. $(3x-2y)+(5z+a)+(3b-2c)$; $(3x-2y+5z)+(a+3b-2c)$.
5. $(2a-3b)+(4c-2d)-(e-5)$; $(2a-3b+4c)-(2d+e-5)$.
6. $(a^6+2a^4)-(3a^3-5a^2)-(3a+1)$; $(a^6+2a^4-3a^3)+(5a^2-3a-1)$.
7. $(4a^2+5b^2)-(3c^2+2x^2)-(3y^2-2z^2)$; $(4a^2+5b^2-3c^2)$
$-(2x^2+3y^2-2z^2)$.
8. $-(5a-2c)-(3d+2z)-(y-3x)$; $-(5a-2c+3a)-(2z+y-3x)$.
9. $-(3x^2+2y^2)-(5z^2+a)+(2b-3c)$; $-(3x^2+2y^2+5z^2)$
$-(a-2b+3c)$.
10. $\{3x-(2y-5z)\}+\{a+(3b-2c)\}$.
11. $\{2a-(3b-4c)\}-\{2d+(e-5)\}$.
12. $\{a^6+(2a^4-3a^2)\}+\{5a^2-(3a+1)\}$.
13. $\{4a^2+(5b^2-3c^2)\}-\{2x^2+(3y^2-2z^2)\}$.
14. $-\{5a-(2c-3d)\}-\{2z+(y-3x)\}$.
15. $-\{3x^2+(2y^2+5z^2)\}-\{a-(2b-3c)\}$.
16. $(a-b+c)x^3-(b-c+d)x^2-(c+d+e)x$.

17. $ax^3+(5b-2)x^2-(a-3b-4)x.$
18. $-3x^3+(13a-3)x^2-(3c-6)x.$
19. $-(4ab-3)x^5-(3ab+3)x^3+(3a-2c^2)x^2.$
20. $(a^2-5b)x^3-(3b-6)x^2+(4a-2)x.$
21. $2(ax-by).$ **22.** $2(a+b)x^2.$ **23.** $(3a-p-1)x^3+(b-2)x^2-rx.$
24. $2(ax+cy)$; $2b(x+y).$ **25.** $(a+p)x^3-(b-q)x^2-(r-1)x.$

Ex. XX. (p. 39).

1. $-6.$ **2.** $15.$ **3.** $-48.$ **4.** $360.$ **5.** $216.$ **6.** $-216.$
7. $\cdot 48.$ **8.** $-162.$ **9.** $-34.$ **10.** $1.$ **11.** $98.$
12. $3.$ **13.** $67.$ **14.** $0.$ **15.** $13.$ **16.** $-32.$
17. $9.$ **18.** $2.$ **19.** (1) $-6.$ (2) $36.$ (3) $-\cdot 72.$
(4) $-40\cdot 5.$ (5) $16.$ **20.** $\frac{5}{11}.$

Ex. XXI. (p. 40).

1. $28ab.$ **2.** $-6ac.$ **3.** $15ab.$ **4.** $10abcd.$
5. $12a^4x^2.$ **6.** $-3a^3b^3.$ **7.** $-12ab^5y^5.$ **8.** $-15a^5b^3c^2.$
9. $-18a^3b^6c^2.$ **10.** $26x^8y^8z^8.$ **11.** $abx^3y^4.$ **12.** $-x^3y^3.$
13. $a^2bc^2.$ **14.** $2a^2cx^2y.$ **15.** $-mnx^5.$ **16.** $12a^6b^4.$
17. $-36a^3bc^3d.$ **18.** $-36a^7b^3.$ **19.** $-60a^4b^2cx^2y.$ **20.** $216x^4y^4.$
21. $-42a^6b^{12}.$ **22.** $16x^2y^2.$ **23.** $81a^4.$ **24.** $144x^8y^8.$
25. $216a^6b^6.$ **26.** $-729a^{12}.$ **27.** $4x^4y^6z^8.$ **28.** $81a^4b^4c^4.$
29. $-243a^5b^5c^5.$ **30.** $-8a^6b^3.$ **31.** $81a^4b^8.$ **32.** $-a^{14}.$
33. $-a^{11}.$ **34.** $-a^9b^9.$ **35.** $60a^4b^3c^6.$

Ex. XXII. (p. 41).

1. $4a^2+12ab.$ **2.** $ax^2+3axyz.$ **3.** $24x+18y.$
4. $-2abx^2-6by^2z.$ **5.** $a^3b^3-ab^4c^3.$ **6.** $-9a^2x-6ab.$
7. $12x^3y^2z+6xy^3z^2.$ **8.** $15a^4b^3d^2+20a^2c^2d^4.$
9. $9a^3b^3-6a^4b^2-21a^6b.$ **10.** $4a^3-6a^4b-4a^6b^2.$
11. $x^3-x^2y+xy^2.$ **12.** $-a^3x+a^2x^2-ax^3.$
13. $-abx^3+a^2bx^2-ab^2x.$ **14.** $-x^4y+3x^3y^2-3x^2y^3-xy^4$
15. $-3a^4b-12a^3b^2+9a^2b^3+3ab^4.$ **16.** $5a^4b^3c^4-15a^2b^3c^3+10a^2b^2c^3d.$
17. $48a^4b^4c.$ **18.** $16x^9y^6.$ **19.** $240x^5y^5z^5.$
20. $42a^8b^4x^8y^5z.$ **21.** $12b^3c^2-34b^2c^3+32c^5.$
22. $6x^2y^2-xy^3-12y^4.$ **23.** $2x^4-3ax^3-a^4.$ **24.** $a^6-8b^6.$
25. $-4a^4+16a^2b^2-4ab^3-8b^4.$

Ex. XXIII. (pp. 43-44).

1. $x^2+7x-78.$ **2.** $x^2+8x+15.$ **3.** $x^2+2x-15.$
4. $x^2-2x-15.$ **5.** $2a^2+3ab-2b^2.$ **6.** $20a^2+23ab+6b^2.$

7. $2a^2+7ab+3b^2$. **8.** $2ac-bc-6ad+3bd$. **9.** $6x^2+13xy+6y^2$.
10. $6a^2b^3-ab^3-12b^4$. **11.** x^3+y^3. **12.** x^3+6x^2+7x-6.
13. $x^3-6x^2+11x-6$. **14.** x^4-16y^4. **15.** $6x^4-96$.
16. $18x^4-17a^2x^2+4a^4$. **17.** $10x^5-19x^4y+13x^2y^3-9y^5$.
18. $81x^4-y^4$. **19.** a^5+32b^5. **20.** a^6-x^6.
21. $a^2-4b^2+12bc+9c^2$. **22.** $a^4+a^3-2a^2+3a-1$; $a^4-a^3-8a^2+a+1$.
23. $x^4-4a^3x+3a^4$. **24.** x^6-a^6.
25. $2x^6-9x^5+21x^3-x^2-6x+3$. **26.** $a^6+2a^3b^3+b^6$.
27. $x^4+2ax^3+3a^2x^2+2a^3x+8a^4$.
28. $6x^6-3x^5-4x^4+x^3+5x^2-6x-15$.
29. $27a^3+b^3+8-18ab$. **30.** $a^3-b^3+c^3+3abc$.
31. $a^3-8b^3-27c^3-18abc$. **32.** $25x^5+16x-64$.
33. $3a^5-8a^4b+11a^3b^2-9a^2b^3+4ab^4-b^5$.
34. $27a^3+8b^3+c^3-18abc$.
35. $6a^5-17a^4b+22a^3b^2-27a^2b^3+32ab^4-21b^5$.
36. $a^5+4a^4-3a^3-20a^2+18$. **37.** $a^5-41a-120$.
38. $a^6\ 1$. **39.** $4x^6-5x^5+8x^4-10x^3-8x^2-5x-4$.
40. $1+2x^2-7x^4-16x^6$. **41.** $a^2b^2-a^2c^2-b^2d^2+c^2d^2$.
42. $a^4-2a^2b^2+b^4+4abc^2-c^4$. **43.** $1+x^2-x^4-x^6$.
44. 4 ; 0. **45.** -138 ; -60. **46.** 0 ; 0.

Ex. XXIV. (p. 45).

1. $x^2-(a+b)x+ab$. **2.** $x^2-(a-b)x-ab$. **3.** $x^2+(a-b)x-ab$.
4. $x^3-(a+c)x^2+(ac+b)x-bc$; $x^4-(a^2-b+c)x^2+a(b+c)x-bc$.
5. $a^2-amx-2m^2x^2+3mnx^3-n^2x^4$;
$a^2+a(m+2n)x-\{a(m+n)-2mn\}x^2-(m^2+2n^2)x^3+mnx^4$.
6. $x^3+(a+b+c)x^2+(ab+ac+bc)x+abc$; $x^4-(a^2+b^2)x^2+a^2b^2$.
7. $(m+n)x^4+(m^2+2mn+n^2-1)x^3+(2m^2n+2mn^2-m-n)x^2$
$+(m+n-2mn)x-1$.
8. $x^3-px^2+qx-a^3+a^2p-aq$.
9. $1-(a-1)x-(a-b+1)x^2+(a+b-c)x^3-(b+c)x^4+cx^5$.
10. $ax^4-(a+b)x^3+(a+b+c)x^2-(b+c)x+c$.
11. $b-ap+q$. **12.** $ac-b^2$.
13. $a^3x^3-a^2(b-c+d)x^2y-(abc-abd+acd)xy^2+bcdy^3$.
14. $4x^4+6(m-n)x^3-(4m^2+9mn+4n^2)x^2+6mn(m-n)x+4m^2n^2$.
15. $x^5-(2a^2+2b^2+ab)x^3+(a^4+a^3b+a^2b^2+ab^3+b^4)x-a^2b^2(a+b)$.

Ex. XXV. (p. 46).

1. $\frac{3}{13}a^3b^7$. **2.** $\frac{3}{4}x^5y^2z^4$. **3.** $-\frac{5}{3}ax^5$. **4.** $-\frac{9}{5}a^2b^3c^4$.
5. $-\frac{1}{5}a^4b^6c^8$. **6.** $\frac{2}{81}x^2y^3$. **7.** $-\frac{9}{5}x^3y^3-\frac{9}{5}x^2y^4+\frac{3}{10}xyz^2$.
8. $3a^2x+8b^2x-9c^2x$.

9. $-\frac{1}{8}a^4bx^4y+\frac{3}{8}a^3b^2x^3y^2-\frac{27}{64}a^2b^3x^2y^3+\frac{243}{512}ab^4xy^4$.
10. $\frac{2}{3}a^6b^2-\frac{5}{6}a^5b^3+\frac{10}{9}a^4b^4+\frac{10}{3}a^3b^2$; $-\frac{2}{3}a^8+\frac{1}{2}a^7b-\frac{2}{3}a^6b^2-2a^4$.
11. $\frac{1}{3}a^6b^7$. 12 $-\frac{25}{3}a^9b^3$. 13. $-\frac{27}{1250}x^8y^7$.
14. $2a^6-a^4b-3a^3b^2+\frac{13}{6}a^2b^3-b^6$. 15. $\frac{1}{8}x^3+\frac{5}{16}x^2y-\frac{1}{24}xy^2-\frac{1}{3}y^3$.
16. $\frac{2}{5}a^4-\frac{136}{75}a^3b+\frac{8}{3}a^2b^2+\frac{8}{15}ab^3-b^4$. 17. $\frac{9}{2}a^4+2a^2+\frac{8}{9}$.
18. $\frac{1}{4}x^4+y^4$. 19. $\frac{3}{2}x^8-\frac{2}{3}x^6+x^5-\frac{27}{8}x^4+\frac{3}{2}$.
20. $1-\frac{1}{4}a^2+\frac{5}{6}a^3-\frac{1}{9}a^4+\frac{1}{16}a^6$.

Ex. XXVI. (pp. 46-47).

1. $a^6-a^5x-a^4x^2+a^2x^4+ax^5-x^6$. 2. $4x^4+4ax^3+a^2x^2-a^4$.
3. x^4-10x^2+9. 4. $x^3+(a+b+c)x^2+(ab+ac+bc)x+abc$.
5. $a^{16}+a^8+1$. 6. $a^4-5a^2b^2+4b^4$. 7. $a^8+a^4b^4+b^8$.
8. $a^4+10a^3+35a^2+50a+24$. 9. $x^4-5a^2x^2+4a^4$.
10. $m^4x^4-13m^2n^2x^2y^2+36x^4y^4$. 11. x^8-16y^8.

Ex. XXVII. (p. 48).

1. $6x^3-25x^2+28x-49$. 2. $12x^3+x^2-25$. 3. $2x^4-5x^3+x^2-$
4. $x^4-6x^2-16x-15$. 5. $27a^3+8b^3$. 6. x^3-8y^3.
7. $4x^5+3x^4-23x^3+25x^2-14x+4$; $16x^5-32x^4+35x^3-23x^2+9x-2$.
8. $5x^6-11x^5+21x^4-13x^3+19x^2-12x+9$.

Ex. XXVIII. (p. 49).

1. $x^2+4xy+4y^2$. 2. $9x^2-6xy+y^2$. 3. $25a^2+30ab+9b^2$.
4. $9a^2-30ab+25b^2$. 5. $a^4+2a^2b^2+b^4$. 6. $a^4-2a^2b^2+b^4$.
7. $16a^2b^2+24ab+9$. 8. $4x^4+12x^2+9$. 9. $25a^2b^2+70ab+49$.
10. $a^2b^2-6abcd+9c^2d^2$. 11. $4x^2+4x+1$. 12. $9x^2-24xy+16y^2$.
13 $4a^4+12a^2+9$. 14. $9+12x+4x^2$. 15. $4x^2-12xy+9y^2$.
16. $a^4-6a^3x+9a^2x^2$. 17. $b^2x^4-2bcx^3y+c^2x^2y^2$.
18. $4a^2b^2+20abc+25c^2$. 19. $1+4abc+4a^2b^2c^2$.
20. $16a^2b^4-24ab^4c+9b^4c^2$. 21. 9801. 22. 7225.
23. 6084. 24. 11025. 25. 1008016. 26. 998001.
27. 1010025. 28. 250300·09. 29. 63·936016. 30. 9994·0009.

Ex. XXIX. (p. 50).

1. $a^2+b^2+c^2-2ab+2ac-2bc$. 2. $x^4+6x^3+11x^2+6x+1$.
3. $4x^4+12x^3-7x^2-24x+16$. 4. $16x^4-16x^3-36x^2+20x+25$.
5. $a^6+b^6+c^6+2a^3b^3-2a^3c^3-2b^3c^3$. 6. $a^4+2a^3b-a^2b^2-4ab^3+4b^4$.
7. $4x^4-12x^3-7x^2+24x+16$. 8. $4+12x-7x^2-24x^3+16x^4$.
9. $a^2+b^2+c^2+d^2+2ab-2ac-2ad-2bc-2bd+2cd$.
10. $a^6-a^5-\frac{7}{4}a^4-a^3+2a^2+2a+1$.
11. $1-6x+15x^2-20x^3+15x^4-6x^5+x^6$.
12. $a^4+b^4+c^4+d^4-2a^2b^2-2a^2c^2+2a^2d^2+2b^2c^2-2b^2d^2-2c^2d^2$.

Ex. XXX. (p. 51).

1. $a^2-1.$ **2.** $x^2-9.$ **3.** $a^2-x^2.$ **4.** $4a^2-1.$
5. $9a^2x^2-b^2.$ **6.** $9x^2-25.$ **7.** $9a^2-25b^2$ **8.** $a^2-49b^2.$
9. $4p^2-q^2.$ **10.** $25x^2-16a^2.$ **11.** $a^4-9b^4.$ **12.** $4a^6-x^2.$
13. $16-a^6.$ **14.** $144-49x^2.$ **15.** $64-25x^2.$ **16.** $a^2-49b^2.$
17. $a^4-b^4.$ **18.** $x^6-a^6.$ **19.** $1-a^4x^4.$ **20.** $9a^2-b^2.$
21. $p^2x^4-q^2.$ **22.** $1-x^4.$ **23.** $x^4-81.$ **24.** $a^4-625.$
25. $81-1296x^4.$ **26** $a^4-b^4.$ **27.** $x^4-4096.$ **28.** $81x^4-16a^4.$
29. $a^8-256c^8.$ **30.** 999999. **31.** 9996. **32.** 39975.
33. 120·75. **34.** 3·9975. **35.** 249856. **36.** 399·9964.
37. 89975. **38.** 14399. **39.** 39936. **40.** 80999999·75.

Ex. XXXI. (pp. 52-53).

1 $9a^2-4b^2+4bc-c^2.$ **2.** $x^4+2x^3+x^2-4.$
3. $a^2-4b^2+12bc-9c^2.$ **4.** $a^4+4b^4.$
5. $a^4-2a^3x+a^2x^2-x^4.$ **6.** $a^4-a^2x^2+2ax^3-x^4.$
7. $x^4-a^4+2a^3x-a^2x^2.$ **8.** $1-4a^2+12ab-9b^2.$
9. $4a^2+12ab+9b^2-25.$ **10.** $a^6-a^4b^2-4a^3b^3+4b^6.$
11. $4a^4-5a^2b^2+b^4.$ **12.** $x^4+4y^4.$
13. $a^2+2ab+b^2-c^2$; $a^2-b^2+2ac+c^2$; $a^2-b^2-2bc-c^2.$
14. $a^2-2ab+b^2-c^2$; $-a^2+2ab-b^2+c^2$; $-a^2+b^2-2bc+c^2.$
15. $4a^2-b^2+6bc-9c^2$; $-4a^2+12ac+b^2-9c^2.$
16. $4a^2-b^2-6bc-9c^2$; $-4a^2+4ab-b^2+9c^2.$
17. $a^2+2ac+c^2-b^2-2bd-d^2$; $a^2+2ad+d^2-b^2-2bc-c^2$;
$b^2+2bc+c^2-a^2-2ad-d^2$; $a^2-b^2-c^2-d^2-2bc-2bd-2cd.$
18. $a^2+2ad+d^2-4b^2+12bc-9c^2$; $9c^2+6cd+d^2-a^2+4ab-4b^2$;
$a^2+6ac+9c^2-4b^2+4bd-d^2$; $a^2-4ab+4b^2-12bc+6ac+9c^2-d^2.$
19. $a^4-b^4-6b^2c^2+4b^3c-c^4+4bc^3.$ **20.** $1+2x^2-7x^4-16x^6.$
21. $a^2b^2+c^2d^2-a^2c^2-b^2d^2.$ **22.** $a^6-b^6.$
23. $a^8x^8+a^4x^4+1.$ **24.** $x^{16}+x^8y^8+y^{16}.$ **25.** $x^8-162x^4+6561.$

Ex. XXXII. (p. 54).

1. $x^2+4x+3.$ **2.** $x^2-2x-24.$ **3.** $x^2-10x+24.$
4. $a^2b^2-ab-6.$ **5.** $4a^2x^2-8abx+3b^2.$ **6.** $x^4+3x^2-4.$
7. $x^4-4x^2y^2+3y^4.$ **8.** $25x^2-5ax-6a^2.$ **9.** $15+8x+x^2.$
10. $a^2b^2-10ab+21.$ **11.** $x^4-x^2y-6y^2.$ **12.** $49x^2+14xy-3y^2.$
13. $9x^2+3ax-2a^2.$ **14.** $16a^4-8a^2-15.$ **15.** $21-8x-4x^2.$
16. 10. **17.** $-13.$ **18.** 5. **19.** $y-z.$ **20.** $-10.$
21. $-65.$ **22.** $-3x.$ **23.** $3(x-2).$
24. $x^4-2x^3-25x^2+26x+120.$

Ex. XXXIII. (p. 55).

1. $x^3+6x^2+11x+6$.   2. $x^3-6x^2+11x-6$.
3. $a^3+a^2-14a-24$.   4. $a^3-16a^2+81a-126$.
5. $x^3-5x^2-26x+120$.   6. $x^3+6x^2-7x-60$.
7. $a^3+5a^2b-12ab^2-36b^3$.   8. $1-3x-13x^2+15x^3$.
9. $x^3-5x^2y-2xy^2+24y^3$ 10. $a^6+4a^4b^2-7a^2b^4-10b^6$.
11. -3; -8.   12. $-18y^2$; $3y$.   13. $19y^2$; $-8y$.   14. -28

Ex. XXXIV. (p. 56).

1 $12x^2+17x+6$.   2. $6x^2-23x+20$.   3. $24x^2-29x-4$.
4. $16x^2+6x-7$.   5. $14x^2-29xy-15y^2$.   6. $8x^2-38x+35$.
7. $6x^2-19x+14$.   8. $8x^2-14x-15$.   9. $6x^2-13x-8$.
10. $28x^2+xy-45y^2$.   11. $12x^2-x-20$.   12. $14x^2-29x+12$.
13. $14x^2-29x-15$.   14. $26x^2-41x+3$.   15. $5+9x-2x^2$.
16. $24x^2-50x+25$.   17. $12x^2-7x-12$.   18. $12x^2-25x+12$.
19. $-13y$.   20. -41.   21. $3y$.   22. -11.   23. -13.   24. 11.

Ex. XXXV. (p. 57).

1. $x^3-9x^2+27x-27$.   2. $8a^3+60a^2+150a+125$.
3. $8+12ax+6a^2x^2+a^3x^3$.   4. $a^6+12a^4b^2+48a^2b^4+64b^6$.
5. $x^6-6x^4y^2+12x^2y^4-8y^6$.   6. $8x^3-36x^2+54x-27$.
7. $27a^6+54a^4b+36a^2b^2+8b^3$.   8. $8a^9-36a^6b^2+54a^3b^4-27b^6$.
9. x^3+27.   10. $8a^3-27b^3$.   11. $1+a^3b^3$.   12. x^6-y^6.
13. $8a^3+b^3$ 14. $8x^3y^3-1$.   15. $64a^3-125b^3$
16. $729a^6+8x^3$.   17. $8x^3-27y^3$.   18. $216a^3-b^3$.
19. a^6-b^6.   20 x^6-64.

Ex. XXXVI. (pp. 59-60).

1. $-3ab$.   2. $-2a^2b$.   3. $2qr$.   4. $-8abxz$.
5. $3a^2byz$.   6. $-2abc^2z$.   7. $-7p^2qr$.   8. $-9abc^3$.
9. $-bc^2$.   10. $5xy^2$.   11. $-35bx$.   12. -2.
13. $-8abc$.   14. $8x$.   15. $-5a^2$.   16. $-4ab^2$.
17. $-8a^3c^4$.   18. $-8a^2b^4c^4$.   19. xyz^2.   20. $-17bx$.
21. $3a^2b^2c$.   22. $-7c^4dy$.

Ex. XXXVII. (p. 60).

1. $x-y$.   2. $-a+b$.   3. $-3a-4b$.   4. $9a-3c$
5. $5a-7ab^2$.   6. $-x^3+x^2-x$.   7. $3xy-2xz+3yz$.
8. $-a^2b^2+7abc^2-4c^4$.   9. $a^2x^2-3ax+3by-b^2y^2$.
10. $-4m^3n+3m^2-2mn^2+n$.   11. $15a^4b^3-5a^3b^2+3a^2b-a-2b^2$.
12. $-2a^4b^3+4ab^2c^3d-8bc^2d$.

13. $6p^3q^2+9p^2q-3pq^2+q^3$; $-6p^3q-9p^2+3pq-q^2$.
14. $-4x^4y^6+3x^3y^4-2x^2y^3+xy^2+6$.
15. $-4a^2+2ab^2c-3b^3c^2+1$. 16. $3xy^2z^4+5x^2yz^3+6x^3z^2-2x^4y^5$.

Ex. XXXVIII. (pp. 62-63).

1. $x+5$. 2. $3a-2b$. 3. m^2-4m+3. 4. $3x+2y$.
5. $2ab-3b^2$. 6. $x+6$. 7. $x-8$. 8. $2a+7b$.
9. $4x-3y$. 10. $2a+3b$. 11. $16a^2-20ab+25b^2$.
12. $4a^2+6ab+9b^2$. 13. $ab-11$. 14. x^3+12
15. $16a^4-4a^2b^2+b^4$. 16. $a^2+2ab+2b^2$. 17. x^2+xy+y^2.
18. $7a-8b$. 19. x^4-x^2+1. 20. $2x^2y^2+2xy+1$.
21. $3a^2+2a+1$. 22. $a^4+2a^3+3a^2+4a+5$. 23. $19x^2+15x+9$.
24. $a^2+2ab-3b^2$. 25. $2x^2+3y-1$. 26. x^3-7x+5.
27. x^3-x^2-x-1. 28. $x^2-4ax+4a^2$. 29 $-2a^2+8ab-5b^2$.
30. $-x^3-2x^2-3x-4$. 31. a^3+b^3. 32. $x^2+2xy+3y^2$.
33. m^2-2m+3. 34. $1-2x+3x^2-4x^3+5x^4$.
35. $a^4-2a^3b+3a^2b^2-2ab^3+b^4$. 36. $x^4+2x^3+3x^2+2x+1$.
37. a^3-2a^2+2a-1. 38. $4x^2-6x+9$. 39. $a^3+2a^2b+3ab^2+4b^3$.
40. $4a^2+14a+9$. 41. $3a+2b+c$. 42. $3x^2-4x+5$.
43. $2a^3+5ax^2-2x^3$. 44. $4x^2-3ax+a^2$. 45. $a^2-5ab+6b^2$.
46. $-1-3xy-13x^2y^2$. 47. $x^4+3x^3y+8x^2y^2-8y^4$.
48. $3x^3+2x^2-4x-10$. 49. $7x^2-7xy+5y^2$.
50. x^2-x-19. 51. $x^7-x^6+x^4-x^3+x-1$.
52. $a^7-a^6x+a^5x^2-a^4x^3+a^3x^4-a^2x^5+ax^6-x^7$.
53. $x^4+x^3y+x^2y^2+xy^3+y^4$.

Ex. XXXIX. (pp. 64-65).

1. $a+b+c$. 2. $a-b-c$. 3. x^2-px+q.
4. ax^2+bx-c. 5. $a+b-c-d$. 6. $a+2b-c$.
7. $3a+2b+c$. 8. $a^2+b^2+c^2+ab-ac+bc$.
9. $a^2+b^2+c^2-ab+ac+bc$. 10. $1-x+2y+x^2+2xy+4y^2$.
11. $1-x-2y+x^2-2xy+4y^2$. 12. $x^2+4y^2+9z^2+2xy-3xz+6yz$.
13. $x^2+y^2+z^2+1$. 14. $a-b-c$.
15. $x+a$. 16. $a^3-3ab(a-b)-b^3$. 17. $x^2+a^2-(x+a)$.
18. $z^2+yz+zx+xy$. 19. $x^2-xy+y^2+x+y+1$.
20. $x^2+4y^2+9z^2+6yz+3xz-2xy$. 21. $a+2b+3c$. 22. $a+b$.

Ex. XL. (pp. 65-66).

1. $-\frac{9}{5}ax$. 2. $-\frac{15}{28}bx$. 3. $\frac{3}{4}a^3b^3$. 4. $-\frac{8}{45}ax$.
5. $\frac{3}{10}b$. 6. $-\frac{1}{3}xy^3$. 7. $\frac{1}{2}ab-\frac{8}{9}$. 8. $-a+3b+\frac{16}{5}c$.
9. $3abc-\frac{9}{2}bc+4a^3c^2$; $\frac{1}{2}ab^3c-\frac{3}{4}b^3c+\frac{2}{3}a^3b^3c^3$.

10. $\frac{1}{2}a^2 - \frac{1}{3}ab + \frac{1}{4}b^2$. **11.** $7ax + 1$. **12.** $3x^2 - \frac{2}{3}xy + \frac{3}{4}y^2$.
13. $2a^2 - ay - \frac{1}{2}y^2$. **14.** $\frac{1}{16}x^2 + \frac{1}{20}xy + \frac{1}{25}y^2$. **15.** $\frac{1}{4}a^2 - \frac{1}{6}ab + \frac{1}{9}b^2$.
16. $a^2 + \frac{3}{4}a + \frac{5}{4}$. **17.** $a - \frac{1}{3}b$. **18.** $\frac{1}{3}x^2 + \frac{2}{3}xy + \frac{2}{7}y^2$.
19. $\frac{3}{2}x^2 - x + \frac{2}{3}$. **20.** $a^3 - \frac{1}{2}a^2b + \frac{1}{3}b^3$. **21.** $x^2 + \frac{2}{3}x + \frac{1}{3}$.
22. $\frac{1}{25}x^2 - \frac{1}{5}xy + \frac{1}{9}y^2$. **23.** $\frac{1}{2}x^2 - \frac{3}{4}x + 6$. **24.** $\frac{3}{2}x^3 - 5x^2 + \frac{1}{4}x + 9$.
25. $3a - \frac{7}{2}b + \frac{1}{3}c$.

Ex. XLI. (p. 67).

1. $a - ax + ax^2 - ax^3$. **2.** $1 + 5x + 15x^2 + 45x^3$.
3. $1 - 5x + 15x^2 - 45x^3$. **4.** $1 + 2a - 8a^3 - 16a^4$.
5. $1 + 2x + 3x^2 + 4x^3$. **6.** $1 - (a+b)x + (a+b)bx^2 - (a+b)b^2x^3$.
7. $-1 - a + 2a^3 + 4a^4$. **8.** $a^3 - ap^2 + aq - r$.

Ex. XLII. (p. 69).

1. $4x^2 - 7$. **2.** $a^2 - 3a + 9$. **3.** $x^3 - 3x + 2$.
4. $x - 3$. **5.** $9x^2 - 3x + 1$. **6.** $x^3 - x^2 + x - 1$.
7. $3x^2 + 2x - 4$. **8.** $x^2 - x$. **9.** $3a - 2b$.
10. $12x^4 - 11x^3 + 10x^2 + 39x + 8$.

Ex. XLIII. (p. 70).

1. $a - x$. **2.** $a^4 + a^3x + a^2x^2 + ax^3 + x^4$.
3. $a^5 - a^4x + a^3x^2 - a^2x^3 + ax^4 - x^5$. **4.** $3x + 1$. **5.** $5x - 1$.
6. $2x - 3$. **7.** $1 - 2x + 4x^2$. **8.** $9x^2 + 3x + 1$. **9.** $1 - 2x + 4x^2 - 8x^3$.
10. $x^3 + 3x^2y + 9xy^2 + 27y^3$. **11.** $a^4 - 2a^3b + 4a^2b^2 - 8ab^3 + 16b^4$.
12. $x^{15} - x^{12}y^2 + x^9y^4 - x^6y^6 + x^3y^8 - y^{10}$. **13.** $\frac{1}{4}a^2 - \frac{1}{2}ab + b^2$.
14. $-x^2y^2z + xyz^2 - z^3$. **15.** $4m^2 - n^2$.
16. $a^4 + 2a^3x + 4a^2x^2 + 8ax^3 + 16x^4$. **17.** $32 - 16a + 8a^2 - 4a^3 + 2a^4 - a^5$.
18. $a^2 - 7a + 49$. **19.** $a + b - c$. **20.** $x^2 + xy - xz + y^2 - 2yz + z^2$.

Revision Papers I.

Paper I.

1. 15. **2.** $4 + x$. **3.** $7a - 7b$. **4.** $2a$. **6.** 14 ; 1 ; −3 ; 3 ; 19.
7. $(ax - by) - (cz + bx) + (cy + az)$; $(ax - by - cz) - (bx - cy - az)$.
8. $42x^2 + 216xy + 30y^2$. **9.** $a^2 + ab - ac - bc$. **10.** $4x^2$.

Paper II.

1. $125\frac{7}{12}$. **2.** $2x^2 - a^3 - 2a^2 + 2a + 7$. **5.** $8a^3 + 27b^3 - c^3 + 18abc$.
6. $6x^3 - 2x^2 - 8x - 16$. **7.** $x - 6$. **8.** 22 ; 9 ; 0 ; −6 ; −3 ; 4.
9. $-35x + 18y + 17z$. **10.** $x^8 - x^6y^2 - x^2y^6 + y^8$.

Paper III.

1. $\frac{1}{3}$. **2.** $a^2+b^2+c^2$. **3.** $12b^2$. **4.** 94.
5. $\cdot 06-\cdot 3x+\cdot 2x^2-x^3$; $\cdot 031$. **6.** $2(x^2+y^2+z^2-yz-zx-xy)$.
7. x^6-a^6. **8.** x^3. **9.** $x^2+5(a-1)x-b$. **10.** $3a-6b$ miles.

Paper IV.

1. $3\sqrt{(133)}$. **2.** $5x^2+6xy+2y^2$. **3.** $a^2d^2-b^3d+bc^2d$.
4. $x^2+3ax+3a^2$ and a *rem* $3a^3$. **5.** $20x^2-5ax$.
6. $3a^2+4a-2$. **7.** $1+3a+4a^2+2a^3$. **8.** $3a^2+4ab+5b^2$.
10. $a+3b+2d$.

Paper V.

1. 2700. **2.** $\frac{8}{5}a^2-\frac{5}{4}b^2+c^2-\frac{27}{7}d^2$. **3.** $a^3-a^2b+3a^2c-4ab^2$
$-9ac^2+12abc+24b^2c+45bc^2+4b^3-27c^3$; $a^2-4b^2+12bc-9c^2$.
4. $x^3-2(a+b+c)x^2+2(ab+ac)x$.
5. $x^4-2qx^3-2(p^2-q^2)x^2+(p^3+p^2q+pq^2-q^3)x-p^2q^2$:
$x^2-(p+q)x+q^2$. **6.** $x^4-(4a^2+9b^2)x^2+36a^2b^2$.
7. $2ab+3ac+6bc$. **8.** $16(x^4-x^2y^2+y^4)-8(x^2+y^2)a^2+a^4$.
9. $3a^2-a+2$.

Paper VI.

1. $\frac{7}{24}$. **2.** $(10a+9b+9c)(a+c+b)=10a^2+19ab+19ac+9b^2+$
$18bc+9c^2$. **3.** $7x^2-17ax-12a^2$. **4.** x^2-ax-b.
5. $2y^2$. **6.** $x^3+3x^2-16x-48$; $6x+24$. **7.** $7x^2+5xy+3y^2$.
8. x^3+y^3 and x^4-y^4 are divisible by $x+y$; x^4-y^4 and
x^5-y^5 are divisible by $x-y$. **9.** x^8+x^4+1.

Paper VII.

1. 1. **2.** (i) $16a^2b^2+12abc+9c^2$. (ii) $9a^2b^2-6abc+4c^2$.
3. (i) a^4+a^2-20. (ii) 14375. (iii) 996004.
4. $x^8-116x^7+1789x^6-10460x^5+2502x^4-5382x^3+4633x^2$
$-708x+1189$.
5. $3a^2-ab+b^2$. **6.** $16ax^3$. **7.** $2x^5-x^4y-\frac{1}{3}x^3y^2-\frac{35}{6}x^2y^3+y^5$.
8. $\frac{8}{9}x^2-\frac{1}{4}xy+\frac{2}{3}y^2$. **9.** $62x^2+23xy-4y^2$. **10.** $x+y+z+xyz$.

Paper VIII.

1. 4. **2.** $3a^2-12b^2-96c^2-3ab+4a-12b+5c$. **3.** $3\sqrt{3}$.
4. (i) x^4-25. (ii) $9x^2-45x+56$. (iii) $12x^2-47x+45$.
5. $x^{10}-x^5y^5+y^{10}$. **6.** $2x^2+x-1$; $x+1$. **7.** $a_1x+b_1y+c_1z$.
8. $5x$. **9.** $2x^2-2(a+b)x+ab$. **10.** 3 miles.

Paper IX.

1. $\frac{5}{6}$. **2.** (i) $9a^8 - 12a^7x + 12a^5x^3 - 20a^3x^5 - 20ax^7 - 25x^8$.
(ii) $x^8 - 2ax^7 - 2a^2x^6 + 6a^3x^5 - 6a^5x^3 + 2a^6x^2 + 2a^7x - a^8$.
3. $x^2 - 2ax - a^2 + 1$. **4.** $9a^2 + 6ac + 4bc - 3ab - 6b^2$.
5. $2 + 4x + 8x^2 + 16x^3 + 32x^4$. **6.** $3(a^2 + b^2 + c^2) - 2(bc + ca + ab)$; 59.
7. $x^7 - 2ax^6 - 16a^3x^4 + 32a^4x^3 + 64a^6x - 128a^7$.
8. $2x^4 - 5x^3 - 4x^2 + x + 2$. **9** $x^4 - x^3y + x^2y^2 - xy^3 + y^4$. **10.** $6a^2 + 2$.

Paper X.

1. 128. **2.** $-2x - 30y + 53z$. **3.** $-60a^5x^3y^6$; $-4a^2x^2y$.
4. $5bc + bc^2 + b^2c^2$; -2. **5.** $3x^2 - 7x + 9$. **6.** $a^2 + 2ab + b^2 - ac - bc + c^2$.
8. $a^4 - 3a^3b + 6a^2b^2 - 3ab^3 + b^4$ **9.** $x^2 + (n+3)ax + 3a^2$. **10.** $2m$; $2n$.

Ex. XLIV. (p. 78).

1. $x(x + 3a)$. **2.** $a^2(a^2 - b)$. **3.** $x^2(x^3 + 5)$. **4.** $x^2(x^4 - 5a)$.
5. $5x^2(x - 3y)$. **6.** $a(a - 2b)$. **7.** $7(3 - 5x)$. **8.** $-2a(a - 3)$.
9. $3x^2(1 + 3x - 4x^2)$. **10.** $4a(a - 4b + 6)$. **11.** $8y(x + 2z)$.
12. $3a^3b^2(b^2 - 6ab + 7a^2)$. **13.** $6xyz(x + 2y - 3x^2y^2)$.
14. $7x^2y^2z(6z^2 + 7xyz - 9x^2y^2)$. **15.** $7xy(2x^2 - xy + 8y^2)$.
16. $5ab(a^3 + 5a^2b - 2ab^2 + 3b^3)$. **17.** $6xyz(6x - 9y + 8z)$.
18. $10a^3b^2c(7 - 6abc + 5a^2b^2c^2 - 4a^3b^3c^3)$.

Ex. XLV. (p. 79).

1. $(a+b)(a+c)$. **2.** $(a-b)(a+c)$. **3.** $(a+b)(c+d)$.
4. $(ac-2b)(ab+c)$. **5.** $(a+b)(x+c)$. **6.** $(a+b)(x-c)$.
7. $(ac-3d)(ac+b)$. **8.** $(a^2+b^2)(c+d)$. **9.** $(x^2+2)(2x-3)$.
10. $(11x^2+7)(x-5)$. **11.** $(5a-3c)(4b-7d)$. **12.** $2(3a+2c)(2a-3b)$.
13. $(3a-4d)(5b+3c)$. **14.** $(x^2+1)(y^2+1)$. **15.** $(x^2+4)(x+5)$.
16. $(x-y)(x+y-5)$. **17.** $z(6x-5)(3y^2-2x)$. **18.** $(xy-5z)(a+bc)$.
19. $(a+2)(ax-3by)$. **20.** $2(a-c)(a+3b)$.

Ex. XLVI. (p. 80).

1. $(x+6)^2$. **2.** $(a-4)^2$. **3.** $(x+7)^2$. **4.** $(2a-1)^2$.
5. $(2x+5y)^2$. **6.** $(2x+1)^2$. **7.** $(3a-5b)^2$. **8.** $(a^2+7)^2$.
9. $(3a^2+b^2)^2$. **10.** $b^2(4ab-c^2)^2$. **11.** $b^2(2a^2-7bc)^2$.
12. $(2abc+1)^2$. **13.** $(2x^3y+5z^4)^2$. **14.** $(a^2-10)^2$.
15. $(\frac{3}{5}a-\frac{2}{3}b)^2$. **16.** $(3a-2b+3c)^2$. **17.** $(2x-y+z)^2$.
18. $(ax-by+2c)^2$. **19.** $(x^2+2x-2)^2$. **20.** $(2a+3b+x+2y)^2$.
21. 1. **22.** 4. **23.** ·0025. **24.** ·0961. **25.** 4.

Ex. XLVII. (p. 81).

1. $(1+2x)(1-2x)$. 2. $(a+3x)(a-3x)$. 3. $(3m+2n)(3m-2n)$.
4. $(5x+4)(5x-4)$. 5. $(x+3)(x-3)$. 6. $(1+x^2)(1+x)(1-x)$
7. $(a+13)(a-13)$. 8. $(ab+1)(ab-1)$. 9. $(5+ab)(5-ab)$.
10. $(3x+4y)(3x-4y)$. 11. $(9x+8)(9x-8)$. 12. $(6+x^4)(6-x^4)$.
13. $(a^2+5)(a^2-5)$. 14. $(a^2b+10)(a^2b-10)$. 15. $(7a+9b)(7a-9b)$.
16. $(3x+7y)(3x-7y)$. 17. $(a+17b)(a-17b)$.
18. $(11a+12b)(11a-12b)$. 19. $(12ab+11c^2)(12ab-11c^2)$.
20. $(a^3+b^2)(a^3-b^2)$. 21. $4(a+2)(a-2)$.
22. $(5+a^2)(5-a^2)$. 23 $(7x^2+1)(7x^2-1)$.
24. $(a^2b^6+3c^4)(a^2b^6-3c^4)$. 25. $(5ax+2y)(5ax-2y)$.
26. $x^2y^2(4x+5y)(4x-5y)$. 27. $xy(ay+x^2)(ay-x^2)$.
28. $2ab^2c(a+2c)(a-2c)$. 29 $x^2(5x+a)(5x-a)$.
30. $a^4(a+3b^3)(a-3b^3)$. 31. $(9x^2+8)(9x^2-8)$.
32. $7x^4(x+3a)(x-3a)$. 33. $3(x^2+10)(x^2-10)$.
34. $11(1+3a)(1-3a)$. 35. $5(3xy+4)(3xy-4)$.
36. $141a^3b^2(a^3b^2+2)(a^3b^2-2)$. 37. $5c(11a+12b)(11a-12b)$.
38. $7(a^2+7b^2)(a^2-7b^2)$. 39. $17(1+2ab)(1-2ab)$.
40. 1840. 41. 54000. 42. 250000. 43. 15600.
44. 149400. 45. 573. 46. 11800. 47. 998000.
48. 14352. 49. 29496. 50. 45584.

Ex. XLVIII. (p. 82).

1. $(a-b+c)(a-b-c)$. 2. $(a+b-c)(a-b+c)$. 3. $(a+3b)(a-b)$.
4. $(a^2+b^2+c^2+d^2)(a^2+b^2-c^2-d^2)$. 5. $(a+b)(a-b+4)$.
6. $(a+b+2)(a-b)$. 7. $(x+2y+4a)(x+y-4a)$.
8. $(5x+a+b)(5x-a-b)$. 9. $(c+a-b)(c-a+b)$.
10. $(4x-5)(2x+1)$. 11. $8ab$. 12. $(a+4x-y)(a-4x+y)$.
13. $7(x+y)(x-y)$. 14. $(x^2+4xy+y^2)(x^2-4xy+y^2)$.
15. $2ab-1$. 16. $(a^2+2ab+2b^2)(a^2-2ab+2b^2)$.
17. $(7x-y)(7y-x)$. 18. $3(a+b+2c+2d)(a+b-2c-2d)$.
19. $(3x+2y+2a)(x+4y)$. 20. $(a+b-2c)(a-b+2c)$.
21. $(a-b+c)(a-b-c)$. 22. $(c+a-b)(c-a+b)$.
23. $(2a^2+3a-1)(2a^2-3a+1)$. 24. $(x-3y+4z)(x-3y-4z)$.
25. $(3a-2b+4x+y)(3a-2b-4x-y)$. 26. $(a+b+c)(a+b-c)$.
27. $(a+b-c+d)(a-b+c+d)$. 28. $(x^2+x+1)(x^2-x-1)$.
29. $(a-b+c+d)(a-b-c-d)$. 30. $(a+b+c)(a-b+c)$.
31. $(a^2+b^2-c^2-d^2)(a^2-b^2+c^2-d^2)$. 32. $(5+a-b)(5-a+b)$.
33. $(2x^2+1)(5-4x)$. 34. $(a+b)^2(a-b)^2$.
35. $(3a+b-4d-1)(3a-b+4d-1)$.

Ex. XLIX. (p. 83).

1. $(2a+b+c)^2$. **2.** $(x^2+x-1)^2$. **3.** $(2x^2-3x-4)^2$. **4.** $(x^2+2x-2)^2$.
5. $(1-\frac{1}{2}b+\frac{1}{4}c)^2$. **6.** $(x^2+\frac{1}{3}x+\frac{1}{4})^2$. **7.** $(x^2-5x+7)^2$. **8.** $(a-b-c+d)^2$.

Ex. L. (pp. 84-85).

1. $(x+1)(x+5)$. **2.** $(x+4)(x+5)$. **3.** $(x+1)(x-6)$.
4. $(x-3)(x-5)$. **5.** $(x+1)(x+7)$. **6.** $(x-1)(x-9)$.
7. $(x-2)(x+3)$. **8.** $(x+2)(x-3)$. **9.** $(x-3)(x+1)$.
10. $(x+5)(x-3)$. **11.** $(x+8)(x-1)$. **12.** $(x+1)(x-9)$.
13. $(x+3)(x+4)$. **14.** $(x-2)(x-7)$. **15.** $(x+2)(x-7)$.
16. $(x-3)(x+4)$. **17.** $(1-x)(1-2x)$. **18** $(x+11)(x-10)$.
19. $(x+7)(x+9)$. **20.** $(x-11)(x-12)$. **21.** $(x-10)(x-20)$.
22. $(x-3a)(x-13a)$. **23.** $(a^2+2b^2)(a^2+7b^2)$. **24.** $(a^3-3)(a^3-4)$.
25. $(xy+3z)(xy+11z)$. **26.** $(ab-11c)(ab-13c)$. **27.** $(7+x)(6-x)$.
28. $(6+x)(11-x)$. **29.** $(x+1)(x-5)$. **30.** $(x+13y)(x-2y)$.
31. $(a^3-9)(a^3+5)$. **32.** $(x-9)(x+8)$. **33.** $(ab+1)(ab-4)$.
34. $(xy-11)(xy+14)$. **35.** $(x+3)(x-16)$. **36.** $(3ab+1)(2ab-1)$.
37. $(5xy-1)(5xy-1)$. **38** $(18x+y)(3x-y)$. **39.** $(x+2a)(x-5b)$.
40. $(x-6a)(x-5b)$. **41.** $(x-13y)(x-13y)$. **42** $(13x-1)(2x+1)$.
43. $(x-1)(43x+1)$. **44.** $(1-x)(1-6x)$.
45. $(x-7a)(x+3b)$. **46.** $a^2(x-a)(x-2a)$.
47. $a(a-3x)(a+2x)$. **48.** $a^2(a^2+8b^2)(a^2-7b^2)$.
49. $p^4(m+7)(m-12)$. **50.** $(x-3mp)(x+5np)$.

Ex. LI. (p. 86).

1 $(2x+3)(2x+1)$. **2.** $(4x+1)(x+3)$. **3.** $(x-2)(3x-7)$.
4. $(3x-1)(4x-1)$. **5.** $2(x+2)(4x+3)$. **6.** $(a+3)(2a+1)$.
7. $(4x-1)(x+3)$. **8.** $(2x-3)(2x+1)$. **9.** $(3x-2)(x+2)$.
10. $(3x+4)(2x-1)$. **11.** $(4x+1)(3x-2)$. **12.** $2(6x-1)(x-1)$.
13. $(4x+1)(3x-1)$. **14.** $(3x-5)(x+1)$. **15.** $(4a^2-x^2)(3a^2+x^2)$.
16. $ab(3a-2b)(a+b)$. **17.** $xy(2x+y)(x+2y)$.
18. $3y^2(3x+2y)(x-y)$. **19.** $a^2(3ax-1)(2ax+1)$.
20. $x^2(2b-3x)(3b+x)$. **21** $(4-a)(2+5a)$.
22. $(7+a)(4-5a)$. **23.** $(7x-4)(2x-3)$. **24** $(13x+2)(x+3)$.
25. $(3x+4)(3x-2)$. **26.** $(2x-7)(2x+9)$. **27.** $(2a-1)(a+5)$.
28. $(3-x)(1+8x)$. **29.** $(3x+5)(x-6)$. **30.** $(6x+1)(3x-2)$.

Ex. LII. (p. 87).

1. $(3x+2)(x+4)$. **2.** $(3x+2)(x-4)$. **3.** $(7x+3)(2x-5)$.
4. $(2x-3a)(2x-a)$. **5.** $(7x-3y)(x+y)$. **6.** $(2x-1)(x+3)$.

7. $(5x+2)(x+3)$. **8.** $(4x-5)(6x-5)$. **9.** $(4-5x)(7+x)$.
10. $(3x+2a)(2x-3a)$. **11.** $(13x-5y)(7x+17y)$.
12 $(11x+13y)(9x-11y)$. **13.** $(15a^2+17b^2)(14a^2-9b^2)$.

Ex. LIII. (pp. 87-88).

1. $(x+y)(x^2-xy+y^2)$. **2.** $(x-y)(x^2+xy+y^2)$.
3. $(1+x)(1-x+x^2)$. **4.** $(1-x)(1+x+x^2)$.
5. $(x^2+y)(x^4-x^2y+y^2)$. **6.** $(x^2-y)(x^4+x^2y+y^2)$.
7. $(a-2)(a^2+2a+4)$. **8.** $(2x+1)(4x^2-2x+1)$.
9. $(ab+1)(a^2b^2-ab+1)$. **10.** $(ab-1)(a^2b^2+ab+1)$.
11. $(a+4b)(a^2-4ab+16b^2)$. **12.** $(3a+1)(9a^2-3a+1)$.
13. $(a-4b)(a^2+4ab+16b^2)$. **14.** $(x^2+1)(x^4-x^2+1)$.
15. $(1-3x^2)(1+3x^2+9x^4)$. **16.** $(2a-3b)(4a^2+6ab+9b^2)$.
17. $(6a-b)(36a^2+6ab+b^2)$. **18** $(9x+2a)(81x^2-18ax+4a^2)$.
19. $(x^2+4)(x^4-4x^2+16)$. **20.** $(1+9a)(1-9a+81a^2)$.
21. $(7a-1)(49a^2+7a+1)$. **22.** $(4x-5y)(16x^2+20xy+25y^2)$.
23. $2(a+4)(a^2-4a+16)$. **24.** $(5a^2+8x)(25a^4-40a^2x+64x^2)$.
25. $x^3(a+3x)(a^2-3ax+9x^2)$. **26.** $3(3-a)(9+3a+a^2)$.
27. $x^2y(a+3y)(a^2-3ay+9y^2)$. **28.** $(2x^3+y^2)(4x^6-2x^3y^2+y^4)$.
29. $(4-a+b)(16+4a-4b+a^2-2ab+b^2)$.
30. $(6+4a-5b)(36-24a+30b+16a^2-40ab+25b^2)$.
31. $x(x-3)(x^2+3x+9)$. **32.** $(7a+9b)(103a^2+180ab+81b^2)$.

Ex. LIV. (p. 88).

1. $(x+2)^3$. **2.** $(a+2x)^3$. **3.** $(2x-3)^3$. **4.** $(5a-2)^3$.
5. $(x-5)^3$. **6.** $(\frac{1}{2}a-\frac{2}{3}b)^3$. **7** $(4x-3)^3$. **8.** $(a-6)^3$.

Ex. LV. (p. 89).

1. $(x^2+3xy-2y^2)(x^2-3xy-2y^2)$. **2.** $(a^2+a+1)(a^2-a+1)$.
3. $(3a^2+4a+5)(3a^2-4a+5)$. **4.** $(x^2+2x-4)(x^2-2x-4)$.
5. $(a^2+4ab-b^2)(a^2-4ab-b^2)$. **6.** $(x^2+3xy+y^2)(x^2-3xy+y^2)$.
7. $(3x^2+2xy+7y^2)(3x^2-2xy+7y^2)$. **8.** $(x^2+x-2)(x^2-x-2)$.
9. $(4x^2+2x+1)(4x^2-2x+1)$. **10.** $(4a^2+6ax+9x^2)(4a^2-6ax+9x^2)$.
11. $(a^2+2ab+2b^2)(a^2-2ab+2b^2)$.
12. $(7a^2+13ab+11b^2)(7a^2-13ab+11b^2)$.
13. $(3x^2+3xy+5y^2)(3x^2-3xy+5y^2)$.
14. $(5a^2+7ab+4b^2)(5a^2-7ab+4b^2)$. **15.** $(x^2+2x+2)(x^2-2x+2)$.

Ex. LVI. (p. 90).

$(x+2)(x-2)(x^2+2x+4)(x^2-2x+4)$.
$(x+2)(x-2)(x^2+4)(x^4+16)$.

3. $(1+3y)(1-3y)(1+3y+9y^2)(1-3y+9y^2)$.
4. $(1+2a)(1-2a)(1+4a^2)$.
5. $(x+a)(x-a)(x^2+ax+a^2)(x^2-ax+a^2)(x^2+a^2)(x^4-a^2x^2+a^4)$.
6. $(x+a)(x-a)(x^2+a^2)(x^4+a^4)(x^8+a^8)$.
7. $(ax+by)(ax-by)(a^2x^2+b^2y^2)$.
8. $(x^2+x+1)(x^2-x+1)(x^4-x^2+1)$.
9. $(x^2+xy+y^2)(x^2-xy+y^2)(x^4-x^2y^2+y^4)$.
10. $(a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2)$.
11. $(x+1)(x^2-x+1)(x-1)(x^2+x+1)$.
12. $80(a+b)(a-b)(a^2+b^2)$.
13. $(a+b+c)(a+b-c)(a-b+c)(b+c-a)$.
14. $(a+2b-2c)(a-2b+2c)(a^2-8bc+4b^2+4c^2)$.

Ex. LVII. (pp. 91-92).

1. $4(x+1)(x-1)$. 2. $3a^3b^3c^2(a^2-6bc+8ab)$. 3. $abc(a+c)^2$.
4. $x^2(x-b)^2$. 5. $(ab^3+c^2)(ab^3-c^2)(a^2b^6+c^4)$.
6. $x^2(x+a)^2(x-a)^2$. 7. $(a-b)(b-c)$. 8. $(a+7)(a-3)$.
9. $(3ab^2-5c)^2$. 10. $3(x-y)(x-6y)$. 11. $2(a+5)(a-5)$.
12. $5a(ab+4c)(ab-3$ 13. $(2ab-3c)(2ab-c)$. 14. $3(2+x)(2-x)$.
15. $(2x+y-z)(4x^2-2xy+2xz+y^2-2yz+z^2)$.
16. $(a+b-5c)(a^2+2ab+b^2+5ac+5bc+25c^2)$.
17. $(a+b+c)(a^2+2ab+b^2-ac-bc+c^2)$.
18. $2x(x^2+3y^2)$. 19. $2y(3x^2+y^2)$. 20. $3(1+a-b)(1-a+b)$.
21. $(ax+1)(bx+1)$. 22. $17(x+1)(x+2)$. 23. $13(3x+1)(3x-1)$.
24. $2(5a+1)(25a^2-5a+1)$. 25. $(a+b+1)(a^2+2ab+b^2-a-b+1)$.
26. $(a^2+ab-b^2)(a^2-ab-b^2)$. 27. $4x^2(x+3)(7x-5)$.
28. $(3+a+b)(3-a-b)$. 29. $(3x-1)(2x-3)$. 30. $(a+15)(a-9)$.
31. $(7a+11b)(7a-11b)$. 32. $(a+1)(b+1)(a-1)(b-1)$.
33. $5(7a+6b-c)(c-a)$. 34. $(3a+4b+c)(5c-a)$.
35. $(x-y)(x^2-5xy+7y^2)$. 36. $(x+7)(11x-2)$. 37. $(7x+5y)(3x-4y)$.
38. $(x^2+2xy+4y^2)(x^2-2xy+4y^2)(x^4-4x^2y^2+16y^4)$.
39. $y(x-3y)(x-12y)$. 40. $(3x+5y)(3x-5y)(9x^2+25y^2)$.
41. $(2x+2y-a-b)(x+y-3a-3b)$. 42. $(5x+13)(6x-11)$.
43. $(7x-6)(6x-17)$. 44. $(21x-5)(3x+7)$.
45. $(x^2+4x+16)(x^2-4x+16)$. 46. $(3x+4)(3x-4)(x-4)(x+2)$.
47. $5a(13a^2+18ab+12b^2)$. 48. $ab(a-b)$.
49. $2(x-y)(4y-x)$. 50. $5b(a-b)$.
51. $2(x+y)(4x-y)$. 52. $4y(x+y)$. 53. $(a+b)(a^2+ab+b^2)$.
54. $(x+1)(x+2)(x-2)(x-3)$. 55. $a^2(b+a)(b-a)$.
56. $x(6x+1)(3x-2)$. 57. $16x(1-x)$. 58. $x(1-2x)(3-2x)$.
59. $7(x+1)(x-1)$. 60. $b(a-5)(a^2+5a+25)$.

61. $(a^2+b^2+c^2)(x^2+y^2+z^2)$.
62. $(x^2+2a^2)(x^2-2a^2)(x^2+2ax+2a^2)(x^2-2ax+2a^2)$.
63. $(a+b-3c)(a+b-3c-1)$. **64.** $2(1+a)(1+c)(a-c)$.
65. $2(a+b+c+d)(a-d)$. **66.** $(a^2+ax+x^2)(a^2-ax+x^2)(a^4-a^2x^2+x^4)$.
67. $(x+y+z+u)(z+u-x-y)(x-y+z-u)(x-y-z+u)$.
68. $(x^2+2x+3)(x^2-2x+3)$. **69.** $(x^2+pxy-y^2)(x^2-pxy-y^2)$.
70. $(x+a)^3(x-a)^3$. **71.** $(7x+8y)(49x^2-56xy+64y^2)$.
72. $(5-x)(12+x)$. **73.** $(x+7)(x+6)$. **74.** $(x+2)(x+6)(x^2+8x+10)$.
75. $(x+7)(x-6)$. **76.** $(x+6a)(x-11a)$. **77.** $(2x+y)(2x-y)(3x^2+y^2)$.
78. $(3x+2)(x-4)$. **79.** $(2x+3)(3x-2)$. **80.** $(5x+1)(2x-5)$.
81. $(13x-11)(3x+2)$. **82.** $2(2a+b)(4a^2+ab+b^2)$. **83.** $(6-x)(x-4)$.
84. $(x+1)^2(x-1)$. **85.** $(a+2b-c-3d)(a-2b+c-3d)$.

Ex. LVIII. (pp. 94-95).

1. 0. **2.** x^8-16y^8. **3.** x^2-1. **4.** $(a+b+c)(a-b+c)$.
5. $(x^2-xy+y^2)^2$. **6.** $(x^4-a^2x^2+a^4)(x^2+ax+a^2)$. **7.** $x+y$.
8. $(a^4+x^4)(a^2+x^2)(a-x)$. **9.** $(x+y)^2+2(x+y)z+4z^2$.
10. $x^4+5x^2y^2+y^4$. **11.** $(ax+by+cz)^2-(ax+by+cz)(cx-by+az)$ $+(cx-by+az)^2$. **12.** x^2-ax+a^2. **13.** a^4-b^4.
14. $7a^2+13b^2+21c^2+19ab+24ac+33bc$. **15.** $x^2+5x-14$.
16. $-52x(x-7)$. **17.** $x(x^2+22)(11x-10)$. **19.** $2x^2-xy-3y^2$.
20. $4(6a^2b^2-a^4-b^4)$. **21.** $(3x^2+y^2)(x^2+3y^2)$. **22.** $4(y-z)^2$.
25. $a^3+2a^2b+2ab^2+b^3$. **26.** $64x^4(9x^2-1)$.
27. $7(x-13)$. **28.** $7x^2-10x+3$.

Ex. LX. (pp. 101-102).

1. 4. **2.** 4. **3.** -4. **4.** 3. **5.** -4. **6.** -7.
7. 0. **8.** 6. **9.** -20. **10.** 15. **11.** -18. **12.** 3.
13. 0. **14.** $2\frac{2}{3}$. **15.** $1\frac{8}{9}$. **16.** 8. **17.** 6. **18.** 2.
19. 16. **20.** -10. **21.** 0. **22.** $-\frac{1}{3}$. **23.** $2\frac{1}{2}$. **24.** 6.
25. 1. **26.** -3. **27.** 5. **28.** 2. **29.** $2\frac{4}{7}$. **30.** $2\frac{1}{2}$.
31. $-2\frac{3}{4}$. **32.** $-\cdot 03$. **33.** $\cdot 4$. **34.** $\cdot 01$. **35.** 5.

Ex. LXI. (pp. 104-105).

1. 5. **2.** 2. **3.** 3. **4.** $\frac{4}{7}$. **5.** $-\frac{1}{2}$. **6.** 7.
7. 8. **8.** 9. **9.** 3. **10.** 2. **11.** 1. **12.** 4.
13. 2. **14.** 5. **15.** 2. **16.** 4. **17.** 28. **18.** 4.
19. $-2\frac{1}{5}$. **20.** $3\frac{1}{2}$. **21.** 0. **22.** 7. **23.** -4. **24.** $\frac{5}{8}$.
25. 6. **26.** 3. **27.** -10. **28.** 10. **29.** 1. **30.** 2.
31. 4. **32.** 7. **33.** -2. **34.** 3. **35.** 2. **36.** $2\frac{1}{3}$.
37. -5. **38.** 3. **39.** $5\frac{4}{9}$. **40.** 3.

Ex. LXII. (p. 105).

1. $\frac{d-a}{m-n}$. **2.** $\frac{2a}{5}$. **3.** $\frac{4b}{3}$. **4.** $-\frac{a}{9}$. **5.** $-\frac{2}{3}$.
6. $-\frac{a}{2}$. **7.** $\frac{m^2}{n}$. **8.** $\frac{a^2-b^2}{b-4a}$. **9.** $a-b$. **10.** $\frac{2ab}{a+b}$.

Ex. LXIII. (pp. 107-109).

1. 8. **2.** 4. **3.** 12. **4.** 42. **5.** 12. **6.** 12.
7. 12. **8.** 5. **9.** 7. **10.** 4. **11.** 5. **12.** $\frac{2}{3}$.
13. 7. **14.** 104. **15.** 42. **16.** 6. **17.** 5. **18.** -5.
19. 12. **20.** 2. **21.** $1\frac{1}{17}$. **22.** 3. **23.** 7. **24.** -8.
25. 4. **26.** 7. **27.** $\frac{3}{4}$. **28.** $-15\frac{2}{7}$. **29.** 3. **30.** $\frac{1}{4}$.
31. 5. **32.** -7. **33.** 12. **34.** 7. **35.** 11. **36.** 11.
37. 8. **38.** 5. **39.** 7. **40.** 1. **41.** 13. **42.** 5.
43. 2. **44.** 10. **45.** -5. **46.** 6. **47.** $-2\frac{39}{78}$. **48.** 1.

Ex. LXIV. (pp. 110-111).

1. 18. **2.** 56. **3.** 8. **4.** 4. **5.** 2. **6.** 18.
7. 8. **8.** $-\frac{45}{37}$. **9.** 9. **10.** 6. **11.** 4. **12.** 10.
13. 7. **14.** $10\frac{3}{8}$. **15.** 24. **16.** 19. **17.** $-a$. **18.** $b-a$.
19. $a-m$. **20.** $2(a+c)$.

Ex. LXV. (pp. 111-112).

1. 3. **2.** 1. **3.** 60. **4.** $-\frac{11}{13}$. **5.** 1·95. **6.** 2.
7. 5. **8.** 8. **9.** 10. **10.** 7. **11.** ·4. **12.** 5.

Ex. LXVI. (pp. 112-113).

1. 1. **2.** 2·15. **3.** -29·1. **4.** 4·667. **5.** 5·42. **6.** -8.
7. 47. **8.** 1·89. **9.** 1·86. **10.** 30·7. **11.** 3·10. **12.** -1.

Ex. LXVII. (pp. 113-115).

1. 3. **2.** 5. **3.** 6. **4.** $-\frac{1}{3}$. **5.** $\frac{1}{7}$. **6.** $1\frac{2}{5}$.
7. 14. **8.** 5. **9.** 12. **10.** 3. **11.** 2. **12.** $2\frac{1}{7}$.
13. $p-q$. **14.** $a+k$. **15.** $\frac{1}{13}(25a-18b)$. **16.** $\frac{b(b-a)}{a+3b}$.
17. 5. **18.** $\frac{6}{7}$. **19.** $\frac{6}{7}a$. **20.** 7. **21.** 4. **22.** 5. **23.** 5.
24. 34. **25.** $4\frac{1}{2}$. **26.** $7\frac{1}{2}$. **27.** 4. **28.** $\frac{8}{27}a$. **29.** 4. **30.** 2.
31. 9. **32.** No root. **33.** 7. **34.** 13·86. **35.** -1.
36. 2. **37.** 8. **38.** 11. **39.** $3\frac{3}{4}$. **40.** $\frac{1}{2}$. **41.** 10.
42. $-\frac{14a}{25(a+1)}$. **43.** 9. **44.** 5. **45.** 10.

Ex. LXVIII. (pp. 117-121).

1. $x+9$. **2.** $9+x$. **3.** $x-16$. **4.** $16-x$. **5.** $x-15$.
6. $20-x$. **7.** $a/6$. **8.** $35/x$. **9.** $20x$. **10.** $x-7$.
11. $x-25$. **12.** $x-12$. **13.** $x-y$. **14.** $x+9$. **15.** $75-x-y$.
16. $80/x$. **17.** $3y/x$. **18** $x/9$ pie. **19.** $15-y$. **20.** $y/9$; $16z/9$.
21. $5x$. **22.** a^7. **23.** m^x. **24.** y/x *Rs.* **25.** $a+2b$.
26. $x+5$ years ; $x-10$ years. **27.** $\frac{3}{2}x$. **28.** $x/9$ miles ; $9/x$ hrs.
29. x/y miles. **30.** $\frac{3}{10}x$. **31.** $\frac{13}{12}x$. **32.** $x/12$ as ; $192y/x$.
33. $12x/5$. **34.** 64. **35.** $140-x$. **36.** 7. **37.** xy shillings.
38. $5x+2y$; $\frac{1}{20}(5x+2y)$. **39** $2y$. **40.** $2x+2y$. **41.** $7b$.
42. x/y. **43.** pq miles ; x hours. **44.** $15/4x$.
45. $x, x+1, x+2$. **46.** $x-2, x-1, x$. **47.** $x, x+1, x+2, x+3$.
48. $x-2, x-1, x, x+1, x+2$ **49.** $x, x+1, x+2$. **50.** $2x+1$.
51. $2x-2$. **52.** $2x-2, 2x, 2x+2$. **53.** $240a+12b+c$.
54. $a-2x$ years ; $a-2x-y$ years. **55.** $2x-16$. **56.** $25x/y$.
57. px/y hours. **58.** $\frac{1}{8}x+\frac{1}{40}y$ hours. **59.** $6y$ miles.
60. x^2 sq. ft. **61.** xy. **62** $xy+2xz+2yz$ sq. ft. **63.** $xy/5$.
64. $x-yz$. **65.** $xy+z$. **66.** $(x-z)/y$. **67.** $16x-13\frac{1}{4}$.
68. $392/x$ days ; $392/xy$ days. **69.** $88x/3$. **70.** $m(y-x)+x$.
71. $x=12y+5$. **72.** $2x-y=m$. **73.** $20x+3=y-3$.
74. $x-50=y$. **75.** $\frac{1}{5}(x-7)=\frac{1}{9}(2x+3)$. **76.** $(x-1)x(x+1)=a^2$.
77. $240a+30b+12c=x$. **78** $ab=9x$. **79** $xy=5(a-b)$.
80. $2x+10=p$. **81.** $y-\frac{1}{4}x=20$. **82.** $x+7y=a$.
83. $x-(\frac{1}{4}x+y+600)=a$. **84.** $3ap/100$. **85.** $xy=16b$.

Ex. LXIX. (pp. 125).

1. 35, 13. **2.** 9. **3.** 513, 466. **4.** 31, 18. **5.** 12.
6. 71, 17. **7.** 76, 24. **8.** 18. **9.** A *Rs.*84, B *Rs.*42, C *Rs.*14.
10. 120. **11.** 90, 60. **12.** 16. **13.** *Rs.*3. 8*a.*
14. 37, 30, 20. **15.** 15. **16** 20. **17.** $79\frac{3}{5}$. **18.** 55.
19. 20, 15. **20.** 5, 6. **21.** 88. **22.** *Rs.*100. **23.** 15.
24. 41. **25.** A *Rs.*85, B *Rs.*35. **26.** A 28, B 14.
27. 29, 17. **28.** 2. **29.** £2. 6*s.* 8*d.* **30.** 168, 72.
31. *Rs.Rs.*360, *Rs.*120, *Rs.*160. **32.** £52, £2. 12*s.*
33. A *Rs.*400, B *Rs.*500, C *Rs.*100. **34.** A *Rs.*30, B *Rs.*10.
35. *Rs.*8333. 5*a.* 4*p.*, *Rs.*1666. 10*a.* 8*p.* **36.** 77.
37. *Rs.*450, *Rs.*570, *Rs.*630, *Rs.*650. **38.** 9 years.
39. 20 years. **40.** 14, 11. **41.** 22, 8. **42.** 58, 42.
43. 24 feet. **44.** £2. 15*s.* **45.** 11. **46.** $8\frac{3}{4}$ hours.
47. 8. **48.** *Rs.*189. **49.** A *Rs.*3. B *Rs.*5, C *Rs.*7.

50. A *Rs.*630, B *Rs.* 810. **51.** 96, 70. **52.** 15, 5.
53. $98\frac{2}{3}$ miles from R, $10\frac{2}{3}$ hrs. **54.** 5. **55.** 22, 7, 12 gals.
56. 1000. **57.** 1 hr. 20 min. from B's starting, $6\frac{1}{3}$ miles.
58. 25 lbs. **59.** 9 h.c., 4 fl. **60.** 20 in., 16 in.
61. 36 years; 18 years. **62.** A *Rs.*30, B *Rs.*20, C *Rs.*16.
63. 52, 53, 54. **64.** A $4\frac{1}{2}$ miles, B 3 miles an hour.
65. B 30, C 15, D 10. **66.** 36 miles. **67.** $44\frac{1}{2}$ miles.
68. 128. **69.** 80. **70.** 14 **71.** $\frac{17}{21}$. **72.** *Rs.*550, *Rs.*450.
73. 480 **74.** 60, 12. **75.** $44\frac{4}{19}$ days.

Ex. LXX. (pp. 136-137).

1. $10\frac{1}{2}$ miles. **2.** 25 ft. **3.** 3 miles an hour nearly. **4.** 4 ft.
5. 32 ft. **6.** 4·5 ft. **7.** 10·77 ft. **8.** 28 ft. **9.** 9·2 miles.
10. 3·4 miles. **11.** 4·5 miles. **12.** 39·2 ft. **13.** 6·55 ft.
14. 36 ft. **15.** 23·3 miles. **16.** 62·5 ft. nearly. **17.** 2·4 miles.
18. (1) 8·64. (2) 16·2. (3) 15·98. **19.** (1) 2·5. (2) 3·6. (3) 5·4.
20. 29 miles. **21.** 5 in. nearly. **22.** 42·4 miles.
23. 1 mile; 18·08 miles. **24.** 9·6 miles. **25.** 3·9 miles.

Ex. LXXI. (p. 139).

1. $4a^2b^4$. **2.** $9a^4b^6$. **3.** $16a^4b^6c^8$. **4.** $25x^4y^8$.
5. $\dfrac{9x^4}{16y^6}$. **6.** $\dfrac{9a^6}{16c^4}$. **7.** $\dfrac{25a^4}{49b^4c^2}$. **8.** $\dfrac{9a^4b^4c^6}{25a^4x^2}$.
9. $8a^6b^9c^{12}$. **10.** $-8a^6b^6c^9$. **11.** $-27a^6b^6c^{12}$. **12.** $-125a^6b^9$.
13. $-\dfrac{8x^6}{a^3b^6c^9}$. **14.** $-\dfrac{27x^9}{64y^3z^6}$. **15.** $-\dfrac{b^6c^3d^9}{27a^9}$. **16.** $\dfrac{1}{a^{12}b^9}$.
17. $\dfrac{81a^4b^8}{256c^{12}}$. **18.** $-\dfrac{x^{10}y^{15}z^{20}}{32}$. **19.** $-\dfrac{1}{243a^{10}}$. **20.** $\dfrac{x^{18}}{y^{18}z^{12}}$.
21. $-x^{10}y^{20}$. **22.** $81a^8b^{12}c^{16}$. **23.** $64a^6b^{12}c^{18}$. **24.** $-128a^{14}$.

Ex. LXXII. (p. 141).

1. $x^3+6x^2+12x+8$. **2.** $x^4-8x^3+24x^2-32x+16$.
3. $x^5+15x^4+90x^3+270x^2+405x+243$.
4. $1+10x+40x^2+80x^3+80x^4+32x^5$.
5. $64a^6-192a^5+240a^4-160a^3+60a^2-12a+1$.
6. $81x^4+108x^3+54x^2+12x+1$. **7.** $16x^4-32ax^3+24a^2x^2-8a^3x+a^4$.
8. $243x^5+810ax^4+1080a^2x^3+720a^3x^2+240a^4x+32a^5$.
9. $4096a^6-18432a^5b+34560a^4b^2-34560a^3b^3+$
$19440a^2b^4-5832ab^5+729b^6$.
10. $a^6x^6-6a^5x^5y^2+15a^4x^4y^4-20a^3x^3y^6+15a^2x^2y^8-6axy^{10}+y^{12}$.
11. $a^4x^4+4a^3x^5+6a^2x^6+4ax^7+x^8$.

12. $32a^5b^5 - 80a^4b^6 + 80a^3b^7 - 40a^2b^8 + 10ab^9 - b^{10}$.
13. $a^3 - 3a^2b + 3a^2c + 3ab^2 - 6abc + 3ac^2 - b^3 + 3b^2c - 3bc^2 + c^3$.
14. $a^3 - 3a^2b - 3a^2c + 3ab^2 + 6abc + 3ac^2 - b^3 - 3b^2c - 3bc^2 - c^3$.
15. $1 - 3x + 6x^2 - 7x^3 + 6x^4 - 3x^5 + x^6$.
16. $1 + 3x + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6$.
17. $a^3 + 3a^2bx + 3a(b^2+ac)x^2 + (6ac+b^2)bx^3 + 3(ac+b^2)cx^4 + 3bcx^5 + c^3x^6$.
18. $1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$.
19. $1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6$.
20. $a^3 - 6a^2b + 3a^2c + 12ab^2 - 12abc + 3ac^2 - 8b^3 + 12b^2c - 6bc^2 + c^3$.
21. $1 - 3x + 5x^3 - 3x^5 - x^6$. **22.** $1 + 9x + 33x^2 + 63x^3 + 66x^4 + 36x^5 + 8x^6$.
23. $2(4 + 25x^2 + 16x^4)$. **24.** $1 + 3x^2 + 6x^4 + 7x^6 + 6x^8 + 3x^{10} + x^{12}$.
25. $2(36x + 171x^3 + 144x^5)$. **26.** $1 + 3x^3 + 3x^6 + x^9$.

Ex. LXXIII. (p. 142).

1. $1 - 4ax + 2a^2x^2 + 4a^3x^3 + a^4x^4$. **2.** $4a^4 - 4a^3 - 7a^2 + 4a + 4$.
3. $a^4 - 4a^3b + 10a^2b^2 - 12ab^3 + 9b^4$. **4.** $1 - 2x + 3x^2 - 4x^3 + 3x^4 - 2x^5 + x^6$.
5. $x^6 - 4x^5 + 10x^4 - 4x^3 - 7x^2 + 24x + 16$.
6. $1 + 4x - 2x^2 - 4x^3 + 25x^4 - 24x^5 + 16x^6$.
7. $1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$.
8. $1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6$.
9. $a^8 - 8a^7x + 28a^6x^2 - 56a^5x^3 + 70a^4x^4 - 56a^3x^5 + 28a^2x^6 - 8ax^7 + x^8$.
10. $a^6 - 4a^5b + 8a^4b^2 - 10a^3b^3 + 8a^2b^4 - 4ab^5 + b^6$.
11. $1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8$.
12. $a^8 - 4a^7x + 6a^6x^2 - 8a^5x^3 + 11a^4x^4 - 8a^3x^5 + 6a^2x^6 - 4ax^7 + x^8$.
13. $1 + 8x + 28x^2 + 56x^3 + 70x^4 + 56x^5 + 28x^6 + 8x^7 + x^8$.
14. 46. **15.** -44.

Ex. LXXIV. (p. 143).

1. $\pm 5xy^2z^3$. **2.** $\pm 11a^3b^2$. **3.** $\pm 12a^2b^3c^4$. **4.** $\pm 2ab^2c^3$.
5. $\pm 7x^2y^3z$. **6.** $\pm 10a^4b^6c^8$. **7.** $\pm 3a^2b^2c^3$. **8.** $\pm 4x^4y^3$.
9. $\pm \frac{3ax^2y^3}{5z}$. **10.** $\pm \frac{7xy^2}{8ab}$. **11.** $\pm \frac{5x^3y^5}{4ab^2}$. **12.** $\pm \frac{7a^5b^3c^4}{4x^7y^8}$.
13. $2x^2y^3$. **14.** $-3a^2b$. **15.** $2x^9$. **16.** $-4a^3b^2$.
17. $-\frac{2ay^2}{3x^3}$. **18.** $\frac{4b^2c^3}{5a^4}$. **19.** $-\frac{6abc^5}{7x^2}$. **20.** $\frac{4a^9b}{x^3}$.
21. $\pm axy^2$. **22.** $\pm a^4x^3$. **23.** $2a^3$. **24.** $-x^4y^6$.
25. $\pm \frac{2xy^2}{5a^3}$. **26.** $\pm \frac{3ab^2c^3}{4x^4}$. **27.** $\frac{3a^2b}{x^5}$. **28.** $\frac{2ab^2}{c^3}$.
29. $\pm \frac{2x^2y}{3z^3}$. **30.** $\frac{a^3b^2}{c}$.

Ex. LXXV. (pp. 145-146).

1. $2x+y$. 2. $5a-3b$. 3. $5x^2+3xy$. 4. $7ab-a^2$.
5. $4xy+5yz$. 6. $5a^2bc+c^4$. 7. $1+2x+3x^2$. 8. $3x^2+2x+3$.
9. $2a^2-3a+4$. 10. $3a+2b+c$. 11. $x^2-4xy+4y^2$.
12. $4x^2-$ [illegible] $ab+2b^2$. 13. x^3-2x^2+3x-4. 14. x^3+4x-1.
15. $x^2-ax+2a^2$. 16. $2x^2+2ax+4b^2$. 17. $1-2x+3x^2-4x^3$
18. $3a-b+5c+d$. 19. $x^3-2x^2y+2xy^2-y^3$.
20. $1-3x+3x^2-x^3$. 21. $2x^3-3x^2+x-4$.
22. $x^3-6x^2+12x-8$. 23. $x^2-(y+z)x-yz$.
24. $p+qx+rx^2+sx^3$. 25. $2-3a-a^2+2a^3$.
26. $x^2+(y+z)x+y^2+z^2$.

Ex. LXXVI. (pp. 147-148).

1. $\frac{x}{3}+\frac{3}{x}$. 2. $\frac{2x}{y}+\frac{2y}{x}$. 3. $\frac{5a}{3b}-\frac{3b}{5a}$. 4. $\frac{x^2}{2}-\frac{2x}{3}-\frac{3}{4}$.
5. $3x^2-\frac{1}{3}xy+3y^2$. 6. $1-\frac{1}{2}xy-2x^2y^2$. 7. $\frac{1}{2}x^2-\frac{1}{3}ax+\frac{1}{5}a^2$.
8. $x^2-1+\frac{1}{x^2}$. 9. $3x-4+\frac{1}{2x}$. 10. $x+2+\frac{3}{x}$.
11. $\frac{x}{a}-1+\frac{a}{x}$. 12. $\frac{x^2}{2}-2x+\frac{a}{3}$. 13. $\frac{2x}{3y}-\frac{4x}{5z}-\frac{3y}{4z}$.
14. $\frac{x^2}{y^2}-2+\frac{y^2}{x^2}$. 15. $1-x$. 16. $a-2$ 17. $2a-3b$.
18. $1-x+x^2$. 19. $x-2$. 20. $a-b$. 21. $1-x-\frac{1}{2}x^2-\frac{1}{2}x^3$.
22. $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}$. 23. $a-\frac{b}{2a}-\frac{b^2}{8a^3}-\frac{b^3}{16a^5}$.
24. $1-\frac{1}{2}x-\frac{3}{8}x^2-\frac{5}{16}x^3$.

Ex. LXXVII. (p. 149).

1. $x-\frac{1}{x}-2$. 2. $2-c+\frac{b}{2}$. 3. $\frac{a}{b}+\frac{b}{a}-1$. 4. $\frac{x}{y}-\frac{y}{x}-\frac{1}{2}$.
5. $x^2+\frac{1}{2}yz-z^2$. 6. x^2-2x+1. 7. $ab-ac+bc$.
8. $a-b+c-d$. 9. $a^2-b^2+c^2-d^2$. 10. $ax+by+cz$.
11. a^2+b^2. 12. $(a^2+b^2)(c^2+d^2)$. 13. $3a^2-ab+5b^2$.
14. $a^2+(2b-c)a+c^2$. 15. $\frac{x^2}{y^2}+\frac{y^2}{x^2}-\left(\frac{x}{y}+\frac{y}{x}\right)+1$.

Ex. LXXVIII (p. 151).

1. $x+2y$. 2. $a-3$. 3. $x+4$. 4. $2a-3b$. 5. $a+8b$.
6. $2x-7y$. 7. $m-4nx$. 8. $ax-5bx$. 9. a^2+2a+1.
10. x^2-4x+2. 11. x^2+2x+3. 12. a^2-ab+b^2. 13. $2x^2+4xy-3y^2$.

14. x^3-x^2+x-1. **15.** $x+a+b$. **16.** $\frac{a}{3b}-\frac{2b}{a}$. **17.** $a^2+\frac{1}{a^2}+2$.
18. $a-b+c$. **19.** $1-2x+3x^2-4x^3$. **20.** $1-x^2$.
21. x^2-y^2. **22.** $b(a+b-c)$. **23** $1+2x$. **24.** $x-2a$.

Ex. LXXIX. (p. 153).

1. x^2y. **2.** x^2y^3. **3.** $3x^2$. **4.** $2ab^2$. **5.** $4y^3z^3$. **6.** $5a^2b^3$.
7. $6a^2bc^2$. **8.** $4ax^3$. **9.** $2a$. **10.** $a^2b^3c^4$. **11.** $4a^2bc$.
12. $5a^2$. **13.** $5x^2z^2$. **14.** $6y^2z^3$. **15.** $11xy^2z^2$.

Ex. LXXX. (pp. 153-154).

1. $3xy$. **2.** ax. **3.** a. **4.** a. **5.** $a(2-b)$. **6.** $a-b$.
7. $a+x$. **8.** $2x^2(a+x)^2$. **9.** $x^2(a+x)^2$. **10.** $ab(a-b)^2$.
11. $2(x-1)$. **12.** $x^2(x+1)$. **13.** $2(x+a)$. **14.** x.
15. $x-3$. **16.** $x+3y$. **17.** $a+x$. **18.** $4(x^2+y^2)$. **19.** $a^2(x+1)$.
20. $3(ax+2)$. **21.** $a+4$. **22.** $x+y$. **23.** $x-y$. **24.** $a-3$.
25. $a-1$. **26.** $a+b-c$.

Ex. LXXXI. (pp. 157-159).

1. $3x-2$. **2.** $2x+3$. **3.** $3x+5$. **4.** $2x+3$.
5. $3x-2$. **6.** $3x-2$. **7.** $8x^2+14x-15$. **8.** $4x-5$.
9. $2(x^2+2x+1)$. **10.** $y-2$. **11.** $x-2a$. **12.** $x+3$.
13. $x-1$. **14.** $x-3$. **15.** $x-y$. **16.** $x+3$.
17. $x+2$. **18.** $3(x+3)$. **19.** x^2+y^2. **20.** $a(a+b)$.
21. x^2-3. **22.** $x+3$. **23.** $x-8$. **24.** $x-a$.
25. $x+4$. **26.** $x+5$. **27.** $x-1$. **28.** $x-3$.
29. $x-2$. **30.** $3x-5$. **31.** x^2+2x-3. **32.** $a(a^3-b^2)$.
33. $x^2-2xy+y^2$. **34.** x^2+4x+4. **35.** $5x^2-1$. **36.** $x-4a$.
37. $2x+5$. **38.** $2(x^2+ax-2a^2)$. **39.** $3x-11$. **40.** x^2-2x+1.
41. x^3-x+1. **42.** $2x^2+3x-2$. **43** $x-4$. **44.** $2x-1$.
45. $x-2a$. **46.** x^2-x+1. **47.** $2x-y$. **48.** $x(x^2+3x+11)$.
49. x^2-3x+4. **50.** $x+\frac{1}{2}$. **51.** $2x^2-x-2$. **52.** x^2-4x+3.
53. $4a^2-3ab+b^2$. **54.** $x^3-5x^2+13x-14$. **55.** $2x^2-3$.
56. $x-1$. **57** $3x^2-2xy+y^2$. **58.** $x(2x^2+2xy-y^2)$.
59. $x^2-2xy+3y^2$. **60.** $x^2-2ax+a^2$. **61.** x^2+2x+3.
62. x^2-4x+1. **63.** x^2+x+41. **64.** $a=6$; $x-2$.
65. $y=5$.

Ex. LXXXII. (p. 159).

1. x^2-2x+5. **2.** x^2+1. **3.** $5x^2-1$. **4.** $x+4$.
5. $x-2$. **6.** $4x+1$. **7.** $x-1$. **8.** x^2+x-3.

Ex. LXXXIII. (p. 164).

1. $12a^2b^2c$. **2.** $36x^3y^3$. **3.** $24a^2b^2x^3y^3$. **4.** $20a^2x^2y^2z^2$.
5. $a^4b^3c^3$. **6.** $12a^2b^2c^2$. **7.** $6a^2b^2cx^2y$. **8.** $120a^4b^2$.
9. $10a^5b^5$. **10.** $1800a^3x^3$. **11.** $924a^5b^6c^8x^7y^2$. **12.** $1001a^8b^7c^8x^9$.

Ex. LXXXIV. (p. 165).

1. $axy(x-y)$. **2.** $a(b^2-d^2)$. **3.** $6(a^2-b^2)$. **4.** $12a(a^2-1)$.
5. a^3+b^3. **6.** $(x-3)(x+4)(x-5)$. **7.** $(x+1)(x+3)(x-4)$.
8. a^4-1. **9.** $120xy(x^2-y^2)$. **10.** $24a^2b^2(a^2-b^2)$.
11. $36xy^2(x^2-y^2)$. **12.** $(x^2-25)(x^2-36)$. **13.** $ab(a^2-b^2)(a^2-4b^2)$.
14. $(x-3)(x-5)(x-7)$. **15.** $(a^3+1)(a+4)$. **16.** $xy(y^2-x^2)$.
17. $(a^2-x^2)(a^2-4x^2)$. **18.** $(x^2-9)(x^2+3x+9)$.
19. $72a^2b^3(a^2-b^2)(a^3-b^3)$. **20.** $72(a^2-b^2)(a^6-b^6)$. **21.** $30x^2(x^2-1)^2$.
22. $2x(x-2)^2(x+2)(x^2+2)$. **23.** $(x^6-a^6)(x^2+a^2)$. **24.** $(x^4-1)(x^6-1)$.
25. $(3x+1)(2x-1)(x+2)$. **26.** $(a+b)(a-b)^2(a+5b)(a+6b)$.
27. x^4-y^4. **28.** $(1+2x+4x^2)(1+2x-4x^2)(1-4x^2)$.
29. $(9x^2-1)(x^2-3)^2(9x^4-1)$. **30.** $(x-1)^2(x+1)$.
31. x^4-16a^4. **32.** $(x-a)(x-b)(x+3a+b)$.

Ex. LXXXV. (pp. 167-168).

1. $(x^2+1)(4x-1)(3x-1)$. **2.** $(x+1)(x+3)^2(x+4)(x+5)$.
3. $(a^2-4b^2)(a^2-b^2)$. **4.** $(x-a)^2(x-3a)(3x-7a)$.
5. $(x+2)(2x-1)(3x-1)(4x^2-3x+1)$.
6. $(x^2+3x+2)(3x^2+8x-3)(2x^2-3x+1)$.
7. $(x^2-4)(x^2+x-2)(x^3-x+1)$. **8.** $(x-1)(x-2)(x-3)(x-4)$.
9. $(x^2-1)(x^4+7x^2+16)$. **10.** $(x-8)^2(x+9)(9x^2-100)$.
11. $(x^2+3x-4)(x+4)(x-3)(3x+4)$.
12. $(x-1)(x+4)(x-5)(x+3)(x+6)(x$ [illegible]$)$.
13. $(x^2+x-3)^2(x^2-x+3)(x^3-x-3)$. **14.** x^3+2x^2-3.
15. a^3-2a^2+2a-1.

Ex. LXXXVI. (pp. 170-171).

1. 2. **2.** 30. **3.** 53. **4.** -6. **5.** -3439. **6.** -2.
7. -4. **8.** -2. **9.** 35. **10.** 23. **11.** 144. **12.** -6.
19. 23. **20.** 4. **21.** 14 or -13. **25.** $b+c+1=0$.
27. $(p+q)^2(p+q+1)=a$.

REVISION PAPERS II.

Paper I.

1. $(y^2-4)^2$. **2.** (i) $(2x+9y^3)(4x^2-18xy^3+81y^6)$.
(ii) $(x^2-6x+18)(x^2+6x+18)$. (iii) $(7x-9y)(8x+11y)$.

3. $-8abc$. **5.** $2(x+4)$. **6.** $-9(x^2+19x-5)$.
7. $2x^3-4x^2+x-1$. **8.** $x^{2m+1}+3x^m-5y^{m-2}$.
9. (i) $x=2$. (ii) $x=8$. **10.** 24.

Paper II.

1. $(bx+a)(x^2-x+1)$. **2.** $x^4+5(a-b)x^3+(6a^2-25ab+6b^2)x^2-30ab(a-b)x+36a^2b^2$. **3.** x^2-2x+2. **4.** $3a^2-ab+5b^2$.
5. (i) $x=7$. (ii) $x=4$. **6.** $6xyz$. **7.** $(x^3-a^3)(x^3+a^3)^2$.
8. $(3x-2)(2x-3)$. **10.** 1800.

Paper III.

1 x^4-x^2-1. **2.** (i) $(2x^2-1)(2x^2+1)(2x^2+2x+1)(2x^2-2x+1)$. (ii) $x(x+2)^2$. (iii) $(x-2)(2x-1)(2x^2-5x+5)$. **3.** $2x^2+2x+3$.
4. x^6-a^6. **5.** $x^3-ax^2+2a^2x+a^3$. **7.** $3x^2-2(a+b)x+a^2+b^2$.
8. $4(a+b)^2x^2$. **9.** (i) $x=5$. (ii) $x=\dfrac{2ab}{a+b}$. **10.** £4680 ; £4720.

Paper IV.

1. $12abc$. **2.** $1+(a+b)x+\frac{1}{2}\{a(a-1)+b(b-1)+2ab\}x^2+\frac{1}{6}\{a(a-1)(a-2)+b(b-1)(b-2)+3ab(a+b-2)\}x^3$; $\frac{1}{2}(a+b)(a+b-1)$. **3.** $5x$. **4.** (i) $(x^2+3xy-y^2)(x^2-3xy-y^2)$. (ii) $(a-b+c+d)(a+b+c-d)(a+b+c+d)(b+c+d-a)$.
5. $abc(a+b+c)(b+c-a)(c+a-b)(a+b-c)$. **6.** $16a^4$.
8. (i) $4x^2+16x+11$. (ii) $x^3+x+2-\dfrac{2}{x^3}$. **9.** $x=2$.
10. $\{d-(a+b)t\}$miles ; $\dfrac{d}{a+b}$ hours.

Paper V.

1. $x^3+x^2y+xy^2+y^3$; $y-x$. **2.** $8a^3$. **3.** $x-1$; 1.
4. $(2x+3y+z)(2x-3y-z)(2x+3y-z)$.
5. (i) $(2x+3y)(2x-3y-3)$. (ii) $(x^2+3xy+4y^2)(x^2-3xy+4y^2)$. (iii) $(x^2+2x-3)(x^2-2x-3)$. (iv) $(8x-1)^3(1-a)(1+a+a^2)$.
6. (i) $ax^3+3bx-c$. (ii) $\dfrac{a^3}{3}+\dfrac{3b^3}{a^3}-\dfrac{5}{b^3}$. **7.** (i) $\dfrac{a}{b}-1-\dfrac{b}{a}$. (ii) $x-4+\dfrac{2}{x}$. **8.** 4. **10.** 234.

Paper VI.

1. $(x^2+y^2)^2=(x^2-y^2)^2+(2xy)^2$. (a) 16^2+30^2. **2.** (i) $(x-11)(x+17)$. (ii) $(x+1)(x+2)(x-1)(x-2)$. (iii) $\{(a-c)^2+b(a+c)\}(a-c)(a+b+c)$.

3. $(x-3)(x-4)(2x-1)^2$. 4. $(a+1)x^2+(a^2+1)x+a^3$; $194\frac{1}{8}$.
5. (i) $a^2+5ab-3b^2$. (ii) $a(x^3+1)+b(x^2+x)$. 6. $6x+3$.
7. $(ax+by)^2+(ay-bx)^2$. 8. x^2-5x+8. 9. $x=15$. 10. 42.

Paper VII.

1. $y^4+11y^3+47y^2+93y+69$. 2. a^2-ab+b^2. 3. $2x^2+5x-3$.
4. (i) $ax+2b+\frac{3c}{x}$. (ii) $(a+b)^2+3c(a+b)+c^2$.
5. (i) $x^2+\frac{3}{x^2}-2$. (ii) $2+3x-x^3$. 6. (i) $(3x-5)(4x+7)$.
(ii) $(2x-3)(4x+9)$. (iii) $(9a^2+12ab+8b^2)(9a^2-12ab+8b^2)$.
(iv) $(x-a)^2(x+2a)$. 7. $(x+y)^4+z^4$. 8. 0.
9. $x=\frac{5}{6}$. 10. 60.

Paper VIII.

1. 0. 2. $3x-1$; $\frac{1}{3}$. 3. (i) $x^3-\frac{1}{4}x+\frac{1}{3}$. (ii) $\frac{2x}{3a}+\frac{3a}{2x}+\frac{3ax}{b^2}$.
4. G. C. M. $=x+3$; L. C. M. $=(3x^3+8x^2+3x-2)(6x^4+7x^3-27x^2+17x-3)$
5. $2x^3-x^2-3$. 6. 36·1 ft. 7. $a^4-4a^2bc+7b^2c^2$.
9. (i) $x=\frac{2}{3}$. (ii) $x=-23$. 10. 1.

Ex. LXXXVII. (pp. 181-182).

1. $\frac{3}{5ab^2c}$. 2. $\frac{5a}{7bcd}$. 3. $\frac{11xz^4}{7a^6y^2}$. 4. $\frac{6a^2x}{7y}$. 5. $\frac{4b^2c^6}{7a}$.
6. $\frac{7c^2x^2}{11ab^3y^2}$. 7. $\frac{a+y}{a}$. 8. $\frac{x}{a}$. 9. $\frac{m}{3(m-2x)}$. 10. $\frac{7x}{5a}$.
11. $\frac{a^2-3ab}{2b(a+2b)}$. 12. $\frac{x^3-3y^2}{y(x-2y)}$. 13. $\frac{2mn}{m+n}$. 14. $\frac{3abc}{a+b+c}$.
15. $\frac{3xy-5y^2}{4x-7y}$. 16. $\frac{c}{2df}$. 17. $\frac{c+y}{f+2x}$. 18. $\frac{x-1}{a}$.
19. $\frac{x^2+a^2}{x^3}$. 20. $\frac{a^4+a^2b^2+b^4}{a^2+b^2}$. 21. $\frac{x^2-bx}{x+b}$. 22. $\frac{x^2+1}{x^4+x^2+1}$.
23. $\frac{a-b}{a+b}$. 24. $\frac{x+b}{x-c}$. 25. $\frac{a+b-c}{a-b-c}$. 26. $\frac{x-1}{x+1}$. 27. $\frac{x-1}{x+2}$.
28. $\frac{a+b}{a-b}$. 29. $\frac{3a-2x}{5a+3x}$. 30. $\frac{2a-3x}{2a+3x}$. 31. $\frac{2a+b-c}{2a-b-c}$.
32. $\frac{cx+d}{ax+b}$. 33. $\frac{x-5}{2x+3}$. 34. $\frac{a^2-3a+9}{3}$. 35. $\frac{x-1}{x+1}$.

36. $\dfrac{3x-2y}{(2x^2+y^2)(x+2y)}$. **37.** $\dfrac{a+b-c-d}{a-b+c-d}$. **38.** $\dfrac{x^2+3a}{x^2-3a}$. **39.** $\dfrac{x+y-1}{x+y+1}$.

Ex. LXXXVIII. (p. 183).

1. $\dfrac{x+4}{x^2-2x+1}$. **2.** $\dfrac{7x-2y}{5x^2-3xy+2y^2}$. **3.** $\dfrac{x^2+x-2}{x^2+5x+5}$.

4. $\dfrac{5a^3(a+x)}{x(a^2+ax+x^2)}$. **5.** $\dfrac{x^2-ax+a^2}{x^2-a^2}$. **6.** $\dfrac{x^2+4x+4}{x^2+x+1}$.

7. $\dfrac{3ax^2+1}{4a^2x^4+2ax^2-1}$. **8.** $\dfrac{3x^2+x}{4x^2+2x+1}$. **9.** $\dfrac{(x-1)^2}{2x^3-4x^2+2x-3}$.

10. $\dfrac{x-5}{x+5}$. **11.** $\dfrac{2x^2+3ax+7a^2}{x^2-6ax+2a^2}$. **12.** $\dfrac{2x+3}{5x-2}$. **13.** $\dfrac{x-1}{x+1}$.

14. $\dfrac{3(x^2-7ax+12a^2)}{2(x^2+7ax+12a^2)}$. **15.** $\dfrac{2x^2+3x-5}{7x-5}$. **16.** $\dfrac{x^2+2x+15}{x-7}$.

17. $\dfrac{x^2+x-12}{x^2-x-12}$. **18.** $\dfrac{x-2}{x+4}$. **19.** $\dfrac{a^3-2a^2+a+4}{a^3-2a^2+12a-18}$.

Ex. LXXXIX. (p. 184).

1. $3x+\dfrac{5x}{7}$. **2.** $2a+\dfrac{5b}{4a}$. **3.** $a-\dfrac{b^2}{a}$. **4.** $3x-6+\dfrac{29}{x+4}$.

5. $a-2x+\dfrac{3x^2}{a+x}$. **6.** $2x+6+\dfrac{23}{x-3}$. **7.** $2a-3x+\dfrac{7x^2}{5a-x}$.

8. $12x+3+\dfrac{19}{4x-1}$. **9.** $x-1-\dfrac{2x-1}{x^2-x+1}$. **10.** $x+3\cdot\dfrac{x-2}{x^2-3x+4}$

11. $\dfrac{6xy+2}{3y}$. **12.** $\dfrac{2a}{a+b}$. **13.** $\dfrac{x^3-2x^2-3x}{x-2}$. **14.** $\dfrac{x^2-10x+30}{x-3}$

15. $\dfrac{a^3+2x^3}{a+2x}$. **16.** $\dfrac{x^2+xy+y^2}{x+a}$. **17.** $\dfrac{x^3-y^3}{x^2-xy+y^2}$.

Ex. XC. (p. 186).

1. $\dfrac{bcx,\ acy,\ abz}{abc}$. **2.** $\dfrac{6cx^2,\ 4by^2,\ 3az^2}{12abc}$.

3. $\dfrac{9x,\ 5ab,\ 45c}{15ax}$. **4.** $\dfrac{40b^3x^2y,\ 45ab^2x^3,\ 48a^2by^3,\ 50a^3xy^2}{60a^3b^3}$.

5. $\dfrac{a^2+2ax+x^2,\ a^2-2ax+x^2}{a^2-x^2}$. **6.** $\dfrac{6ax-2bx,\ 20a-4b,\ 2ax^2-bx^2}{8x^3}$.

7. $\dfrac{a^2x^2-b^2x^2,\ a^2y^2+b^2y^2}{a^4-b^4}$. **8.** $\dfrac{8ax^2-8bx^2,\ xy}{6(a^2-b^2)}$. **9.** $\dfrac{a-x,\ a+x,\ 2a}{4a^3(a^2-x^2)}$.

10. $\dfrac{x^3+x^2-x-1,\ ax(x+1)^2,\ 3a(x^3-x^2-x+1),\ 4b(x-1)^2, 5(x^2-1)}{(x^2-1)^2}$.

Ex. XCI. (pp. 188-189).

1. $\dfrac{13x}{12}$. 2. $\dfrac{13a}{12x}$. 3. $\dfrac{8ayz+5bxz+9cxy}{12xyz}$. 4. $\dfrac{7-2x}{12}$

5. $\dfrac{x+7}{30x}$. 6. $\dfrac{2x^3+x^2-2}{8x^2}$. 7. $\dfrac{71a-51b}{50a}$. 8. $\dfrac{25a-20b}{12}$

9. $\dfrac{25x-20y}{12}$. 10. $-\dfrac{3}{4a}$ 11. $\dfrac{5a^6b-4a^4b+3a^3b^2+5a^3b^3}{a^4b^2}$.

12. $\dfrac{a^2+b^2}{2(a+b)b}$. 13. $\dfrac{3a^2-ab+2b^2}{6(a-b)b}$. 14. $\dfrac{a^2+b^2}{a^2-b^2}$. 15. $\dfrac{a^2+b^2}{a^2-b^2}$.

16. $\dfrac{ab}{a-b}$. 17. $\dfrac{2a^2-2ab+2b^2}{a^2-b^2}$. 18. $\dfrac{2ab}{a^2-b^2}$. 19. $\dfrac{a^2-ab+b^2}{a^2-b^2}$.

20. $\dfrac{x-y}{x}$. 21. $\dfrac{4}{a^2-10a+21}$. 22. $\dfrac{20x}{1-25x^2}$. 23. $\dfrac{a+bx}{c+dx}$.

24. $\dfrac{a^2+x^2}{a^2(a+x)}$. 25. $\dfrac{a^2+x^2}{a^2(x-a)}$. 26. $\dfrac{8x-2}{x(x-1)(x+2)}$. 27. $\dfrac{2}{x(x^2-1)}$.

28. $\dfrac{2a^4}{a^4-x^4}$. 29. o. 30. $\dfrac{1}{x^2(x^2-1)}$. 31. $\dfrac{a}{4a^2-b^2}$. 32. $\dfrac{1+x^3+x^6}{x^2(x^2+1)^2}$.

33. $\dfrac{2x}{x+y}$. 34. $\dfrac{a+bx}{b+ax}$. 35. $\dfrac{2x^4+4x^2y^2-2y^4}{x^4-y^4}$. 36. $\dfrac{x-3x^2+3x^3}{(1-x)^3}$.

37. $\dfrac{4a}{a+x}$. 38. $\dfrac{17a}{a^2-16}$. 39. 1. 40. $\dfrac{3a^2-ab}{a^3+b^3}$. 41. $\dfrac{18}{(x-1)(x+2)(x+5)}$.

Ex. XCII. (pp. 191-193).

1. $\dfrac{16x^2-5x-13}{(3x+2)^3}$. 2. $\dfrac{2y^2}{x^3-y^3}$. 3. $\dfrac{6}{(x+4)(x^2-9)}$. 4. $\dfrac{x^2+2x+4}{x(x^3+1)}$.

5. o. 6. o. 7. $\dfrac{x}{(x-2a)^2}$. 8. $\dfrac{b}{27a^3+b^3}$. 9. $\dfrac{14}{x^2-49}$.

10. $3a-5b$. 11. $\dfrac{2a^3}{a^4-x^4}$. 12. $\dfrac{4x^3y-x^2y^2-y^4}{x^4-y^4}$. 13. $\dfrac{8(a+6)}{a^4-16}$.

14. $\dfrac{1+2x+3x^2}{4(1-x^4)}$. 15. $\dfrac{16x}{1-x^4}$. 16. $\dfrac{4a^2}{a^2-b^2}$. 17. $\dfrac{x^2-ax-13a^2}{x^2-ax-12a^2}$

18. $\dfrac{3a^2}{a^2-1}$ 19. o. 20. $\dfrac{1}{x^2-1}$. 21. $\dfrac{1}{(x-1)(x-2)(x-3)}$.

22. $-\dfrac{4}{(x-1)(x-3)(x-5)}$. 23. $\dfrac{y^2-ax}{y^2-a^2}$. 24. $\dfrac{x^3+2x^2+2x-1}{x^4-1}$.

25. $\dfrac{17x}{(x-3)(x+4)(x-7)}$. **26.** $\dfrac{b^2}{(a+b)(a^2+b^2)}$. **27.** 0. **28.** $\dfrac{3}{a+c}$.

29. $\dfrac{x^2+y^2}{x^2-y^2}$. **30.** $\dfrac{2(x^2+1)}{x(x^2-1)}$. **31.** 1. **32.** $\dfrac{a^2+b^2}{a^2-b^2}$. **33.** $\dfrac{2a}{a+b}$. **34.** 2.

35. $\dfrac{x+c}{(a-x)(x-b)}$. **36.** $\dfrac{1}{x+y}$. **37.** $\dfrac{1}{(x-1)(x-2)(x-3)}$.

38. $\dfrac{2(a+x)}{a^2+ax+x^2}$. **39.** $\dfrac{1}{a^4-x^4}$. **40.** $\dfrac{2x^3}{x^4-1}$. **41.** $\dfrac{a^3}{(a-x)(a^3+x^3)}$.

42. $\dfrac{4x^4+8}{1+x^4+x^8}$. **43.** $-\dfrac{16x^7}{1-x^{16}}$. **44.** 1. **45.** 1. **46.** 1.

47. $\dfrac{a+b+c}{(b+c-a)(c+a-b)(a+b-c)}$. **48.** $\dfrac{24b^4}{a(a^2-b^2)(a^2-4b^2)}$.

49. $\dfrac{3}{x^2+5x+4}$. **50.** $-\dfrac{3b}{(2a-3b)(a-4b)}$.

Ex. XCIII. (pp. 194-195).

1. $\dfrac{c}{9b}$. **2.** $\dfrac{2b^2}{5c}$. **3.** $\dfrac{25ax}{28by}$. **4.** $9ax$. **5.** $\dfrac{a}{12b^2}$. **6.** $\dfrac{x(a+x)}{b(a-x)}$.

7. $\dfrac{abc}{xyz}$. **8.** $\dfrac{a^2-5a+6}{a^2}$. **9.** $\dfrac{2ax^2(x-y)}{c}$. **10.** $\dfrac{a^2+b^2}{a}$.

11. $\dfrac{ab}{a^2+4b^2}$. **12.** $\dfrac{x}{x^2+y^2}$. **13.** x^2+a^2. **14.** $\dfrac{a-6}{a-3}$.

15. $\dfrac{a+4}{a+5}$. **16.** $\dfrac{x+5}{x-5}$. **17.** $\dfrac{y}{x-y}$. **18.** $\dfrac{x(x+3a)}{a(x+2a)}$.

Ex. XCIV. (pp. 195-196).

1. $\dfrac{40y}{49z}$. **2.** $\dfrac{3a^4b^3}{4x^3y}$. **3.** $\dfrac{a^2+b^2}{(a-b)^2}$. **4.** 1. **5.** $\dfrac{a(a^2+b^2)(a+b)^2}{(a-b)^2}$.

6. $b(a+b)$. **7.** $\dfrac{x-6}{x-3}$. **8.** $\dfrac{2xy^2}{3}$. **9.** $\dfrac{x^2-y^2}{xy}$. **10.** $\dfrac{3a}{2b}$. **11.** $\dfrac{x-4}{x+4}$.

12. $\dfrac{a+b-c}{a-b+c}$. **13.** x. **14.** $\dfrac{x^2+b^2}{x-b}$. **15.** $\dfrac{x^2-ax+a^2}{x^2+ax+a^2}$.

Ex. XCV. (p. 197).

1. $\dfrac{a^4-x^4}{a^2x}$. **2.** $a-1$. **3.** $\dfrac{y}{x}$. **4.** $\dfrac{3a^2(a-b)}{b}$. **5.** $\dfrac{a^4x(ax-1)}{a-b}$.

6. $\dfrac{x-1}{x^2}$. **7.** $\dfrac{(a^2-b^2)b}{a^3}$. **8.** $\dfrac{1}{x^2+y^2}$. **9.** 1. **10.** $\dfrac{a^2-x^2}{a}$. **11.** $\dfrac{1}{x+y}$.

12. $\frac{x-3}{x-4}$. **13.** $x^2+1+\frac{1}{x^2}$. **14.** $\frac{x^4}{a^4}+\frac{x^2}{a^2}+1$. **15.** $\frac{x^3}{y^3}-\frac{y^3}{x^3}$.

16. $x^2+1+\frac{1}{x^2}$. **17.** $\frac{a^2}{b^2}+\frac{b^2}{a^2}-1$. **18.** $\frac{a}{b}-1$. **19.** $\left(\frac{y}{x}-\frac{x}{y}\right)^3$.

20. $x^4+x^3y-xy^3-y^4-\frac{y^6}{x}+\frac{y^7}{x^3}+\frac{y^8}{x^4}$.

Ex. XCVI. (pp. 199-200).

1. $\frac{4-3x}{10}$. **2.** $\frac{3x}{15-2x}$. **3.** $\frac{6-2x}{2x+5}$. **4.** $\frac{18x+14}{21}$. **5.** $\frac{6x}{3x-1}$.

6. $\frac{27-4x}{2(4x-9)}$. **7.** $\frac{12x-40}{33-2x}$. **8.** $\frac{10-13x}{6}$. **9.** $x-1$. **10.** $\frac{20-3x}{2x-25}$.

11. $\frac{14-20x}{9(x+1)}$. **12.** $\frac{1+a}{1+a^2}$. **13.** $\frac{1}{x}$. **14.** 1. **15.** $\frac{b^2}{a^2}$. **16.** $\frac{a^2+x^2}{2ax}$.

17. y. **18.** $\frac{a-4}{a-5}$. **19.** $\frac{6+7x-x^3}{3-3x^2+x^3}$. **20.** $\frac{4}{3x}$. **21.** $\frac{x(1+x+x^3)}{1+x^2}$

22. x. **23.** 1. **24.** $\frac{4}{3a}$. **25.** $\frac{4a^4}{a^4-x^4}$.

Ex. XCVII. (pp. 201-202).

1. $\frac{x^3}{x^3+a^3}$. **2.** $\frac{x(1-x)}{2+x}$. **3.** $\frac{x+1}{(x-1)(2x-1)}$. **4.** $-\frac{a^4+a^2b^2+b^4}{ab(a-b)^2}$.

5. $\frac{(a+b+c)^2}{2bc}$. **6.** $\frac{(m-n)(m-s)}{2m-n-s}$. **7.** $\frac{1}{1-x^2}$. **8.** $\frac{-4xy(x^2+y^2)}{x^4+x^2y^2+y^4}$.

9. a^2. **10.** $\frac{3}{5}$. **11.** $\frac{b}{a}$. **12.** $\frac{4a^2x}{x^4-a^4}$. **13.** $\frac{xy}{x^2+y^2}$.

14. $\frac{(x-4)^2}{(x-7)(3x-5)}$. **15.** $\frac{5(a+x)}{(2a-x)^2}$.

Ex. XCVIII. (p. 208).

1. 1 ; 3. **2.** 19 ; 18. **3.** 1 ; 1. **4.** 5 ; 2. **5.** 1 ; -1.
6. 1 ; 2. **7.** 6 ; 7. **8.** 8 ; 2. **9.** 7 ; 17. **10.** 5 ; 6.
11. 14 ; 15. **12.** $\frac{1}{2}$; $1\frac{1}{2}$. **13.** 2 ; 1. **14.** 12 ; 3. **15.** 5 ; 1.
16. 2 ; 3. **17.** 7 ; 5. **18.** $3\frac{1}{2}$; $2\frac{1}{3}$. **19.** $\frac{1}{2}$; $-\frac{1}{2}$. **20.** 2 ; -1.
21. 2 ; 1. **22.** 13 ; $9\frac{1}{3}$. **23.** 3 ; 6. **24.** $1\frac{1}{3}$; $2\frac{1}{2}$. **25.** 4 ; 5.
26. 1 ; 7. **27.** 3 ; 2. **28.** 7 ; 10.

Ex. XCIX. (pp. 209-211).

1. 10 ; 24. **2.** $\frac{2}{3}$; $-\frac{3}{4}$. **3.** 2 ; 3. **4.** 7 ; 2. **5.** 7 ; 9.
6. 5 ; 2. **7.** $\frac{3}{14}$; $\frac{5}{7}$. **8.** 18 ; 48. **9.** $2\frac{1}{6}$; $3\frac{5}{9}$.

10. 6 ; 8. **11.** 8 ; 5. **12.** 17 ; 11. **13.** 40 ; 60. **14.** 6 ; 4.
15. 5 ; 9. **16.** 3 ; 2. **17.** 3 ; 2. **18.** 5 ; 5. **19.** 12 ; 6.
20. 11 ; 7. **21.** -2 ; $-\frac{1}{2}$. **22.** 7 ; 8. **23.** 7 ; 9. **24.** ·02 ; 2·9.
25. 10 ; 8. **26.** 8 ; -15. **27.** $-2·5$; $-3·5$. **28.** 1·95 ; ·675.

Ex. C. (pp. 212-213).

1. 3 ; 6. **2.** -1 ; $-\frac{1}{2}$. **3.** $\frac{2}{5}$; $-\frac{3}{5}$. **4.** $\frac{1}{15}$; 18.
5. 4 ; 10. **6.** 2 ; 3. **7.** 3 ; 4. **8.** $\frac{1}{5}$; $\frac{1}{7}$. **9.** $\frac{1}{2}$; $\frac{1}{3}$.
10. $3\frac{1}{2}$; $-2\frac{1}{2}$. **11.** 3 ; 2. **12.** 7 ; 4. **13.** 5 ; 11.
14. -2 ; 5. **15.** 5 ; 2. **16.** 7 ; 3. **17.** $\frac{3}{2}$; $\frac{1}{2}$.
18. 10 ; 15. **19.** 144 ; 216. **20.** 14 ; 9. **21.** 6 ; 9.

Ex. CI. (pp. 213-215).

1. $-b$; $a+b$. **2.** $\dfrac{a-b^2}{1-ab}$; $\dfrac{b-a^2}{1-ab}$. **3.** 1 ; 0.

4. $\dfrac{bc}{a+b}$; $\dfrac{ac}{a+b}$. **5.** $\dfrac{bc_1-b_1c}{ab_1-a_1b}$; $\dfrac{ac_1-a_1c}{a_1b-ab_1}$. **6.** $\dfrac{c(c-b)}{a(a-b)}$; $\dfrac{c(c-a)}{b(b-a)}$.

7. $\dfrac{cq-md}{nq-mp}$; $\dfrac{cp-dn}{mp-nq}$. **8.** a ; b. **9.** $\dfrac{ac+bd}{a^2+b^2}$; $\dfrac{bc-ad}{a^2+b^2}$.

10. $\dfrac{ac(dn+bm)}{ad+bc}$; $\dfrac{bd(cn-am)}{ad+bc}$. **11.** $\dfrac{bc^2}{a^2+c^2}$; $\dfrac{a^2c}{a^2+c^2}$.

12. $\dfrac{ab(a+b)}{a^2+b^2}$; $\dfrac{ab(a-b)}{a^2+b^2}$. **13.** a ; b. **14.** $\dfrac{mp-nq}{ap}$; $\dfrac{mp-nq}{aq}$.

15. $\dfrac{12abm}{a+b}$; $\dfrac{m(7b-5a)(a-b)}{a+b}$. **16.** $\dfrac{b}{a(b-a)}$; $\dfrac{a}{b(a-b)}$.

17. $\dfrac{pr}{p^2-q^2}$; $\dfrac{qr}{q^2-p^2}$. **18.** $\dfrac{b^2+c^2-a^2}{bm-an-cn}$; $\dfrac{b^2+c^2-a^2}{bn+cm-am}$.

19. $\frac{1}{2}$; $\frac{1}{3}$. **20.** $a+b$; $a-b$. **21.** $\frac{1}{2}\left(\dfrac{a}{b}+\dfrac{b}{a}\right)$; $\frac{1}{2}\left(\dfrac{b}{a}-\dfrac{a}{b}\right)+1$.

22. $\dfrac{b^2+c^2-a^2}{2a}$; $\dfrac{a^2+c^2-b^2}{2b}$. **23.** c ; b. **24.** $\dfrac{am-bn}{a^2-b^2}$; $\dfrac{an-bm}{a^2-b^2}$.

25. $\dfrac{m^2-n^2}{am-bn}$; $\dfrac{m^2-n^2}{bm-an}$. **26.** $\dfrac{1}{a}$; $\dfrac{1}{b}$.

27. $\dfrac{abc(bc-ab-ac)}{b^2c^2-c^2a^2-a^2b^2}$; $\dfrac{abc(ab+bc-ac)}{b^2c^2-a^2b^2-a^2c^2}$. **28.** $\dfrac{2a+b}{2}$; $\dfrac{2a-b}{2}$.

29. $(a+b)^2$; $(a-b)^2$. **30.** $\dfrac{b+c-a-d}{4(bc-ad)}$; $\dfrac{c+d-a-b}{2(bc-ad)}$.

Ex. CII. (pp. 216-218).

1. 3 ; 2 ; 1. **2.** 1 ; 2 ; 3. **3.** 4 ; 5 ; 6. **4.** 10 ; 20 ; 5.
5. 7 ; 10 ; 9. **6.** 5 ; 6 ; 7. **7.** 6 ; 11 ; 6. **8.** -28 ; 10 ; 9.

9. 6 ; 7 ; 8. **10.** 4 : -5 ; 6 ; **11.** -5 ; 6 ; -2. **12.** 1 ; 2 ; 3.
13. 1 ; -2 ; 3. **14.** 2 ; -3 ; 4. **15.** 12 ; 12 ; 12. **16.** 1 ; $\frac{1}{2}$; $\frac{1}{3}$.
17. 6 ; 6 ; 6. **18.** a ; b ; c. **19.** $\frac{1}{4}(a+b+2c)$; $\frac{1}{4}(a+2b+c)$;
$\frac{1}{4}(2a+b+c)$. **20** $\dfrac{bc}{(a-b)(a-c)}$; $\dfrac{ca}{(b-c)(b-a)}$; $\dfrac{ab}{(c-a)(c-b)}$.
21. $1\frac{5}{7}$; $2\frac{2}{5}$; -12. **22.** 3 ; 3 ; 3. **23.** $\dfrac{2apqr}{(q+r)p-qr}$;
$\dfrac{2bpqr}{(p+r)q-pr}$; $\dfrac{2cpqr}{(p+q)r-pq}$. **24.** 5 ; 7 . -3.
25. abc ; $bc+ca+ab$; $a+b+c$.

Ex. CIII. (pp. 222-226).

72 and 52. **2.** 65 and 35. **3.** *Rs.*4. 8*a.* and *Rs.*6. **4.** $\frac{9}{15}$.
5. 21 and 40. **6.** $\frac{1}{8}$. **7.** *Rs.* 925 and *Rs.*500.
8. **A** *Rs.*500, **B** *Rs.*400 and **C** *Rs.*200. **9.** *Rs.*400. **10.** $\frac{2}{3}$.
11. **A** 5*s.* and **B** 3*s.* **12.** 17 yds. and 13 yds.
13. 640, 720 and 840. **14.** 108 sq. ft. **15.** 10, 8 and 6.
16. 84 for and 63 against. **17.** *Rs.*2. 8*a* and *Re.*1. 8*a.*
18. **A** 49 years and **B** 21 years. **19.** **A** 20 yrs. and **B** 64 yrs.
20. 24, 12 and 4 years. **21.** 30, 50 and 70, 20 , or 60, 20 and 40, 50.
22. $\frac{1}{75}$. **23.** 48. **24** 3. **25.** 10 yards and 7 yards
26. 150 mangoes and 80 apples. **27.** 23 **28** 91. **29.** 63.
30. 54. **31.** $4\frac{1}{2}$, $3\frac{1}{2}$ and 24. **32.** 12 persons ; 5*s.*
33. 40 lbs. tea and 90 lbs coffee. **34** *Rs.*9 ; 90 passengers.
35. 253. **36.** 646. **37.** $2\frac{1}{2}$ and $7\frac{1}{2}$ miles per hour.
38. 17 florins ; 7 half-crowns. **39.** $\frac{6}{14}$. **40.** 222.
41. 72 apples : 60 pears. **42.** 12 men ; 12 women.
43. 12 men ; 10 women. **44.** 3*s* **45.** **A** *Rs* 70 ; **B** *Rs* 50.
46. 7. **47.** 15 miles ; 2 miles per hour. **48.** 3miles per hour, $8\frac{1}{2}$miles
49. 65. **50.** 15 miles per hour ; 90 miles.

Ex. CIV. (pp 232-233).

4. The Fig. is a rectangle, of which one side $=15$ units and the other $=18$ units ; area $=270$ sq. units. **5.** 48.
6. (1) (3, 0). (2) (8, 5). (3) (-4, -5). (4) (-4, 4).
7. (1) 10. (2) 17. (3) 25. (4) 8 5.
8. (1) 22. (2) 17. (3) 34. (4) 17. (5) 25. (6) 37.
9. 74·6. **10.** The Fig. is a rectangle, of which one side$=15$ units and the other $=28$ units ; area$=420$ sq units.
11. (1) 375 sq. units. (2) 286 sq. units.
(3) 96 sq. units. (4) 52 sq. units.

13. (1) 2·3″ ; 4·2″ ; 9·66 sq. in (2) 4·2″ ; 2″ , 8·4 sq. in.
14. (5, 8). **16.** 226 sq. units. **17.** (1) 102. (2) 52.
18. 2 and 5 ; −2 and 6.

Ex. CV. (p. 238).

3. 3·2 sq. in. **4.** A straight line (i) parallel to the y axis, (ii) parallel to the x-axis. **5.** 1·5 sq. in.

Ex. CVI. (p. 241).

1. (1) $10y=9x+15$. (2) $5y=26-2x$. (3) $y=7$. (4) $3x+4y=18$
(5) $2y=3x+12$. (6) $y=2x-5$. (7) $y=7-2x$. (8) $x+13y+46=0$.
2. $x+y=2$. **3.** $y=3x+4$. **4.** (3, 2), (−2, −2), (8, 6).
5. 4·24 ; −7. **6.** ·58$\dot{3}$; 4·5$\dot{3}$. **7.** (0, 2), (−4, −4), (2, 5), (4, 8).
8. $y+5=2x$. **9.** $y=-4x+7$, $y=-x+1$, $4y=-x+13$.
10. (1) 4·4. (2) 2·4.

Ex. CVII. (pp. 243-244).

1. (1) $x=2$; $y=1$. (2) $x=8$; $y=6$. (3) $x=-3$; $y=4$.
(4) $x=4$; $y=0$. (5) $x=3$, $y=-2$. (6) $x=-2·25$; $y=3·5$.
(7) $x=6$; $y=5$. (8) $x=9$; $y=12$. (9) $x=2·8$; $y=3·2$.
(10) $x=6$; $y=11$. (11) $x=8$; $y=3$. (12) $x=7$; $y=8$.
(13) $x=11$; $y=1$. (14) $x=5$; $y=2$. (15) $x=2$; $y=3$.
2. $x=1$, $y=6$. **3.** (−2, 1) ; (1, −2) ; (2, 3)
5. (−3, 2) ; (4, 1) ; (3, 4). **6.** $a=5$.

Ex. CVIII. (pp. 261—266).

1. 55 lbs. ; 84 lbs. ; 14·8 kilogrammes ; 17·3 kilogrammes.
2. 39·3 in. ; 91·6 cms. ; $y=0·393x$.
3. 12·57, 34·57, 62·86 in. ; 15, 10 in. **4.** (i) 76 ; (ii) 53.
5. 60°C. **6.** £3 ; £4. 10*s*. **7.** 52°·1. **9.** 2·2 in. ; 12·45 cms.
10. 87, 78. 67, 51, 46, 42, 39, 38, 36, 17.
12. *Rs*.199 ; *Rs*.410 ; *Rs* 574. **14.** £1. 15*s*. 1*d*. *nearly* ; 615 copies to the nearest 5.
15. 58, 38, 29 ; $y=\frac{20}{11}x-60$.
16. 2·60, 5·63, 4·16, 5·77. **17.** *Rs*.46. 8*a*.
18. 9*s*. 6*d*. **19.** 167° ; 5°. **20.** £350 ; 4250 copies.
21. 1 A. M. ; 17 and 14 miles. **22.** 10·8 miles. **23.** ·28 yds.
24. 30 miles ; 12 miles. **25.** 70 miles ; $2\frac{1}{3}$ and 2 hrs.
26. 3 P. M. **27.** In 10 secs. from A's start, 33·3 yds. from the starting point. **28.** (i) 16·4 min. after 4. (ii) 5·5 and 27·3 min. after 4. **29.** 9 secs. **30.** 22 miles ; 48 min.
31. 6·5 miles. **32.** ·3 of a mile per hour.

33. (i) 27·3 min. after 5. (ii) 10·9 and 43·6 min. after 5. (iii) 6 o'clock. **34.** 1 : 2. **35.** $9\frac{3}{4}$ hrs. from **A**'s start; $7\frac{1}{2}$ hrs. and $12\frac{1}{4}$ hrs. from **A**'s start. **36.** 4 miles. **37.** $5\frac{5}{6}$ miles.
38. **B** 11·4 secs., **C** 10·5 secs. **39.** 4·8 lbs. **40.** (i) *Rs.*80. (ii) *Rs.*240.

Ex. CIX. (p. 271).

1. $x^{\frac{3}{2}}+x^{\frac{4}{3}}+x^{\frac{5}{4}}+x^{\frac{6}{5}}$. **2.** $ab^{\frac{2}{3}}+a^{\frac{2}{3}}b^{\frac{4}{3}}+a^{\frac{1}{3}}b^{2}+a^{\frac{5}{3}}b$.

3. $ab^{\frac{5}{3}}+a^{\frac{3}{2}}+a^{2}b^{\frac{1}{3}}+ab^{\frac{2}{3}}$. **4.** $a^{\frac{1}{2}}b^{2}+ab^{2}+a^{\frac{3}{2}}b^{2}+a^{\frac{5}{2}}b^{2}$.

5. (i) $a^{-1}+2b^{-2}+3c^{-3}+4ab^{-1}+5a^{-1}b$.

(ii) $\frac{1}{a}+\frac{2}{b^{2}}+\frac{3}{c^{3}}+\frac{4}{a^{-1}b}+\frac{5}{ab^{-1}}$.

6. (i) $a^{3}b^{-3}+3a^{2}b^{-1}+5ab^{-2}+4a^{-2}b+2a^{-3}b^{2}$.

(ii) $\frac{1}{a^{-3}b^{3}}+\frac{3}{a^{-2}b}+\frac{5}{a^{-1}b^{2}}+\frac{4}{a^{2}b^{-1}}+\frac{2}{a^{3}b^{-2}}$.

7. (i) $\frac{1}{3}a^{3}b^{-2}c^{-2}+4a^{-2}b^{-1}c^{2}+2a^{-1}bc+\frac{1}{3}a^{-1}b^{-1}c^{-1}$.

(ii) $\frac{1}{3a^{-3}b^{2}c^{2}}+\frac{4}{a^{2}bc^{-2}}+\frac{2}{ab^{-1}c^{-1}}+\frac{1}{3bca}$.

8. (i) $\frac{1}{2}abc^{-\frac{1}{4}}+\frac{2}{3}a^{-\frac{1}{2}}b^{2}c^{2}+\frac{3}{4}a^{-\frac{2}{3}}b^{-\frac{1}{3}}c^{-\frac{2}{3}}+5a^{-\frac{7}{4}}c$.

(ii) $\frac{1}{2a^{-1}b^{-1}c^{\frac{1}{4}}}+\frac{2}{3a^{\frac{1}{2}}b^{-2}c^{-2}}+\frac{3}{4a^{\frac{2}{3}}b^{\frac{1}{3}}c^{\frac{2}{3}}}+\frac{5}{a^{\frac{7}{4}}c^{-1}}$.

9. $\sqrt{a}+2\sqrt[3]{a^{2}}+3\sqrt[4]{a^{3}}+4\sqrt[5]{a}+\sqrt[4]{a^{3}}$.

10. $\frac{\sqrt[3]{a}}{\sqrt[3]{b^{2}}}+\frac{\sqrt{(a^{2}b)}}{2\sqrt{c}}+\frac{2\sqrt[4]{(ac^{3})}}{3\sqrt{b^{3}}}+\frac{\sqrt[5]{(b^{2}c^{3})}}{4\sqrt[5]{a}}+\frac{\sqrt[4]{(bc^{5})}}{5\sqrt[4]{a^{3}}}$.

11. $\frac{bc}{a}+\frac{ac}{b^{2}}+\frac{1}{abc}+\frac{c^{3}}{ab^{2}}$. **12.** $\frac{1}{\sqrt{a^{2}}}+\frac{\sqrt{a}}{\sqrt[3]{b^{4}}}+\frac{\sqrt[3]{b^{2}}}{\sqrt{a^{3}}}+\frac{1}{\sqrt[3]{b^{5}}}$.

13. $\frac{c}{a^{2}b^{2}}+2abc+\frac{3a^{3}}{bc^{2}}+ab^{2}c^{3}$. **14.** $\frac{\sqrt[3]{b}}{a^{2}}+\frac{\sqrt{a^{3}}}{\sqrt[3]{b^{2}}}+\frac{\sqrt[3]{a^{2}}}{\sqrt[4]{b^{3}}}+\frac{b^{2}}{\sqrt{a}}$.

Ex. CX. (pp. 272-273).

1. $\frac{1}{4}$. **2.** $\frac{1}{9}$. **3.** 8. **4.** $\frac{1}{8}$. **5.** 125. **6.** $3b^{-2}c$.

7. $\frac{1}{2}x^{3}y^{-2}$. **8.** $\frac{1}{16}$. **9.** $\frac{1}{49}$. **10.** $ab^{-\frac{4}{3}}$. **11.** $a^{\frac{1}{2}}$. **12.** $a^{-\frac{1}{4}}$.

13. $a^{2}b$. **14.** $x^{-4}y^{4}$. **15.** $x^{2}y^{\frac{1}{4}}$. **16.** $x^{3}y^{2}z$. **17.** 1. **18.** $a^{2}b^{6}c$.

19. x^{-m}. **20.** x^{a}. **21.** x^{2abc}. **22.** $(a^{4}-b^{4})^{m}$. **23.** $x^{-b^{2}}$.

24. $\frac{1}{x^{p^{2}+q^{2}}}$. **25.** $\left(\frac{p}{q}\right)^{m+n}$. **26.** $\left(\frac{p}{q}\right)^{p+q}$. **27.** y^{2}. **28.** 1.

29. 1. **30.** $\left(\frac{a}{b}\right)^{mn}$. **31.** 256. **32.** [illegible]. **33.** [illegible].

Ex. CXI. (pp. 275-277).

1. $x^{\frac{3}{2}} - 2xy^{\frac{1}{2}} + 2x^{\frac{1}{2}}y - y^{\frac{3}{2}}$. **2.** $a - b^{2}$. **3.** $x^{\frac{7}{3}}y + x^{\frac{1}{3}}y^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{4}{3}} - y$

4. $42x^{\frac{5}{6}} - 18x^{\frac{1}{3}}y^{\frac{1}{3}} - 9x^{\frac{2}{3}}y^{\frac{2}{3}} - 14x^{\frac{1}{2}}y^{\frac{2}{3}} + 6y - 4x^{\frac{1}{3}}y^{\frac{4}{3}} + 49x^{\frac{7}{6}}y^{\frac{1}{3}} + 14xy$.

5. $a^2x^{\frac{1}{4}} - ax^{\frac{3}{4}} + 16x^{\frac{5}{4}}$. **6.** $x^2 - 4y + 6xz^{\frac{1}{3}} + 9z^{\frac{2}{3}}$.

7. $a^2 - 64b^2$. **8.** $x^{\frac{3}{2}} + y^{\frac{3}{2}} + z^{\frac{3}{2}} - 3x^{\frac{1}{2}}y^{\frac{1}{2}}z^{\frac{1}{2}}$. **9.** $x - y$. **10.** $x^{-1} - 1$

11. $a^2 - a^{\frac{4}{3}} + 2a^{\frac{2}{3}} - 2 - a^{-\frac{2}{3}} + a^{-2}$. **12.** $x + x^{\frac{3}{4}}y^{-\frac{1}{4}} - x^{\frac{1}{4}}y^{-\frac{3}{4}} - y^{-1}$.

13. $8x^{\frac{3}{4}} + 4x^{\frac{1}{2}}y^{\frac{1}{2}} + 2x^{\frac{1}{4}}y + y^{\frac{3}{2}}$. **14.** $x^{-\frac{2}{3}} + x^{-\frac{1}{3}}y^{-\frac{1}{3}} + y^{-\frac{2}{3}}$.

15. $a^{-\frac{5}{2}} - 2a^{-2}b^{\frac{1}{3}} + 4a^{-\frac{3}{2}}b^{\frac{2}{3}} - 8a^{-1}b + 16a^{-\frac{1}{2}}b^{\frac{4}{3}} - 32b^{\frac{5}{3}}$.

16. $x^{\frac{2}{3}} + 2x^{\frac{1}{2}} + 3x^{\frac{1}{3}} + 2x^{\frac{1}{6}} + 1$. **17.** $x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$.

18. $x - a^{\frac{1}{3}}x^{\frac{2}{3}} + a^{\frac{2}{3}}x^{\frac{1}{3}} - a$. **19.** $x^{\frac{2}{3}} - a^{\frac{1}{3}}x^{\frac{1}{3}} + a^{\frac{2}{3}}$.

20. $x^{\frac{4}{5}} + 5a^{\frac{1}{5}}x^{\frac{3}{5}} + 6a^{\frac{2}{5}}x^{\frac{1}{5}} + a^{\frac{4}{5}}$ **21.** $4a - 2a^{\frac{1}{2}}b^{-\frac{1}{2}} + 2a^{\frac{1}{2}}c^{\frac{1}{3}} + b^{-1} +$
$b^{-\frac{1}{2}}c^{\frac{1}{3}} + c^{\frac{2}{3}}$. **22.** $x^{\frac{4}{3}}y^{-\frac{4}{3}} + x^{\frac{2}{3}}y^{-\frac{2}{3}} + x^{-\frac{2}{3}}y^{\frac{2}{3}} + x^{-\frac{4}{3}}y^{\frac{4}{3}}$.

23. $a^{\frac{2}{3}} + b^{\frac{2}{3}} + c^{\frac{2}{3}} - 2b^{\frac{1}{3}}c^{\frac{1}{3}} + 2a^{\frac{1}{3}}c^{\frac{1}{3}} - 2a^{\frac{1}{3}}b^{\frac{1}{3}}$.

24. $a^{\frac{4}{3}} - 4a + 10a^{\frac{2}{3}} - 16a^{\frac{1}{3}} + 19 - 16a^{-\frac{1}{3}} + 10a^{-\frac{2}{3}} - 4a^{-1} + a^{-\frac{4}{3}}$.

25. $ab^{-3} + 3a^{\frac{1}{3}}b^{-1} + 3a^{-\frac{1}{3}}b + a^{-1}b^3$. **26.** $\frac{2}{27}x^2y^{-1} - 2x + \frac{9}{2}y - \frac{27}{8}x^{-1}y^2$

27. $a^2 - 6a^{\frac{5}{3}}b^{\frac{1}{6}} + 21a^{\frac{4}{3}}b^{\frac{1}{3}} - 44ab^{\frac{1}{2}} + 63a^{\frac{2}{3}}b^{\frac{2}{3}} - 54a^{\frac{1}{3}}b^{\frac{5}{6}} + 27b$.

28. (i) $x^3 - 4x^{\frac{9}{4}}y^{\frac{5}{2}} + 6x^{\frac{3}{2}}y^5 - 4x^{\frac{3}{4}}y^{\frac{15}{2}} + y^{10}$.
(ii) $x^{\frac{15}{4}} - 5x^3y^{\frac{5}{2}} + 10x^{\frac{9}{4}}y^5 - 10x^{\frac{3}{2}}y^{\frac{15}{2}} + 5x^{\frac{3}{4}}y^{10} - y^{\frac{25}{2}}$.

29. (i) $a^2b^{-2} - 4ab^{-1} + 6 - 4a^{-1}b + a^{-2}b^2$.
(ii) $a^{\frac{5}{2}}b^{-\frac{5}{2}} - 5a^{\frac{3}{2}}b^{-\frac{3}{2}} + 10a^{\frac{1}{2}}b^{-\frac{1}{2}} - 10a^{-\frac{1}{2}}b^{\frac{1}{2}} + 5a^{-\frac{3}{2}}b^{\frac{3}{2}} - a^{-\frac{5}{2}}b^{\frac{5}{2}}$.

30. (i) $a^6 - 4a^3 + 6 - 4a^{-3} + a^{-6}$.
(ii) $a^{\frac{15}{2}} - 5a^{\frac{9}{2}} + 10a^{\frac{3}{2}} - 10a^{-\frac{3}{2}} + 5a^{-\frac{9}{2}} - a^{-\frac{15}{2}}$. **31.** $\frac{1}{4}x^{\frac{3}{2}} - \frac{1}{3}x^{\frac{1}{2}} - \frac{1}{2}$.

32. $a^{-\frac{3}{5}}x^{\frac{7}{5}} - x^{\frac{4}{5}} - a^{\frac{4}{5}}$. **33.** $1 - \frac{3}{4}\sqrt{x} + x$. **34.** $ab^{-1} + 1 + ab^{-1}$.

35. $a^{\frac{2}{3}} - \frac{3}{2}a^{\frac{1}{3}} + 3 - 6a^{-\frac{1}{3}} + 9a^{-\frac{2}{3}}$. **36.** $\sqrt{a} + \sqrt{(2b)} + 2\sqrt{(2c)}$.

37. $\frac{x}{y}+\frac{y}{x}-\frac{1}{\sqrt2}$. **38.** $y^{-\frac12}(x+y)-\frac12 x^{\frac14}y^{\frac14}$.

39. $3x^{\frac23}+3x^{-\frac13}-\sqrt2 x^{-\frac13}$. **40.** $x^{\frac23}y^{\frac12}-2x^{-\frac12}y^{-\frac23}+x^{\frac16}y^{-\frac16}$.

41. $a^{-\frac12}x^{\frac12}-1+a^{\frac12}x^{-\frac12}$. **42.** $xy^{-\frac13}-x^{-\frac12}y^{\frac13}$. **43.** $2x^{\frac12}-3y^{\frac14}$.

44. $(e^x+1)(x+1)$. **45.** $x-\frac12\sqrt{(x+1)}$. **46.** $3ax(x^2-a^2)$.

47. $a^{3n}+x^{3n}$. **48.** $a^{4m}-a^{3m}b^n+a^{2m}b^{2n}-a^mb^{3m}+b^{4n}$. **49.** $a^{\frac23}+b^{\frac23}$.

Ex. CXII. (pp. 280-281).

1. $64^{\frac14}$. **2.** $81^{\frac15}$. **3.** $(\frac19)^{\frac13}$. **4.** $(\frac18)^{\frac14}$. **5.** $(\frac32)^{\frac14}$. **6.** $8^{\frac15}$.

7. $25^{\frac12}$; $125^{\frac13}$. **8.** $(\frac{25}{4})^{\frac12}$; $(\frac{125}{8})^{\frac13}$. **9.** $(\frac14a^2)^{\frac12}$; $(\frac{8}{27}a^3)^{\frac13}$.

10. $(\frac14a^4)^{\frac12}$; $(\frac18a^6)^{\frac13}$ **11.** $\{\frac14(a^2+2ab+b^2)\}^{\frac12}$; $\{\frac18(a^3+3a^2b+3ab^2+b^3)\}^{\frac13}$.

12 $(\frac{1}{729})^{\frac13}$; $6561^{-\frac14}$. **13.** $(\frac{27}{1000})^{\frac13}$. $(\frac{10000}{81})^{-\frac14}$ **14.** $(1/a^6)^{\frac12}$; $(a^8)^{-\frac14}$.

15. $\left(\frac{a^3}{b^3c^6}\right)^{\frac13}$, $\left(\frac{b^4c^8}{a^4}\right)^{\frac14}$. **16.** $\sqrt{(125)}$. **17.** $\sqrt3$. **18.** $\sqrt{(12)}$.

19. $\sqrt{\frac14}$. **20.** $\sqrt{\frac14}$. **21.** $\sqrt{(320)}$. **22.** $\sqrt[3]{(54)}$. **23.** $\sqrt[3]{(256)}$.

24. $\sqrt[4]{(2048)}$. **25.** $\sqrt[4]{3}$. **26.** $\sqrt[3]{\frac13}$. **27.** $\sqrt[4]{\frac{1}{8}}$. **28.** $\sqrt{(4a)}$.

29. $\sqrt{(98a^2x)}$. **30.** $\frac16$. **31.** $\sqrt{\left(\frac{a+b}{a-b}\right)}$. **32.** $\frac{1}{a^2+ab+b^2}$.

33. $\sqrt[n]{(x^ny^{2n})}$. **34.** $\sqrt{(2ab)}$. **35.** $\sqrt{(6a^3x)}$. **36.** $\sqrt[3]{\left(\frac{4a^2}{9b^2}\right)}$.

37. $\sqrt{\left(\frac{2a}{3}\right)}$. **38.** $\sqrt{(5xy)}$. **39.** $\sqrt{(a^2-x^2)}$. **40.** $\frac{yz}{x}$. **41.** $\sqrt{\left(\frac{a+x}{a-x}\right)}$.

42. $\sqrt[3]{\left(\frac{ax}{b^2}\right)}$. **43.** $3\sqrt5$. **44.** $5\sqrt5$. **45.** $36\sqrt3$. **46.** $3\sqrt[3]{5}$.

47 $18\sqrt[3]{2}$. **48.** $\frac12\sqrt6$. **49.** $\sqrt[3]{(12)}$. **50.** $\sqrt[4]{(54)}$. **51.** 6.

52. $4\sqrt[4]{2}$. **53.** $8\sqrt[3]{2}$. **54.** $6\sqrt[3]{(48)}$. **55.** $\frac43\sqrt2$. **56.** $\frac{8}{27}\sqrt2$.

57. $\frac29\sqrt[3]{2}$. **58** $\frac17\sqrt{(21)}$. **59.** $\frac12\sqrt[3]{(150)}$. **60.** $\sqrt[4]{(375)}$.

61. $a^2b\sqrt{(ab)}$. **62.** $a^2bc^2\sqrt[3]{(ab^2)}$. **63.** $\frac17\sqrt{(42)}$. **64.** $\frac13\sqrt6$.

65. $\frac34\sqrt2$. **66.** $\frac12\sqrt[3]{6}$. **67.** $\frac13\sqrt[3]{3}$. **68.** $\sqrt[6]{(125)}$, $\sqrt[6]{(121)}$.

69. $\sqrt[12]{(2401)}$, $\sqrt[12]{(729)}$. **70.** $\sqrt[6]{8}$, $\sqrt[6]{5}$. **71.** $4\sqrt7$.

72. $3\sqrt[3]{3}$. **73.** $\sqrt5$. **74.** $\frac13\sqrt[4]{(27)}$. **75.** $3\sqrt[3]{5}$.

76. $3(4\frac12)^{-\frac16}$. **78.** 1·341640... **79.** ·816496... **80.** 1·133893...

Ex. CXIII. (p 282).

1. $\sqrt2$. **2.** $3\sqrt5$. **3.** $\frac{29}{2}\sqrt3$. **4.** $9\sqrt[3]{9}$. **5.** $\frac{23}{3}\sqrt2$.

6. $\left(\frac{2a}{b}+\frac{y}{3c}-\frac{x}{2b}\right)\sqrt[3]{\left(\frac{2ax}{3b}\right)}$. **7.** $24\sqrt3$. **8.** $120\sqrt3$. **9.** 36.

10. $5-\sqrt{6}$. **11.** $6\sqrt{3}+3\sqrt{(30)}$. **12.** $\frac{1}{3}\sqrt{3}+3\sqrt{2}-\sqrt{6}$.
13. $2+\frac{5}{6}\sqrt{6}$. **14.** $216\sqrt[12]{6}$. **15.** $288\sqrt[12]{(72)}$. **16.** 16.
17. $x^4+2x^3-8x^2-6x-1$. **18.** $\frac{2}{3}\sqrt{(21)}$. **19.** $\frac{1}{14}\sqrt{5}$. **20.** $\frac{1}{6}\sqrt[6]{(36000)}$.
21. $\frac{1}{3}(\sqrt{2}+\sqrt{3}+\sqrt{5})$. **22.** $\frac{1}{3}\sqrt{6}+\frac{1}{2}\sqrt[6]{(32)}+\frac{1}{6}\sqrt[4]{(120)}$.
23. $x+y+2\sqrt{(x+y)}+4$.

Ex. CXIV. (pp 284-285).

1. $\frac{1}{27}(58+8\sqrt{7})$. **2.** $\frac{1}{19}(23+8\sqrt{5})$. **3.** $\frac{1}{6}(3-\sqrt{6})$.
4. $\frac{1}{5}(2\sqrt{2}-\sqrt{3})$. **5.** $\sqrt{5}+1$. **6.** $\sqrt{5}-\sqrt{2}$ **7.** $4+\sqrt{2}$.
8. $\frac{1}{10}(4+\sqrt{6})$. **9.** $\frac{1}{2}(7+3\sqrt{5})$. **10.** $\frac{1}{288}(297+85\sqrt{21})$.
11. $\frac{1}{11}(7\sqrt{14}-13)$. **12.** 23·3137. **13.** 2. **14.** 1·1992. **15.** 1·3197.
16. 5. **17.** $\frac{1}{18}(9\sqrt{15}-11\sqrt{6})$. **18.** $1+\sqrt{2}+\sqrt{3}$. **19.** $16\sqrt{(15)}$.
20. $2\sqrt{3}$. **21.** $\dfrac{a+\sqrt{(a^2-x^2)}}{x}$. **22.** $\dfrac{2\sqrt{(a^2-x^2)}}{x^2}$.
23. $4x\sqrt{(x^2-1)}$. **24.** $2x^3$. **25.** $\sqrt{\dfrac{a}{b}}$. **26.** $\dfrac{1}{1-x^2}$.
28. $\frac{1}{2}(2+\sqrt{2}+\sqrt{6})$. **29.** $\frac{1}{3}(2\sqrt{3}-3)$. **30.** $1\frac{1}{3}$.

Ex. CXV. (pp. 287-288).

1. $a^2+b^2+c^2$. **2.** $4(a^2+b^2)$. **3.** $4(a^2+b^2+c^2+d^2)$.
4. $-(b-c)(c-a)(a-b)$. **5.** $4(b^2c^2+c^2a^2+a^2b^2)$. **6.** $2(a^4+b^4+c^4)$.
7. $24abc$. **8.** $1000c^3$. **9.** abc. **10.** 1. **11.** $4a(a^2+3b^2+3c^2)$.
12. $6abc$. **13.** 0. **14.** $2(a+b+c)^3+2abc$. **15.** $4(ax+by+cz)$.
16. $a^3-b^3+8+6ab$. **17.** $x^3-8y^3-27-18xy$. **18.** 0.
19. $(b-c)(c-a)(a-b)$. **20.** $(1-abc)(1-a^2-b^2-c^2+2abc)$. **21.** $8x^3$.

Ex. CXVI. (p. 289).

1. (i) 74. (ii) 109. (iii) 97. **2.** (i) 141. (ii) 41. (iii) 112.
3. (i) 13. (ii) 13. (iii) 36. **4.** (i) 246. (ii) 6. (iii) 335.
5. $(3a+2b)^2$. **6.** $(a-4b)^2$. **7.** $(x^2+5x+5)^2-1^2$.
8. $(x^2+5x+7)^2-(2x+2)^2$. **9.** $(4x^2-2x-1)^2-(2x^2-3x+4)^2$.
10. $(x^2+12ax-31a^2)^2-(4a^2)^2$.

Ex. CXVII. (p. 290).

1. (i) 280. (ii) 1188. (iii) 610 **2.** (i) −10. (ii) 259. (iii) −972.
3. 364. **4.** 198. **5.** (i) 9. (ii) −25. **6.** (i) 73. (ii) −7209.
7. $4x^2$. **8.** $(a+b+c)^3$.

Ex. CXVIII. (p. 292).

1. $(x-4)(x-8)$. **2.** $(x+8)(x-5)$. **3.** $(x-1)(x-102)$.
4. $(x+3)(x+7)$. **5.** $(x-3)(x-9)$. **6.** $(6x-11)(x+2)$.

7. $(3x-7)(7x+12)$. 8. $(25x+43)(x-2)$. 9. $(5x-9)(2x+1)$.
10. $(7x-3)(x+5)$. 11. $(5x+13)(6x-11)$. 12. $(21x-5)(3x+7)$.
13. $(2x+5y)(x-y)$. 14. $(x-19a)(x+10a)$. 15. $(4x+9y)(2x-3y)$.
16. $(8a-9x)(3a+8x)$. 17. $\{2(x+y)-(a+b)\}\{(x+y)-4(a+b)\}$.
18. $(2x^2+3y^2)(2x^2-y^2)$. 19. $(3x^2+y^2)(2x^2-y^2)$.
20. $(xy^2-17)(xy^2+16)$. 21. $(x+\sqrt{a})(x-\sqrt{a})$.
22. $(x+a\sqrt{2})(x-a\sqrt{2})$. 23. $(a^2+ab\sqrt{3}+b^2)(a^2-ab\sqrt{3}+b^2)$.
24. $(x^2+a^2\sqrt{3})(x+a\sqrt[4]{3})(x-a\sqrt[4]{3})$.

Ex. CXIX. (p. 293).

1. $(a-b+c)(a^2+b^2+c^2+bc-ca+ab)$.
2. $(a-b-c)(a^2+b^2+c^2+ab-bc+ca)$.
3. $(x-y+1)(x^2+xy+y^2-x+y+1)$.
4. $(x+y+1)(x^2-xy+y^2-x-y+1)$.
5. $(x-2y+3z)(x^2+4y^2+9z^2+2xy+6yz-3xz)$.
6. $(2a-3b-1)(4a^2+9b^2+1+2a+6ab-3b)$.
7. $(a-b+2)(a^2+b^2+4+ab-2a+2b)$.
8. $(2a+b-1)(4a^2+b^2+1-2ab+b+2a)$.
9. $(a+2b+3c)(a^2+4b^2+9c^2-2ab-6bc-3ca)$.
10. $(2x+y)(x-y)^2$. 11. $(x+2y-3z)(x^2+4y^2+9z^2-2xy+6yz+3xz)$.
12. $(2x-y)(7x^2+8xy+4y^2)$.
13. $(a-5b+3)(a^2+25b^2+9-3a+5ab+15b)$.
14. $(x^2+2x-4)(x^4-2x^3+8x^2+8x+16)$.
15. $2(c-b)(3a^2+b^2+c^2-3ab-3ac+bc)$.

Ex. CXX. (pp. 295-296).

1. $(x+y)(x-1)(1-y)$. 2. $(a^{-1}+a^{-2})(a^{-1}+a^3)(a^{-2}+a^3)$.
3. $(a+b)(a+c-d)(b+c-d)$. 4. $(a+b+c)(bc+ca+ab)$.
5. $(b+c)(c+a)(a+b)$. 6. $(c-b)(a-c)(a+b)$. 7. $(b+c)(c-a)(a-b)$.
8. $(b+3c)(3c+a)(a+b)$. 9. $(a+b+c)(bc+ca+ab)$.
10. $(y+z)(z+x)(x+y)$. 11. $(a+b+c)(bc+ca+ab)$.
12. $(a+b+c)(bc+ca+ab)$. 13. $(a+b+c)(a^2+b^2+c^2)$.
14. $(b+c-a)(c+a-b)(a+b-c)$. 15. $3(2a+b+c)(a+2b+c)(a+b+2c)$.
16. $(a+b+c)(a+b-c)(a-b+c)(b+c-a)$.

Ex. CXXI. (pp. 297-298).

1. $-(b-c)(c-a)(a-b)$. 2. $(b-c)(c-a)(a-b)$.
3. $-(b-c)(c-a)(a-b)(a+b+c)$. 4. $(b-c)(c-a)(a-b)(a+b+c)$.
5. $-(b-c)(c-a)(a-b)(bc+ca+ab)$.
6. $-(b+c)(c+a)(a+b)(b-c)(c-a)(a-b)$.
7. $-(b-c)(c-a)(a-b)(bc+ca+ab)$.
8. $-(b-c)(c-a)(a-b)(a^2+b^2+c^2+bc+ca+ab)$.

9. $-(b-c)(c-a)(a-b)\{b^2c^2+c^2a^2+a^2b^2+abc(a+b+c)\}$.
10. $-(b-c)(c-a)(a-b)(a^2+b^2+c^2+bc+ca+ab)$.
11. $-(b+c)(c+a)(a+b)(b-c)(c-a)(a-b)$
12. $(b+c)(c+a)(a+b)(b-c)(c-a)(a-b)$.
13. $(b-c)(c-a)(a-b)\{a^3+b^3+c^3+a^2(b+c)+b^2(c+a)+c^2(a+b)+abc\}$
14. $-(b-c)(c-a)(a-b)\{a^3+b^3+c^3+bc(b+c)+ca(c+a)+ab(a+b)+abc\}$.
15. $(b-c)(c-a)(a-b)(a+b+c)$.
16. $(b-c)(c-a)(a-b)(bc+ca+ab)$.
17. $5(b-c)(c-a)(a-b)(a^2+b^2+c^2-bc-ca-ab)$.
18. $-(b-c)(c-a)(a-b)$. 19. $-(b-c)(c-a)(a-b)$.
20 $-(b-c)(c-a)(a-b)(a+b+c+3)$. 21. $-(b-c)(c-a)(a-b)$.
22. $(b-c)(c-a)(a-b)$. 23. $-(b-c)(c-a)(a-b)(a+b+c)$.
24. $-(b-c)(c-a)(a-b)x^2$. 25. $-2(b-c)(c-a)(a-b)(a+b+c)$.

Ex. CXXII (p. 300).

1. $(x+1)^2(x-3)(x-8)$ 2 $(x-1)^2(x+4)(x-9)$.
3. $(x-1)^3(x+4)(3x-1)$. 4. $(x+1)^3(3x^2-9x+8)$.
5. $(x-1)^2(x-4)(4x+1)$. 6. $(x+1)^2(x+2)(2x-3)$.
7. $(x-1)^2(x+1)(x-2)(x+3)$. 8. $(x+1)(x-1)(x^2-ax+b)$
9. $(3a+2b+c)(2a+b+3c)$. 10. $(2a+2b+c)(a+2b+2c)$.
11. $(2a+2b+c)(a-6b+4c)$. 12. $(a-b+2c)(a-2b-2c)$.
13. $(a-b-c)(2a+3b+c)$. 14 $(a-7b+5c)(a-3b)$.
15. $(2a+b-3c)(2a-3b+3c)$. 16. $(x^2+4x+1)(x^2-3x+1)$.
17. $(x^2+2x+2)(x^2-7x+2)$. 18 $(x+1)^4(x^2-6x+1)$.
19. $(x^2+1)(x+1)^2(x^2-2x-1)$. 20. $(x^2+1)^2(x^4-5x^2+1)$.
21. $(x^2+1)^2(x^4-7x^2+1)$. 22. $(x-1)(x+1)^4$.
23. $(3x^2-2x-3)(4x^2+x-4)$. 24 $(x-a)(x^2+ax+a^2)(x^2-ax+a^2)$.

Ex. CXXIII. (pp. 302-303).

1. $(x-1)(x^2+x+4)$. 2. $(x-2)(x^2+x+2)$. 3. $(x-1)^2(x+2)$.
4. $(x-2)(x-3)(x+5)$ 5. $(x-1)(x^2+3x+3)$.
6. $(x+1)(x+2)(x-3)$. 7. $(x-8)^2(x+9)$. 8. $(x+4)(x-6)^2$.
9. $(x-4a)(x^2+2ax+3a^2)$. 10. $(x-1)(x+3)(2x+5)$.
11. $(3x-11)(x^2-2x-1)$. 12 $(2x+5)(2x^2-x+4)$.
13. $(x-1)(x-2)(x-3)(x-4)$. 14. $(x+3)(x-4)(x^2-x+11)$.
15. $(x+1)(x-4)(x^2-3x+1)$. 16. $(x+2)(x+4)(x-4)(x+10)$.
17. $(x+1)(x+2)(x-9)(x-10)$. 18. $(x^2+3x-5)(x^2-3x+5)$.
19. $(x^2-3x-6)(x^2-3x-16)$. 20. $(x^2+7x+5)(x^2+7x+17)$.
21. $(2x^2-3x+6)(2x^2-3x-8)$. 22. $(2x^2-4x-3)(2x^2-6x+3)$.
23. $(x+2a-b-c)(x+2b-a-c)$.

24. $(a-3b-2)(a^2+9b^2+4+2a+3ab-6b)$.
25. $(a-4)(a^2+5a+21)$. 26. $(1+a^2)(1+b^2)(1+a)(1+b)(1-a)(1-b)$.
27. $(a+b)(a-b+c)(b+c-a)$. 28. $(ab+ac-b+c)(ab-ac+b+c)$.
29. $(bx-a)(cx^2+bx-a)$. 30. $(x^2+x+1)(x^3-x+1)$.

Miscellaneous Factors. (Harder).

1. $(x+1)(x-1)(x+a-1)(x-a+1)$. 2. $4(a-b)^2(a+b+1)$.
3. $(4a+b+c)(a+4b+c)$.
4. $(x+1)(x-1)(x+1-xy+y)(x-1-xy-y)$.
5. $(x-y)(x-y-1)$. 6. $(a+b+c)(a+b-c)(b+c-a)(c+a-b)$.
7. $(a^2+b^2-ab)(a^2+b^2+a+b)$. 8. $(x-y+1)(x+ay-1)$.
9. $(b-c)(c-a)(a-b)$.
10. $(a+b+c+d)(a+b-c-d)(a-b-c+d)(a-b+c-d)$.
11. $(1+a)(1-a)(b+c+ab-ac)(b+c-ab+ac)$.
12. $(x-a)\left(x-\frac{1}{a}\right)$. 13. $(a+1)(a-1)\left(x-\frac{2}{y}\right)\left(x^2+\frac{2x}{y}+\frac{4}{y^2}\right)$.
14. $(x-3)(x-4)(x+5)$. 15. $(x^2+7)(x^3-5)$.
16. $(x+1)(x-1)(x+3)(x+5)$. 17. $(x+1)(6x^2-8x+3)$.
18. $(b+c)(c+a)(a+b)$. 19. $3(2a-b)(a-2b)(a+b)$.
20. $3(b+c)(c+a)(a+b)$. 21. $(x-1)^3(x-2)$. 22. $(x-1)(x+2)(x-3)^2$.
23. $-(b-c)(c-a)(a-b)(a+b+c)$. 24. $(a+b+c-1)^2(a+b+c+2)$.
25. $(a+2)(a+3)(a^2+7a+14)$.

Ex CXXIV. (pp. 309-313).

50. (1) $pq-r$. (2) $pq-3r$. (3) $p^3-3(pq-r)$. (4) $q^3-3r(pq-r)$.

Ex CXXV. (pp. 314-315).

1. $\dfrac{a+b}{c^2-(a-b)^2}$. 2. $\dfrac{7(x^2+xy+y^2)}{5}$. 3. $\dfrac{(c^x-1)(x^2-x+1)}{(c^x+1)(x-1)}$.
4. $x^4+5x^2y^2+y^4$. 5. $\dfrac{x-y-z}{x-4y+4z}$. 6. 1. 7. -1. 8. 3.
9. $\dfrac{(a+b)^2+2c(a+b)+c^2}{(a+b)^2-c(a+b)+c^2}$. 10. $\dfrac{x^{m-1}}{b(a+bx)}$. 11. $a^2+b^2+c^2$.
12. $\left(\dfrac{ab+1}{ab-1}\right)^2$. 13. $1+abc$. 14. $\dfrac{3x-(a+b)}{x^2-(a+b)x+ab}$. 15. $\dfrac{a-b}{b-c}$.
16. $\dfrac{a+b+c}{2}$. 17. 1. 18. a. 19. $\dfrac{a^2+b^2+c^2}{bc+ca+ab}$. 20. $\dfrac{a-c}{b-d}$.

Ex. CXXVI. (pp. 320-323).

1. 2. $\dfrac{4x^3(2x^2+1)}{x^6-1}$. 3. $\dfrac{x^8-x^4+1}{x^8+x^4+1}$. 4. a. 5. 4. 6. 0.

7. 0. **8.** ab. **9.** $\frac{4ax}{x^2-a^2}$. **10.** $\frac{x+c}{(1+ax)(1+bx)}$. **11.** 1.

12. $-\frac{x^n}{(x+a)^n}$. **13.** $2(ac+bd)(ad+bc)$. **14.** 1.

15. $\frac{8abc}{(a-b)(b-c)(c-a)}$. **16.** 0. **17.** 2. **18.** $\frac{3}{(x-1)(x^2+1)}$.

19. 0. **20.** 1. **21.** $a^2+b^2+c^2$. **22.** 0; 0, 1; $a+b+c$; $a^2+b^2+c^2+bc+ca+ab$. **23.** $\frac{1}{abc}$. **24.** $\frac{1}{x(x-a)(x-b)}$.

25. 0. **26** 0. **27.** 0. **28.** 1. **29.** 1. **30** 1.

31. 4. **32.** 1. **33.** -3. **34.** 0. **35.** 0. **36** x^2. **37.** 1.

38. abc. **39.** x. **40.** d. **41** $\frac{(-x)^m}{(x+a)(x+b)(x+c)}$.

42. $\frac{(-x)^m}{(x-a)(x-b)(x-c)}$. **43.** $\frac{x^2-x+1}{(x+a)(x+b)(x+c)}$.

44. $\frac{a+b+c}{(b+c)(c+a)(a+b)}$. **45.** pq.

Ex. CXXVII. (pp. 324-325)

1. $\frac{4b}{a}$. **2.** 1. **3.** $\frac{ab}{a+b}$. **4.** $\frac{(x^2+2)(1+x^4)}{x}$. **5.** $\frac{1}{x}$. **6.** $\frac{1}{2}$.

7. $a-b$. **8.** $\frac{2a^4}{(a^2+b^2)^2}$. **9.** $\frac{c}{a}$. **10.** $\frac{x}{(a+c-x)(c+bx)}$.

11. $a+b+c$. **12.** 1. **13.** b^4. **14.** $\frac{1}{2}x$.

Ex. CXXVIII. (p. 327).

1. $\frac{b}{a}$. **2.** 0. **3.** $\frac{a^4-10a^3b-6ab^3-b^4}{a^4+10a^3b+6ab^3-b^4}$. **4.** $n(n-1)$. **5.** 0.

6. $\frac{10}{21}$. **7.** $1\frac{36}{49}$. **8.** $\frac{1}{b}$. **9.** a. **10.** $\frac{1}{n}$.

REVISION PAPERS III.

Paper I.

2. $2x^2-3x+1$. **3.** $\frac{a^2-b^2}{x^2-y^2}$. **4.** $\left(x+\frac{1}{x}\right)^3$. **5.** $\frac{3x-8}{x^2-7}$.

6. (i) $x=2, y=-2, z=5$. (ii) $x=1\frac{1}{2}, y=-\frac{2}{3}, z=1$

7. $\frac{7x^2-59x+18}{(x+1)(x-2)(x-6)}$. **8.** $\frac{13}{21}$.

Paper II.

1. x^2+2x+3. **2.** 0. **3.** $(x+y)^3+2z(x+y)+4z^2$. **4.** $\frac{2a-3b+c}{2a-3c}$.

5. $\frac{1}{a+2}$. **6.** (i) $x=y=15$. (ii) $x=1,\ y=\frac{1}{4},\ z=\frac{1}{2}$

7. (i) $x^{\frac{(mp+rn)(mn-rp)}{mn}}$. (ii) $\frac{2^n-1}{2^n}$. **8.** 5 miles.

Paper III.

1. $\frac{3x^2+19x+14}{(x^2+5x-6)(x^2+3x-10)}$; $\frac{1}{x^2-3x+2}$ **2.** $\frac{x+1}{x-4}$. **3.** x^2-2x+3.

5. a. **6.** 1. **7.** (1) $x=2,\ y=3,\ z=4$. (2) $x=3,\ y=4,\ z=5$.

Paper IV.

1. (i) $\frac{x-3}{x+3y}$. (ii) $\frac{a^2+b^2}{a}$. **2.** $(a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}})^2$. **4.** $a+b$.

5. (i) $x=3,\ y=2$. (ii) $x=4\frac{2}{7},\ y=3\frac{2}{7},\ z=3\frac{2}{7}$.

6. $\sqrt{x}+\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{2}}$. **7.** x^2-5x+1. **8.** 432.

Paper V.

1. $x^6+x^{-6}+3(x^2+x^{-2})$. **2.** $4(a^3+b^3+c^3-3abc)$.

3. $x-2a$; $(2x^2+4x+1)(2x^2-x-6)$. **5.** $\frac{x^3}{(x-a)(x-b)(x-c)}$.

6. $(b+c)(c+a)(a+b)$. **7.** (i) $x=\frac{a}{a+b},\ y=\frac{b}{a-b}$.

(ii) $x=\frac{1}{2}a,\ y=\frac{1}{2}b,\ z=\frac{1}{2}c$. **8.** 324

Paper VI.

1. 0. **2.** x^2+7x+1. **3.** (i) $\frac{4}{x^4-16}$. (ii) $2x$. **4.** $3\frac{1}{3}$ hours.

5. (i) x^2+5x+5. (ii) a^2+b^2. **6.** (i) $x=y=1$. (ii) $x=y=c$.

7. $a=11,\ b=-4$. **8** 57.

Paper VII

1. (i) 1. (ii) a^2+b^2. **2.** 0. **3.** $\sqrt{(a+x)}$. **5.** $3y=x+5$.

6. (i) $x=\frac{ab}{a+b},\ y=\frac{ab}{a-b}$. (ii) $x=\frac{a^2b}{a-b},\ y=\frac{ab^2}{a+b}$.

7. $5x^2-7xy+5y^2$. **8** $4\frac{1}{2}$, 4, $3\frac{3}{5}$ miles per hour.

Paper VIII.

1. $x+y+z+xyz$. **2.** $\frac{x^2(x^2-2x+1)}{256}$; $\frac{x(x-1)}{16}$. **3.** $x-1$.

4. 1. **5.** x^2+3x+2; $(x^2-4x+3)(x^2-4)(x+4)$. **6.** 10.

7. (i) $x=a(a-b)$, $y=b(a-b)$. (ii) $x=\dfrac{a_1b_2-a_2b_1}{c_1b_2-c_2b_1}$, $y=\dfrac{b_1a_2-b_2a_1}{c_1a_2-c_2a_1}$.
(iii) $x=1\frac{2}{5}$, $y=2\frac{1}{2}$. **8.** *Rs.*1996, 1st year ; *Rs.*2000, 2nd year.

Paper IX.

1. x^2+x+1. **2.** $3x-a-b-c$. **3.** (i) $\dfrac{4}{2x-7}$. (ii) 1. **4.** $4(a+b)^2$.

5. (i) $x=\dfrac{p}{1-p}$, $y=\dfrac{1}{1+p}$. (ii) $x=\dfrac{a^2-b^2}{ac-bd}$, $y=\dfrac{a^2-b^2}{ad-bc}$.

6. $1\frac{1}{2}$ mds. **7.** 36 ft **8.** $127\frac{1}{2}$.

Paper X.

1. $1+abc$. **2.** $a+\dfrac{x^2}{2a}-\dfrac{x^4}{8a^3}+\dfrac{x^6}{16a^5}$; 10·0498756...

3. $\dfrac{(a-b+c-d)(a-b-c+d)(b-c+d-a)(b-c-d+a)}{(a-b)(b-c)(c-d)(d-a)}$.

5. 180. **6.** (i) $x=\dfrac{a^2bc}{a^2+b^2}$, $y=-\dfrac{ab^2c}{a^2+b^2}$. (ii) $x=\dfrac{a}{a+b}$, $y=\dfrac{b}{a-b}$.

7. 2 hrs. **8.** 2*s.* $2\frac{1}{2}d.$; 31 articles.

Paper XI.

1. $3x+4y=13$. **2.** $x=5$, $y=6$. **3.** $x-1$. **4.** $\frac{1}{6}$.

5. (i) $x=40$, $y=60$. (ii) $x=2$, $y=3$. **6.** $\dfrac{4x^3}{1+x^4+x^8}$.

7. 34 ; -7. **8.** 26·7 miles.

Paper XII.

1. 226. **2.** (6, 8). **3.** (i) 1·49 in. (ii) 5·7 cms. **4.** 81 ft.
5. *Rs.*530. **6.** 162. **7.** 11*s.* **8.** $4\frac{1}{2}$ hrs. **9.** $(2\frac{1}{2}, \frac{1}{4})$.
10. (i) 1 P. M., 28 miles from P. (ii) 20 miles. (iii) 11-30 A. M.

Ex. CXXX. (pp. 351-353).

1. No. **2** Yes ; $a^3-a^2b+ab^2-b^3$. **3.** No. **4.** Yes ; $a^4-a^3b+a^2b^2-ab^3+b^4$. **5.** Yes ; $a^4+a^3b+a^2b^2+ab^3+b^4$. **6.** No.
7. No. **8.** No. **9.** No. **10.** Yes ; $a^7+a^6b+a^5b^2+a^4b^3+a^3b^4+a^2b^5+ab^6+b^7$. **11.** No. **12.** No. **13.** Yes ; $a^{10}-a^8b^2+a^6b^4-a^4b^6+a^2b^8-b^{10}$. **14** No. **15.** No. **19.** $m=2pn$, where p is any positive integer. **25.** $x^4+x^3y+x^2y^2+xy^3+y^4$.
26. x^3+2x^2+3x+4. **31.** $1+a+a^2+a^3+a^4+a^5+a^6+a^7+a^8+a^9$.
40. $1+a+a^2+a^3+a^4+\ldots\ldots\ldots\ldots+a^{30}+a^{31}$.

Ex. CXXXI. (p. 355).

1. $-7\frac{3}{7}$. 2. $-2\frac{9}{34}$. 3. $11\frac{7}{10}$. 4. 15. 5. $(a^2+ab+b^2)(a+b)$.
6. 1. 7. 0. 8. $2\frac{1}{4}$. 9. $a+b$. 10. 6. 11. $\frac{3}{4}(a+c)$. 12. $4\frac{8}{9}$.

Ex. CXXXII. (pp. 360-363).

1. 1. 2. $\frac{2}{5}$. 3. 4. 4. 9. 5. $5\frac{1}{10}$. 6. $\frac{11}{28}$. 7. 6.
8. $\frac{1}{5}$. 9. 2. 10. 3. 11. 3. 12. 3. 13. 3. 14. 6.
15. $-\frac{3}{5}$. 16. 1. 17. $1\frac{1}{3}$. 18. $\frac{3}{5}$. 19. 1. 20. $-1\frac{1}{4}$.
21. 1. 22. 20. 23. 8. 24. 14. 25. -107. 26. 7.
27. 3. 28. $\frac{1}{4}$. 29. $-\frac{4}{5}$. 30. 4. 31. $3\frac{3}{8}$. 32. 0.
33. 5. 34. $-\frac{1}{5}$. 35. $1\frac{1}{13}$. 36. $-1\frac{1}{2}$. 37. $4\frac{1}{2}$. 38. 8.
39. 4. 40. $3\frac{1}{2}$. 41. $-\frac{5}{6}$. 42. 4. 43. $5\frac{3}{8}$. 44. $2\frac{1}{2}$.
45. $6\frac{1}{3}$. 46. 2. 47. 8. 48. 0. 49. $\frac{1}{2}$. 50. 2.
51. $-2\frac{1}{2}$. 52. [illegible] 53. 19. 54. $4\frac{1}{4}$. 55. $-2\frac{1}{2}$. 56. $1\frac{1}{11}$.
57. $8\frac{3}{4}$. 58. $-5\frac{3}{4}$. 59. $1\frac{3}{4}$. 60. $-5\frac{1}{2}$. 61. 3.
62. $-5\frac{1}{2}$. 63. -8. 64. 4. 65. $6\frac{1}{2}$. 66. $-1\frac{3}{5}$.
67. 2. 68. 1. 69. 7. 70. $4\frac{1}{2}$. 71. $\pm\sqrt{5}$.

Ex. CXXXIII. (pp. 366-368)

1. $\frac{a(ac+b^2)}{a^2+bc}$. 2. $\frac{ace}{cd-be}$. 3. $\frac{abc}{a+b}$. 4. $\frac{ace+bcd-b^2c}{b(ac-c^2)}$.

5. $\frac{d}{c}$. 6. $\frac{1}{ab}$. 7. $\frac{bfh}{af+2bc-bfg}$. 8. $\frac{abc}{a^2-ab+b^2}$. 9. $\frac{b}{a}(a+c-b)$.

10. $\frac{b^2-a^2}{4a-b}$. 11. $3a$. 12. $-\frac{ab}{2a+b}$. 13. $\frac{2pr}{p^2-2pq-q^2}$.

14. $\frac{ac}{b}$. 15. $\frac{a+b}{2}$. 16. $\frac{ab}{a+b}$. 17. $\frac{(3a-b)c}{2b}$. 18. $\frac{p}{a+b}$.

19. $\frac{ab}{a+b-c}$. 20. $\frac{ab(a+b-2c)}{(a+b)c-a^2-b^2}$. 21. b. 22. $-\frac{a^2+b^2}{a+b}$.

23. $\frac{b^2+ac}{b^2+c^2}$. 24. $\frac{bc^2+ca^2+ab^2-a-b-c}{bc+ca+ab-1}$. 25. $-\frac{a+b}{2}$.

26. $\frac{a+b}{2}$. 27. $\frac{a+b+3}{2}$. 28. $\frac{ab}{a+b}$. 29. $\frac{ab-cd}{a+b-c-d}$.

30. $\frac{ab(c+d)-cd(a+b)}{cd-ab}$. 31. $\frac{ab}{a+b}$. 32. $\frac{2ab}{a-b}$.

33. $\frac{b(a+c)}{a-c}$. 34. $\frac{a^2+b^2}{a+b}$. 35. $-\frac{2ab}{c}$. 36. $a+3b+5c$.

37. a. 38. $\frac{r^3-mn}{m+n-2r}$. 39. $\frac{a}{3}$. 40. 0 or $-\frac{1}{2}(a+b)$.

41. $-b$. 42. $\frac{a+b+c+d}{m+n}$.

Ex. CXXXIV. (pp. 369-370.)

1. 9. **2.** 15. **3.** $\frac{3}{4}$. **4.** $\frac{2}{5}$. **5.** $\frac{9}{20}$. **6.** 2. **7.** 16. **8.** $-\frac{2}{5}$.

9. 4. **10.** $\frac{1}{2}$. **11.** $\frac{\sqrt{a}}{\sqrt{a+2}}$. **12.** $\frac{(n-1)^2 n}{2n-1}$.

13. $\frac{ab}{1-2\sqrt{b}}$. **14.** $\frac{1}{b-2}$. **15.** a^2-b.

16. $\frac{b(2a-b)}{2(a-b)}$. **17.** $\frac{(a-b)^2}{2b}$. **18.** $\frac{b(b-2a)}{3b-2a}$.

19. $\frac{9}{10}a$. **20.** 4. **21.** $-\frac{3}{28}$. **22.** $1\frac{4}{7}$. **23.** 8. **24.** 4.

25. $\frac{m(m+2k)}{2(m+k)}$. **26.** $\frac{1}{4}a$. **27.** $2a$. **28.** $\frac{4}{3}a$. **29.** 16. **30.** $1\frac{2}{3}$.

31. 5. **32.** -3. **33.** $\sqrt[4]{\{\frac{1}{4}d^2(4a^2-d^2)\}}$. **34.** $2\sqrt{(}$[illegible]$)$. **35.** $4a$.

Ex. CXXXV. (p. 372.)

1. 6. **2.** 25. **3.** 4. **4.** 5. **5.** $\frac{4a^2c}{4b^2+c^2}$. **6.** $\pm\frac{1}{2}\sqrt{3}$.

7. $1\frac{1}{4}$. **8.** $\frac{16}{47}$. **9.** $\frac{41}{30}$. **10.** $\frac{a(1+\sqrt{b})^2}{1+b}$.

11. $a\left\{1-\left(\frac{2\sqrt{b}}{b+1}\right)^2\right\}$. **12.** $\pm\sqrt{\{(a-2)^2-1\}}$. **13.** 2.

14. ± 4. **15.** $n\sqrt{(a-\frac{1}{4}n^2)}$.

Ex. CXXXVI. (pp. 375-377.)

1. $\frac{1}{1-a}$. **2.** $\frac{81}{a}$. **3.** $\frac{1}{a}\left\{b-\frac{nc}{n-1}\right\}^2$. **4.** $\frac{(b-2c)^2-ak}{a+3b-4c}$.

5. $\frac{1}{3}(a^2+b^2+c^2)$. **6.** 1. **7.** $\pm\sqrt{\left\{a^2-\frac{(b^3-2a)^3}{27b^3}\right\}}$.

8. $\pm\sqrt{(a^2+\frac{1}{3}b^2)}$. **9.** $\pm\frac{a^2}{a-b}\sqrt{\left(3-\frac{2b}{a}\right)}$. **10.** $-\frac{b^2}{4a}$.

11. $\frac{5a}{4}$. **12.** $\frac{1}{25}$. **13.** 8. **14.** $\frac{ab^{\frac{4}{5}}}{a^{\frac{4}{5}}-b^{\frac{4}{5}}}$.

15. $-a$. **16.** 0. **17.** $\frac{41a^2}{40b}$. **18.** $\pm\frac{1}{15}$.

19. $\frac{a-b}{2\sqrt{(ab)}}$. **20.** $\frac{1}{4}$. **21.** 9. **22.** 3. **23.** $\frac{1}{3}\sqrt{a}$.

24. $\frac{1}{1-a}$. **25.** $1\frac{2}{3}$. **26.** $\frac{ab}{a+b}$. **27.** $\frac{4ac^2}{b^2}$.

Ex. CXXXVII. (p. 378.)

26. **1.** 3. **2.** 1. **3.** [illegible] **4.** p, m. **5.** 4. **6.** 4.

40. **7.** 1. **8.** $2\frac{8}{11}$; $2\frac{4}{11}$. **9.** 3; 5. **10.** 1; 3. **11.** 4; [illegible]